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Research Article

Remarks on Extensions of the Himmelberg Fixed Point Theorem

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Recently, Jafari and Sehgal obtained an extension of the Himmelberg fixed point theorem based on the Kakutani fixed-point theorem. We give generalizations of the extension to almost convex sets instead of convex sets. We also give generalizations for a large class $\mathfrak{B}$ of better admissible multimaps instead of the Kakutani maps. Our arguments are based on the KKM principle and some of previous results due to the second author.

1. Introduction

In 1972, Himmelberg [1] derived the following from the Kakutani fixed point theorem.

Theorem 1.1. *Let T be a nonvoid convex subset of a separated locally convex space L. Let $F : T \to T$ be a u.s.c. multimap such that $F(x)$ is closed and convex for all $x \in T$, and $F(T)$ is contained in some compact subset C of T. Then F has a fixed point.*

Recall that Theorem 1.1 is usually called *the Himmelberg fixed point theorem* and is a common generalization of historically well-known fixed point theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, and Hukuhara (see [2]). Recall also that the multimap F is usually called a *Kakutani map*.

Recently, Jafari and Sehgal [3] obtained an extension of Theorem 1.1 based on the Kakutani fixed point theorem. Our aim in this paper is to give generalizations of the extension to almost convex sets instead of convex sets. We also give generalizations for a large class $\mathfrak{B}$ of better admissible multimaps instead of the Kakutani maps. Our arguments are based on the KKM principle and some results in [3–5].

2. Preliminaries

Recall that, for topological spaces X and Y, a multimap (simply, a map) $F : X \multimap Y$ is u.s.c. (resp., l.s.c.) if, for any closed (resp., open) subset $A \subset X$,

$$F^{-1}(A) := \{x \in X \mid F(x) \cap A \neq \varnothing\} \tag{2.1}$$

is closed (resp., open) in X. If Y is regular, F is u.s.c. and has nonempty closed values, then F has a closed graph.

Himmelberg [1] defined that a subset A of a t.v.s. E is said to be *almost convex* if, for any neighborhood V of the origin 0 in E and for any finite set $\{w_1,\ldots,w_n\}$ of points of A, there exist $z_1,\ldots,z_n \in A$ such that $z_i - w_i \in V$ for all i, and $\operatorname{co}\{z_1,\ldots,z_n\} \subset A$.

As the second author once showed in [6], the classical KKM principle implies many fixed point theorems. In [4], the following almost fixed point theorem was obtained from the KKM principle.

Theorem 2.1. *Let X be a subset of a t.v.s. and Y an almost convex dense subset of X. Let $T : X \multimap E$ be an l.s.c. (resp., a u.s.c.) map such that $T(y)$ is convex for all $y \in Y$. If there is a totally bounded subset K of $\overline{X}$ such that $T(y) \cap K \neq \varnothing$ for each $y \in Y$, then for any convex neighborhood V of the origin 0 of E, there exists a point $x_V \in Y$ such that $T(x_V) \cap (x_V + V) \neq \varnothing$.*

Note that a t.v.s. is not necessarily Hausdorff in Theorem 2.1. It is routine to deduce Theorem 1.1 from Theorem 2.1. In fact, in 2000, we had the following in [7].

Theorem 2.2. *Let X be a subset of a locally convex Hausdorff t.v.s. E and Y an almost convex dense subset of X. Let $T : X \multimap X$ be a compact u.s.c. map with nonempty closed values such that $T(y)$ is convex for all $y \in Y$. Then T has a fixed point.*

In particular, for $Y = X$, we obtain the following generalization [7] of Theorem 1.1.

Theorem 2.3. *Let X be an almost convex subset of a locally convex Hausdorff t.v.s. Then any compact u.s.c. map $T : X \multimap X$ with nonempty closed convex values has a fixed point in X.*

A *polytope* P in a subset X of a t.v.s. E is a subset of X homeomorphic to a standard simplex.

We define "better" admissible class $\mathfrak{B}$ of maps from a subset X of a t.v.s. E into a topological space Y as follows.

$F \in \mathfrak{B}(X,Y) \Leftrightarrow F : X \multimap Y$ is a map such that, for each polytope P in X and for any continuous function $f : F(P) \to P$, the composition $f(F|_P) : P \multimap P$ has a fixed point.

There is a large number of examples of better admissible maps (see [5]). A typical example is an *acyclic map*, that is, a u.s.c. map with compact acyclic values. It is also known that any u.s.c. map with compact values having a *trivial shape* (i.e., contractible in each neighborhood) belongs to $\mathfrak{B}(X,Y)$, see [8].

For a subset C of a t.v.s. E, we say that a multimap $F : C \multimap C$ has an *E-almost fixed point* if, for each neighborhood V of the origin 0 in E, there exist points $x_V \in C$ and $y_V \in F(x_V)$ such that $x_V - y_V \in V$ as in Theorem 2.1.

The following generalization of Theorems 1.1 and 2.3 is a consequence of the main theorem of [5], where $\mathfrak{B}^p$ should be replaced by $\mathfrak{B}$.

Theorem 2.4. *Let X be an almost convex subset of a locally convex t.v.s. E.*

(1) *If $F \in \mathfrak{B}(X,X)$ is compact, then F has an E-almost fixed point.*

(2) *Further, if E is Hausdorff and F is closed, then F has a fixed point.*

In what follows, let $E = (E,\tau)$ be a t.v.s. with topology τ, $\hat{E} = (E,\tau)^\wedge$ the completion of E, and E^* the topological dual of E. Recall that if E^* separates points of E, then (E,τ) is Hausdorff and (E,τ_w) with the weak topology is Hausdorff and locally convex. We will use the following lemmas in [3].

Lemma 2.5 [3, Lemma 2]. *If $(\hat{E})^*$ separates points of $\hat{E}$, then E^* separates points of E.*

Lemma 2.6 [3, Lemma 4]. *Let $(\hat{E})^*$ separate points of $\hat{E}$. Let K be a compact subset of E whose $\overline{\text{co}}K$ in $\hat{E}$ is $\hat{E}$-compact. If a net $\{x_\alpha\} \subset \text{co}K$ is such that for some $u \in K, \{x_\alpha\} \to u$ in (E,τ_w), then there is a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ with $\{x_\beta\} \to u$ in (E,τ).*

3. New fixed point theorems

Motivated by [3], we obtain the following main result of this paper.

Theorem 3.1. *Let E be a t.v.s., C an almost convex subset of E, and K a compact subset of C such that $\overline{\text{co}}K$ is $\hat{E}$-compact. Let $F : C \multimap K$ be a u.s.c. multimap such that*

(1) *for each $x \in C$, $F(x)$ is a nonempty closed subset of K;*

(2) *F has an (E,τ_w)-almost fixed point in K.*

If $(\hat{E},\tau_w)^*$ separates points of $\hat{E}$, then F has a fixed point in K.

Proof. We follow that of [3, Theorem 5]. Since $(\hat{E},\tau_w)^*$ separates points of $\hat{E}$, by Lemma 2.5, (E,τ_w) is a Hausdorff locally convex t.v.s. Let $\mathcal{U}$ be a neighborhood basis of the origin 0 of (E,τ_w) consisting of (E,τ_w)-closed convex and symmetric subsets of E. For each $V \in \mathcal{U}$, there exist points $x_V \in K$, $y_V \in F(x_V)$ such that $x_V - y_V \in V$. Partially order $\mathcal{U}$ by inclusion. Then $\{x_V - y_V \mid V \in \mathcal{U}\} \to 0$ in (E,τ_w). Since $\{y_V \mid V \in \mathcal{U}\} \subset K$, there exists a subnet $\{y_{V'} \mid V' \in \mathcal{U}' \subset \mathcal{U}\}$ and a $u \in K$ such that $\{y_{V'} \mid V' \in \mathcal{U}'\} \to u$ in E. Since $x_V - y_V \in V$, the net $\{x_{V'} \mid V' \in \mathcal{U}'\} \to u$ in (E,τ_w). Since $u \in K$ and $\{x_{V'} \mid V' \in \mathcal{U}'\} \subset \text{co}K$, it follows by Lemma 2.6 that there is a subnet $\{x_{V''}\}$ of $\{x_{V'}\}$ with $\{x_{V''}\} \to u$ in E. Hence $\{y_{V''}\} \to u$ in E also. Since K is regular and F is u.s.c. with closed values, F has a closed graph. Since, for each V'', $y_{V''} \in F(x_{V''})$, we have $u \in F(u)$. This completes our proof. □

From Theorem 2.1, we immediately have the following.

Theorem 3.2. *Let E be a t.v.s. such that (E,τ_w) is a locally convex t.v.s., C an almost convex subset of E, and K a (E,τ_w)-totally bounded subset of C. Let $F : C \multimap K$ be a u.s.c. (resp., an l.s.c.) multimap such that for each $x \in C$, $F(x)$ is a nonempty convex subset of K. Then F has an (E,τ_w)-almost fixed point in K.*

Combining Theorems 3.1 and 3.2, we have the following.

Theorem 3.3. *Let E be a t.v.s., C an almost convex subset of E, and K a compact subset of C such that $\overline{\text{co}}K$ is $\hat{E}$-compact. Let $F : C \multimap K$ be a u.s.c. multimap such that for each $x \in C$, $F(x)$ is a nonempty closed and convex subset of K. If $(\hat{E},\tau_w)^*$ separates points of $\hat{E}$, then F has a fixed point in K.*

When C is convex, Theorem 3.3 reduces to the main theorem of Jafari and Sehgal [3]. As noted in [3], if E is a locally convex Hausdorff t.v.s., then so is $\hat{E}$ and hence $(\hat{E})^*$ separates points of $\hat{E}$. Consequently, Theorem 2.2 follows from Theorem 3.3.

From Theorem 2.4, we immediately have the following.

Theorem 3.4. *Let E be a t.v.s. such that (E,τ_w) is a locally convex t.v.s., C an almost convex subset of E, and K an (E,τ_w)-compact subset of C. If $F \in \mathfrak{B}(C,K)$, then F has an (E,τ_w)-almost fixed point in K.*

Combining Theorems 3.1 and 3.4, we have the following.

Theorem 3.5. *Let E be a t.v.s., C an almost convex subset of E, and K a compact subset of C such that $\overline{\text{co}}K$ is $\hat{E}$-compact. Let $F \in \mathfrak{B}(C,K)$. If $(\hat{E},\tau_w)^*$ separates points of $\hat{E}$, then F has a fixed point in K.*

As noted in [3], if E is a locally convex Hausdorff t.v.s., then so is $\hat{E}$ and hence $(\hat{E})^*$ separates points of $\hat{E}$. Consequently, Theorem 2.4 follows from Theorem 3.5.

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