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Research Article

Fixed Points of Weakly Contractive Maps and Boundedness of Orbits

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We discuss weakly contractive maps on complete metric spaces. Following three methods of generalizing the Banach contraction principle, we obtain some fixed point theorems under some relatively weaker and more general contractive conditions.

1. Introduction

The Banach contraction principle is one of the most fundamental fixed point theorems.

Theorem 1.1 (Banach contraction principle). *Let (X,d) be a complete metric space, and let $f : X \to X$ be a map. If there exists a constant $c \in [0,1)$ such that*

$$d(f(x),f(y)) \le c \cdot d(x,y), \tag{C.1}$$

then f has a unique fixed point u, and $\lim_{n\to\infty} f^n(y) = u$ *for each $y \in X$.*

Since the publication of this result, various authors have generalized and extended it by introducing weakly contractive conditions. In [1], Rhoades gathered 25 contractive conditions in order to compare them and obtain fixed point theorems. Collaço and Silva [2] presented a complete comparison for the maps numbered (1)–(25) by Rhoades [1].

One of the methods of alternating the Banach contractive condition is not to compare $d(f(x),f(y))$ with $d(x,y)$, but compare $d(f^p(x),f^q(y))$ with the distances between any two points in $O_p(x,f) \cup O_q(y,f)$, where $p \ge 1$ and $q \ge 1$ are given integers, and $O_p(x,f) \equiv \{x,f(x),\dots,f^p(x)\}$ (e.g., see [3–6]).

The generalized banach contraction conjecture was established in [7–10], of which the contractive condition is $\min\{d(f^k(x), f^k(y)) : 1 \le k \le J\} \le c \cdot d(x,y)$, where J is a positive integer.

A further method of alternating the Banach contractive condition is to change the constant $c \in [0,1)$ in the contractive condition into a function (e.g., see [11–14]).

The third method of alternating the Banach contractive condition is to compare not only $d(f^p(x), f^q(y))$ with the distances between any two points in $O_p(x,f) \cup O_q(y,f)$, but also $d(f^p(x), f^q(y))$ with the distances between any two points in $O(x,f) \cup O(y,f)$, where $O(x,f) \equiv \{f^n(x) : n = 0,1,2,\dots\}$ (e.g., see [6, 15, 16]).

Following the above three methods of generalizing the Banach contraction principle, we present some of fixed point theorems under some relatively weaker and more general conditions.

2. Weakly contractive maps with the infimum of orbital diameters being 0

Throughout this paper, we assume that (X,d) is a complete metric space, and $f : X \to X$ is a map. Given a subset X_0 of X, denote by $\operatorname{diam}(X_0)$ the diameter of X_0, that is, $\operatorname{diam}(X_0) = \sup\{d(x,y) : x,y \in X_0\}$. For any $x \in X$, write $O(x) = O(x,f) = \{x, f(x), f^2(x), \dots\}$. $O(x)$ is called the orbit of x under f. $O(x)$ is usually regarded as a set of points, while sometimes it is regarded as a sequence of points. Denote by $\mathbb{Z}_+$ the set of all nonnegative integers, and denote by $\mathbb{N}$ the set of all positive integers. For any $n \in \mathbb{N}$, write $\mathbb{N}_n = \{1,\dots,n\}$. For $n \in \mathbb{Z}_+$, write $\mathbb{Z}_n = \{0,1,\dots,n\}$, and $O_n(x) = O_n(x,f) = \{x, f(x), \dots, f^n(x)\}$.

For any given map $f : X \to X$, define $\rho : X \to [0,\infty]$ as follows:

$$\rho(x) = \operatorname{diam}(O(x,f)) = \sup\{d(f^i(x), f^j(x)) : i,j \in \mathbb{Z}_+\} \quad \text{for any } x \in X. \qquad (*)$$

Definition 2.1 (see [16]). Let (X,d) be a metric space, and let $f : X \to X$ be a map. If for any sequence $\{x_n\}$ in X, $\lim_{n\to\infty} \rho(x_n) = \rho(x)$ whenever $\lim_{n\to\infty} x_n = x$, then ρ is called to be closed, and f is called to have closed orbital diametral function.

That f has closed orbital diametral function means $\rho : X \to [0,\infty]$ is continuous. It is easy to see that "f is continuous" and "f has closed orbital diametral function" do not imply each other.

Theorem 2.2. *Let (X,d) be a complete metric space, and suppose that $f : X \to X$ has closed orbital diametral function or $f : X \to X$ is continuous. If there exist a nonnegative real number s, an increasing function $\mu : (0,\infty) \to (0,1]$, and a family of functions $\{\gamma_{ij} : X \times X \to [0,1) : i,j = 0,1,2,\dots\}$ such that, for any $x, y \in X$,*

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \gamma_{ij}(x,y) \le 1 - \mu(d(x,y)), \tag{2.1}$$

$$d(f(x), f(y)) \le s \cdot [\rho(x) + \rho(y)] + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \gamma_{ij}(x,y) d(f^i(x), f^j(y)), \tag{2.2}$$

then f has a unique fixed point if and only if $\inf\{\rho(x) : x \in X\} = 0$.

Proof. The necessity is obvious. Now we show the sufficiency.

For each $n \in \mathbb{N}$, since $\inf\{\rho(x) : x \in X\} = 0$, we can choose a point $v_n \in X$ such that $\rho(v_n) < 1/n$. We claim that $v_1, v_2, \ldots$ is a Cauchy sequence of points. In fact, if $v_1, v_2, \ldots$ is not a Cauchy sequence of points, then there exists $\delta > 0$ such that, for any $k \in \mathbb{N}$, there are $i, j \in \mathbb{N}$ with $i > j > k$ satisfying $d(v_i, v_j) > 3\delta$. Let $\mu_0 = \mu(\delta)$. Choose $k \in \mathbb{N}$ such that $2(s+1)/k < \delta\mu_0/2$, and choose $n > m > k$ such that $d(v_n, v_m) > 3\delta$. Then for any $x \in O(v_n)$ and any $y \in O(v_m)$, we have

$$d(x,y) \geq d(v_n, v_m) - \rho(v_n) - \rho(v_m) > 3\delta - \frac{1}{n} - \frac{1}{m} > \delta, \tag{2.3}$$

this implies $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\gamma_{ij}(x,y) \leq 1 - \mu(d(x,y)) \leq 1 - \mu_0$, and hence

$$\begin{aligned}
d(f(x), f(y)) &\leq s \cdot [\rho(x) + \rho(y)] + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\gamma_{ij}(x,y)[d(x,y) + \rho(x) + \rho(y)] \\
&< (s+1)[\rho(x) + \rho(y)] + (1-\mu_0)d(x,y) \\
&\leq (s+1)[\rho(v_n) + \rho(v_m)] + (1-\mu_0)d(x,y) \\
&< \frac{2(s+1)}{k} + (1-\mu_0)d(x,y) < \frac{\delta\mu_0}{2} + (1-\mu_0)d(x,y) \\
&< \left(1 - \frac{\mu_0}{2}\right)d(x,y).
\end{aligned} \tag{2.4}$$

It follows from (2.4) that $\lim_{i\to\infty} d(f^i(v_n), f^i(v_m)) = \lim_{i\to\infty}(1-\mu_0/2)^i \cdot d(v_n, v_m) = 0$. But this contradicts (2.3).

Thus $v_1, v_2, \ldots$ must be a Cauchy sequence of points. We may assume that it converges to w.

Case 1. If f has closed orbital diametral function, then the function ρ is closed. Noting that $\rho(v_n) < 1/n$, we have $\rho(w) = \lim_{n\to\infty}\rho(v_n) = 0$, which implies that w is a fixed point of f.

Case 2. If f is continuous, then $\lim_{n\to\infty} f(v_n) = f(w)$. Since $d(v_n, f(v_n)) \leq \rho(v_n) < 1/n$, we get $\lim_{n\to\infty} d(v_n, f(v_n)) = 0$, and then $d(w, f(w)) = 0$. Hence w is a fixed point of f.

Thus in both cases w is a fixed point of f.

Suppose u is also a fixed point of f. If $u \neq w$, then by (2.2) and (2.1) we can obtain $d(u,w) = d(f(u), f(w)) \leq s \cdot (0+0) + [1 - \mu(d(u,w))] \cdot d(u,w) < d(u,w)$, which is a contradiction. Hence $u = w$, and w is the unique fixed point of f. Theorem 2.2 is proved. □

Theorem 2.3. *Let (X,d) be a complete metric space, and suppose that $f : X \to X$ has closed orbital diametral function or $f : X \to X$ is continuous. If there exist $s \geq 0$ and $t \in [0,1)$ such that, for any $x, y \in X$,*

$$d(f(x), f(y)) \leq s \cdot [\rho(x) + \rho(y)] + t \cdot \max\{d(f^i(x), f^j(y)) : i \in \mathbb{Z}_+, j \in \mathbb{Z}_+\}, \tag{2.5}$$

then f has a unique fixed point if and only if $\inf\{\rho(x) : x \in X\} = 0$.

The proof of Theorem 2.3 is similar to that of Theorem 2.2, and is omitted.

In [16], Sharma and Thakur discussed the condition

$$\begin{aligned} d(f(x),f(y)) \le\ & ad(x,y)+b[d(x,f(x))+d(y,f(y))] \\ & +c[d(x,f(y))+d(y,f(x))]+e[d(x,f^2(x))+d(y,f^2(y))] \\ & +g[d(f(x),f^2(x))+d(f(y),f^2(x))], \end{aligned} \tag{C}$$

where a, b, c, e, g are all nonnegative real numbers with $3a+2b+4c+5e+3g \le 1$.

In Theorem 2.2, set $s=b+e+g$, $\mu \equiv 1-(a+2c+g)$, $\gamma_{00} \equiv a$, $\gamma_{01}=\gamma_{10} \equiv c$, $\gamma_{21} \equiv g$, and $\gamma_{ij} \equiv 0$, otherwise. Then (C) implies (2.2). In Theorem 2.3, set $s=b+e+g$, and $t=a+2c+g$. Then (C) implies (2.5), too. Thus, by each of Theorems 2.2 and 2.3, we can obtain the following theorem, which improves the main result of Sharma and Thakur [16].

Theorem 2.4. *Suppose that (X,d) is a complete metric space, and $f: X \to X$ has closed orbital diametral function. If (C) holds for any $x,y \in X$ with $a+2c+g<1$, then $\inf\{\rho(x): x \in X\}=0$ if and only if f has a fixed point.*

3. Weakly contractive maps with an orbit on which the moving distance being bounded

In Theorems 2.2 and 2.3, to determine whether f has a fixed point or not, we need the condition that the infimum of orbital diameters is 0. In the following, we will not rely on this condition and discuss some contractive maps whose contractive conditions are still relatively weak. Throughout this section, we assume that $f: X \to X$ is continuous.

Let $f: X \to X$ be a given map. For any integers $i \ge 0$, $j \ge 0$, and $x, y \in X$, write

$$\begin{aligned} d_{ij}(x) &= d_{ijf}(x) = d(f^i(x), f^j(x)), \\ d_{ij}(x,y) &= d_{ijf}(x,y) = d(f^i(x), f^j(y)). \end{aligned} \tag{$*'$}$$

Definition 3.1. Let $Y \subset X$, $k \in \mathbb{N}$, and $g: X \to X$ be a self-mapping. If $\sup\{d(g^k(y),y): y \in Y\} < \infty$, then the moving distance of g^k on Y is bounded.

Obviously, we have the following.

Proposition 3.2. *Let $m \in \mathbb{N}$. If $g(Y) \subset Y$ and the moving distance of g on Y is bounded, then the moving distance of g^m on Y is also bounded.*

However, the converse of the above proposition does not hold. In fact, we have the following counterexample.

Example 3.3. Let $\mathbb{R}=(-\infty,+\infty)$. Define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = -x \quad \text{for } x \in \mathbb{R}. \tag{3.1}$$

It is easy to see that the moving distance of f^2 on $\mathbb{R}$ is bounded (equal to 0), while the moving distance of f on $\mathbb{R}$ is unbounded.

Theorem 3.4. *Let m, n be two given positive integers, and let $d_{ij}(x)$ be defined as in $(*')$. Suppose there exist nonnegative real numbers $a_0, a_1, a_2, \ldots$ with $\sum_{i=0}^{\infty} a_i < 1$ such that*

$$d_{n+m,n}(x) \le \sum_{i=0}^{\infty} a_i d_{i+m,i}(x) \quad \forall x \in X. \tag{3.2}$$

Then the following statements are equivalent:

(1) *f has a periodic point with period being some factor of m;*
(2) *there is an orbit $O(v,f)$ such that the moving distance of f^m on $O(v,f)$ is bounded;*
(3) *f has a bounded orbit.*

Proof. (1)⇒(3)⇒(2) is clear. Now we prove (2)⇒(1). Let $a = \sum_{i=0}^{\infty} a_i$, then $a \in [0,1)$. If $a = 0$, then (2)⇒(1) holds obviously, and hence we may assume $a \in (0,1)$. Let $b_i = a_i/a$, then $\sum_{i=0}^{\infty} b_i = 1$. By (3.2) we get

$$d_{n+m,n}(x) \le a \cdot \sum_{i=0}^{\infty} b_i d_{i+m,i}(x) \quad \text{for any } x \in X. \tag{3.3}$$

Assume $\{d(f^m(y),y) : y \in O(v,f)\}$ is bounded. We claim that

$$d_{n+m,n}(v) \le a \cdot \max\{d_{i+m,i}(v) : i \in \mathbb{Z}_{n-1}\}. \tag{3.4}$$

In fact, if (3.4) does not hold, then by (3.3) there exists $j > n$ such that

$$\begin{gathered} d_{j+m,j}(v) \ge \frac{1}{a} \cdot d_{n+m,n}(v) > 0, \\ d_{i+m,i}(v) < \frac{1}{a} \cdot d_{n+m,n}(v), \quad i = 0,1,\ldots,j-1. \end{gathered} \tag{3.5}$$

Combining (3.5) we obtain

$$d_{j+m,j}(v) > a \cdot \max\{d_{i+m,i}(v) : i \in \mathbb{Z}_{j-1}\}. \tag{3.6}$$

Similarly, we can obtain an infinite sequence of integers $j_0 < j_1 < j_2 < \cdots$ satisfying

$$d_{j_k+m,j_k}(v) \ge \frac{1}{a} \cdot d_{j_{k-1}+m,j_{k-1}}(v), \quad k = 1,2,3,\ldots. \tag{3.7}$$

However, this contradicts to that $\{d(f^m(y),y) : y \in O(v,f)\}$ is bounded. Therefore, (3.4) must hold.

For any $k \in \mathbb{Z}_+$, $O(f^k(v),f) \subseteq O(v,f)$. Replacing v in (3.4) with $f^k(v)$, we have

$$d_{n+m+k,n+k}(v) \le a \cdot \max\{d_{i+m+k,i+k}(v) : i \in \mathbb{Z}_{n-1}\}. \tag{3.8}$$

Write $b = \max\{d_{i+m,i}(v) : i \in \mathbb{Z}_{n-1}\}$. For $j = 0,1,2,\ldots,n-1$, by (3.8) we can successively get

$$\begin{gathered} d_{n+j+m,n+j}(v) \le ab, \\ d_{2n+j+m,2n+j}(v) \le a^2 b, \\ \vdots \end{gathered} \tag{3.9}$$

In general, we have

$$d_{kn+j+m,kn+j}(v) \le a^k b, \quad k = 1,2,\ldots. \tag{3.10}$$

Therefore, it follows from $0 < a < 1$ and (3.10) that $v, f^m(v), f^{2m}(v), f^{3m}(v),\ldots$ is a Cauchy sequence. We may assume it converges to $w \in X$. Then $f^m(w) = w$, and hence w is a periodic point of f with period being some factor of m. Theorem 3.4 is proved. □

As a corollary of Theorem 3.4, we have the following.

Theorem 3.5. *Let n be a given positive integer, and let $d_{ij}(x)$ be defined as in $(*')$. Suppose there exist nonnegative real numbers $a_0, a_1, a_2, \ldots$ with $\sum_{i=0}^{\infty} a_i < 1$ such that*

$$d_{nn}(x,y) \le \sum_{i=0}^{\infty} a_i d_{ii}(x,y) \quad \text{for any } x, y \in X. \tag{3.11}$$

Then the following statements are equivalent:

(1) *f has a fixed point;*

(2) *f has an orbit $O(v,f)$ such that for some $m \in \mathbb{N}$ the moving distance of f^m on $O(v,f)$ is bounded; and*

(3) *f has a bounded orbit.*

Proof. (1)⇒(3)⇒(2) is clear. It remains to prove (2)⇒(1). Suppose the moving distance of f^m on $O(v,f)$ is bounded. Let $x = f^m(v)$, $y = v$, then (3.11) implies (3.2). Therefore, by Theorem 3.4, there exists $w \in X$ such that $f^m(w) = w$.

Since $O(w,f)$ is a finite set, there exist $p, q \in \mathbb{N}$ such that $d_{pq}(w) = \rho(w)$. By (3.11) we have

$$\begin{aligned} \rho(w) &= d_{pq}(w) = d_{nn}\left(f^{(m-1)n+p}(w), f^{(m-1)n+q}(w)\right) \\ &\le \sum_{i=0}^{\infty} a_i d_{ii}\left(f^{(m-1)n+p}(w), f^{(m-1)n+q}(w)\right) \le \left(\sum_{i=0}^{\infty} a_i\right) \cdot \rho(w). \end{aligned} \tag{3.12}$$

Therefore, it follows from $\sum_{i=0}^{\infty} a_i < 1$ that $\rho(w) = 0$. Hence w is a fixed point of f. Theorem 3.5 is proved. □

Remark 3.6. In Theorem 3.5, from (3.11) it follows that f has at most one fixed point, and f has no other periodic point except this point.

Remark 3.7. Equation (3.11) implies that

$$d_{nn}(x,y) \le \left(\sum_{i=0}^{\infty} a_i\right) \operatorname{diam}\big(O(x,f) \cup O(y,f)\big) \quad \text{for any } x,y \in X, \tag{3.13}$$

which is still a particular case of the condition $(C3)$ introduced by Walter [6]. However, all orbits of f are assumed to be bounded in Walter's [6, Theorem 1], while it suffices to assume that f has a bounded orbit in Theorem 3.5. Thus, Theorem 3.5 cannot be deduced from [6, Theorem 1] as a particular case.

Example 3.8. Let $X = [0,+\infty) \subset \mathbb{R}$, and let $f(x) = 2x$ for any $x \in X$. It is easy to see that $O(0,f)$ is the unique bounded orbit of f, and for $n = 1$, (3.11) is satisfied with $a_i = (1/2^{2i+1})$ $(i = 0,1,2,\ldots)$.

Theorem 3.9. *Let m, n be two given positive integers, $v \in X$, and let $d_{ij}(x)$ be defined as in $(*')$. Suppose there exist nonnegative real numbers $a_0,a_1,a_2,\ldots,a_{n-1}$ with $\sum_{i=0}^{n-1} a_i \le 1$ such that*

$$d_{n+m,n}(x) \le \sum_{i=0}^{n-1} a_i d_{i+m,i}(x) \quad \textit{for any } x \in O(v,f). \tag{3.14}$$

Then the moving distance of f^m on $O(v,f)$ is bounded.

Proof. Write $b = \max\{d_{i+m,i}(v) : i \in \mathbb{Z}_{n-1}\}$. Let $a = \sum_{i=0}^{n-1} a_i$, then $a \in [0,1]$. Without loss of generality, we may assume, by increasing one of the numbers $a_0,a_1,a_2,\ldots,a_{n-1}$ if necessary, that $a = 1$. For $j = n,n+1,n+2,\ldots$, by (3.14) we can successively get

$$d_{j+m,j}(v) \le b. \tag{3.15}$$

By (3.15) we have $d(f^m(y),y) \le b$ for any $y \in O(v,f)$. Therefore, the moving distance of f^m on $O(v,f)$ is bounded. Theorem 3.9 is proved. □

By Theorems 3.9 and 3.4, we can immediately obtain the following.

Corollary 3.10. *Let m, n be two given positive integers, and let $d_{ij}(x)$ be defined as in $(*')$. Suppose there exist nonnegative real numbers $a_0,a_1,a_2,\ldots,a_{n-1}$ with $\sum_{i=0}^{n-1} a_i < 1$ such that*

$$d_{n+m,n}(x) \le \sum_{i=0}^{n-1} a_i d_{i+m,i}(x) \quad \textit{for any } x \in X. \tag{3.16}$$

Then f has a periodic point with period being some factor of m.

Corollary 3.11. *Let m, n be two given positive integers, $v \in X$, and let $d_{ij}(x)$ be defined as in $(*')$. Suppose there exist nonnegative real numbers $a_0,a_1,a_2,\ldots,a_{n-1}$ and $b_0,b_1,b_2,\ldots$*

with $\sum_{i=0}^{n-1} a_i \leq 1$ *and* $\sum_{j=0}^{\infty} b_j < 1$ *such that, for any* $x \in O(v,f)$,

$$d_{n+m,n}(x) \leq \min\left\{\sum_{i=0}^{n-1} a_i d_{i+m,i}(x), \sum_{j=0}^{\infty} b_j d_{j+m,j}(x)\right\}. \tag{3.17}$$

Then the moving distance of f *on* $O(v,f)$ *is bounded.*

Proof. It follows from (3.17) and Theorem 3.9 that the moving distance of f^m on $O(v,f)$ is bounded. Therefore, by (3.17) and the proof of Theorem 3.4, v, $f^m(v), f^{2m}(v), \ldots$ converges to a k-period point w of f, where k is a factor of m. Hence $(v, f(v), f^2(v), \ldots)$ (regarded as a sequence of points) converges to the periodic orbit $O(w,f)$. Thus $O(v,f)$ is bounded, and the moving distance of f on $O(v,f)$ is bounded. Corollary 3.11 is proved. □

Coefficients in the preceding contractive conditions (3.2), (3.11), (3.14), (3.16), and (3.17) are all constants. Now we discuss the cases in which coefficients are variables.

Theorem 3.12. *Let* m, n *be two given positive integers, and let* $d_{ij}(x)$ *be defined as in* $(*')$. *If there exists a decreasing function* $\gamma_i : [0,\infty) \to [0,1]$ *for each* $i \in \mathbb{Z}_+$ *satisfying*

$$\sum_{i=0}^{\infty} \gamma_i(t) < 1 \quad \text{for any } t > 0, \tag{3.18}$$

such that

$$d_{n+m,n}(x) \leq \sum_{i=0}^{\infty} \gamma_i(d_{i+m,i}(x)) \cdot d_{i+m,i}(x) \quad \text{for any } x \in X, \tag{3.19}$$

then $\lim_{i\to\infty} d_{i+m,i}(v) = 0$ *for any* $v \in X$ *if and only if the moving distance of* f^m *on* $O(v,f)$ *is bounded.*

Proof. The necessity is obvious. Now we show the sufficiency. For any $i \in \mathbb{Z}_+$, we may assume $\gamma_i(0) = \lim_{t\to+0} \gamma_i(t)$, and

$$\gamma_i(t) \geq \frac{\gamma_i(0)}{2} \quad \text{for any } t > 0. \tag{3.20}$$

In fact, if it is not true, we may define $\gamma_i' : [0,\infty) \to [0,1]$ by $\gamma_i'(0) = \lim_{t\to+0} \gamma_i(t)$ and $\gamma_i'(t) = \max\{\gamma_i(t), \gamma_i'(0)/2\}$ (for any $t > 0$), and replace γ_i' with γ_i, then both (3.18) and (3.19) still hold.

Let $c = \limsup_{i\to\infty} d_{i+m,i}(v)$. Since $\{d(f^m(y), y) : y \in O(v,f)\}$ is bounded, $c < \infty$. Assume $c > 0$. Let $a_i = \gamma_i(c/2)$, and $a = \sum_{i=0}^{\infty} a_i$, then $a < 1$. Choose $\delta > 0$ such that $a(c+\delta) < c - \delta$. Choose an integer $k > n$ such that $d_{k+m,k}(v) > c - \delta$ and $\sup\{d_{j+m,j}(v) : j \geq k - n\} < c + \delta$. Write

$$\begin{aligned} M_1 &= \left\{ i \geq 0 : d_{i+m,i}(f^{k-n}(v)) > \frac{c}{2} \right\}, \\ M_2 &= \left\{ i \geq 0 : d_{i+m,i}(f^{k-n}(v)) \leq \frac{c}{2} \right\}. \end{aligned} \tag{3.21}$$

By (3.19) we get

$$c-\delta < d_{k+m,k}(v) = d_{n+m,n}\big(f^{k-n}(v)\big) \le \sum_{i=0}^{\infty} \gamma_i d_{i+m,i}\big(f^{k-n}(v)\big)\cdot d_{i+m,i}\big(f^{k-n}(v)\big)$$
$$= \left(\sum_{i\in M_1} + \sum_{i\in M_2}\right)\gamma_i\big(d_{i+m,i}\big(f^{k-n}(v)\big)\big)\cdot d_{i+m,i}\big(f^{k-n}(v)\big)$$
$$\le \sum_{i\in M_1}\gamma_i\left(\frac{c}{2}\right)\cdot(c+\delta) + \sum_{i\in M_2}\gamma_i(0)\cdot\frac{c}{2} \le \sum_{i=0}^{\infty}\gamma_i\left(\frac{c}{2}\right)\cdot(c+\delta) = a(c+\delta) < c-\delta, \tag{3.22}$$

which is a contradiction. Thus we have $c=0$. Theorem 3.12 is proved. □

Remark 3.13. In Theorem 3.12, if (3.2) does not hold, then only by (3.18) and (3.19) it is not enough to deduce that f has periodic points. Now we present such a counterexample.

Example 3.14. Let $X=\{\sqrt{n}: n\in\mathbb{N}\}$, then X is a complete subspace of the Euclidean space $\mathbb{R}$. Define $f: X\to X$ by $f(\sqrt{n})=\sqrt{n+1}$ (for any $n\in\mathbb{N}$), then f is uniformly continuous. For any $k\ge 1$, take $\gamma_k(t)\equiv 0$ (for any $t>0$). Let $c_k=(\sqrt{m+n+k}-\sqrt{n+k})/(\sqrt{m+k}-\sqrt{k})$, then $\{c_k\}_{k=1}^{\infty}$ is an increasing sequence. Choose arbitrarily a decreasing function $\gamma_0:[0,\infty)\to[0,1]$ such that $\gamma_0(\sqrt{m+k}-\sqrt{k})=c_k$, then both (3.18) and (3.19) hold for any $x\in X$. However, it is clear that f has no periodic points.

4. Weakly contractive maps with bounded orbits

Throughout this section, we assume that (X,d) is a complete metric space, and $f: X\to X$ is a continuous map. For any given f, let $d_{ij}(x,y)$ be defined as in $(*')$.

Theorem 4.1. *Let p, q be two given positive integers. Assume there exist decreasing functions $\gamma_{ij}:[0,\infty)\to[0,1]$ for all $(i,j)\in\mathbb{Z}_+^2$ satisfying*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\gamma_{ij}(t) < 1 \quad \text{for any } t>0, \tag{4.1}$$

such that

$$d_{pq}(x,y) \le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\gamma_{ij}\big(d_{ij}(x,y)\big)\cdot d_{ij}(x,y) \quad \text{for any } x,y\in X. \tag{4.2}$$

Then f has at most one fixed point, and f has a fixed point if and only if f has a bounded orbit.

Proof. It follows from (4.2) that f has at most one fixed point. If f has a fixed point w, then $O(w,f)$ is bounded. Conversely, suppose f has a bounded orbit $O(v,f)$. Write $v_i=f^i(v)$. Let $c=\lim_{i\to\infty}\rho(v_i)$, then $c<\infty$. If $(v,v_1,v_2,\ldots)$ is not a Cauchy sequence of points, then $c>0$. Analogous to the proof of Theorem 3.12, we may assume $\gamma_{ij}(t)\ge\gamma_{ij}(0)/2$ for any $(i,j)\in\mathbb{Z}_+^2$ and $t>0$. Let $a=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\gamma_{ij}(c/2)$, then $a<1$. Choose $\delta>0$ such that

$a(c+\delta) < c-\delta$. Choose $n > k > p+q$ such that $d(v_n, v_k) > c-\delta$ and $\rho(O(v_{k-p-q}, f)) < c+\delta$. By (4.2) we get

$$c-\delta < d(v_n, v_k) = d_{pq}(v_{n-p}, v_{k-q}) \leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \gamma_{ij}\big(d_{ij}(v_{n-p}, v_{k-q})\big) \cdot d_{ij}(v_{n-p}, v_{k-q}). \tag{4.3}$$

Furthermore, similar to (3.22), splitting the sum on the right of (4.3) into two sums according to whether $d_{ij}(v_{n-p}, v_{k-q})$ is greater than $c/2$ or not, we get

$$c-\delta < \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \gamma_{ij}\left(\frac{c}{2}\right) \cdot (c+\delta) = a(c+\delta) < c-\delta, \tag{4.4}$$

which is a contradiction. Thus $v, v_1, v_2, \ldots$ is a Cauchy sequence of points. Assume it converges to w. By (4.2) we have

$$\lim_{i\to\infty} d(v_{i+1}, v_i) = \lim_{i\to\infty} d_{p,q}(v_{i+1-p}, v_{i-q}) = 0. \tag{4.5}$$

Therefore, by the continuity of f we conclude that w is a fixed point of f. Theorem 4.1 is proved. □

Appendix

Weakly contractive maps with the infimum of orbital diameters being 0 were also discussed in [17], of which the following two theorems are the main results.

Theorem A.1 (see [17, Theorem 2]). *Suppose that (X,d) is a complete metric space, and $f : X \to X$ is a continuous map. Assume there exist $a_i \geq 0$ $(i = 0,1,\ldots,10)$ satisfying*

$$3a_0 + a_1 + a_2 + 2a_3 + 2a_4 + 2a_5 + 3a_6 + a_7 + 2a_8 + 4a_9 + 6a_{10} \leq 1 \tag{A.1}$$

such that, for any $x, y \in X$,

$$\begin{aligned} d(f(x), f(y)) \leq{}& a_0 d(x,y) + a_1 d(x, f(x)) + a_2 d(y, f(y)) + a_3 d(x, f(y)) \\ &+ a_4 d(y, f(x)) + a_5 d(x, f^2(x)) + a_6 d(y, f^2(x)) + a_7 d(f(x), f^2(x)) \\ &+ a_8 d(f(y), f^2(x)) + a_9 d(f^2(y), f^3(x)) + a_{10} d(f^3(y), f^4(x)). \end{aligned} \tag{A.2}$$

Then the following three statements are equivalent:

(1) f *has a fixed point;*

(2) $\inf\{d(x,f(x)) : x \in X\} = 0$*;*

(3) $\inf\{\rho(x) : x \in X\} = 0$.

Theorem A.2 (see [17, Theorem 4]). *Suppose that* (X,d) *is a complete metric space, and* $f : X \to X$ *is a continuous map. Assume there exist* $c_i \geq 0$ $(c_i = 0,1,\dots,6)$ *and* $b_j \geq 0$ $(j = 0,1,\dots,k)$ *satisfying*

$$3c_0 + c_1 + c_2 + 2c_3 + 2c_4 + c_5 + 3c_6 + 2b_0 + 2\sum_{j=1}^{k} jb_j \leq 1 \tag{A.3}$$

such that, for any $x, y \in X$,

$$\begin{aligned} d(f(x),f(y)) \leq{} & c_0 d(x,y) + c_1 d(x,f(x)) + c_2 d(y,f(y)) \\ & + c_3 d(x,f(y)) + c_4 d(x,f^2(x)) + c_5 d(f(x),f^2(x)) \\ & + c_6 d(y,f^2(x)) + \sum_{j=0}^{k} b_j d(f^j(y),f^{j+1}(x)). \end{aligned} \tag{A.4}$$

Then the following three statements are equivalent:

(i) f *has a fixed point;*

(ii) $\inf\{d(x,f(x)) : x \in X\} = 0$*;*

(iii) $\inf\{\rho(x) : x \in X\} = 0$.

The equivalence of (1) (or (i)) and (3) (or (iii)) follows from our Theorem 2.2 or Theorem 2.3. However, (2) (or (ii)) is not equivalent to each of (1) (or (i)) and (3) (or (iii)). Thus, there are some mistakes in the main results of [17]. In fact, we have such a counterexample.

Example A.3. Let $X = \{x_{ij} : i, j \in \mathbb{N}\}$. Define $f : X \to X$ by $f(x_{ij}) = x_{i+1,j}$ (for any $i, j \in \mathbb{N}$). Define a metric d on X as follows:

$$d(x_{ij},x_{mn}) = d(x_{mn},x_{ij}) = \begin{cases} 0, & \text{if } i = m,\ j = n; \\ \dfrac{1}{n}, & \text{if } j = n,\ i = 1,\ m = 2; \\ 1, & \text{otherwise.} \end{cases} \tag{A.5}$$

Then (X,d) is a discrete space. Thus X is complete, f is continuous, and $\inf_{x \in X} d(x, f(x)) = \inf_{n \in \mathbb{N}} d(x_{1n},x_{2n}) = \inf_{n \in \mathbb{N}} 1/n = 0$. Let $c_5 = a_7 \geq 0$ be a real number, and let other coefficients a_i, c_j, and b_k be all 0, then both (A.1) and (A.3) hold. For the given (X,d) and $f : X \to X$, since $d(f(x),f(y)) \leq 1$, and $c_5 d(f(x),f^2(x)) = a_7 d(f(x),f^2(x)) = 1$, both (A.2) and (A.4) hold, too. However, it is clear that f has no fixed points, and each of its orbital diameter is 1. Thus, in [17], the condition (2) (or (ii)) in Theorems 2 and 4 does not imply each of (1) (or (i)) and (3) (or (iii)).

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