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Research Article

Iterative Algorithm for Approximating Solutions of Maximal Monotone Operators in Hilbert Spaces

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We first introduce and analyze an algorithm of approximating solutions of maximal monotone operators in Hilbert spaces. Using this result, we consider the convex minimization problem of finding a minimizer of a proper lower-semicontinuous convex function and the variational problem of finding a solution of a variational inequality.

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space and $T : H \to 2^H$ is a maximal monotone operator. A well-known method for solving the equation $0 \in Tv$ in a Hilbert space H is the proximal point algorithm: $x_1 = x \in H$ and

$$x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, \ldots, \tag{1.1}$$

where $\{r_n\} \subset (0, \infty)$ and $J_r = (I + rT)^{-1}$ for all $r > 0$. This algorithm was first introduced by Martinet [1]. Rockafellar [2] proved that if $\liminf_{n\to\infty} r_n > 0$ and $T^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ defined by (1.1) converges weakly to an element of $T^{-1}0$. Later, many researchers have studied the convergence of the sequence defined by (1.1) in a Hilbert space; see, for instance, [3–6] and the references mentioned therein. In particular, Kamimura and Takahashi [7] proved the following result.

Theorem 1.1. *Let $T : H \to 2^H$ be a maximal monotone operator. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = u \in H$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, \ldots, \tag{1.2}$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\lim_{n\to\infty} r_n = \infty$. *If* $T^{-1}0 \neq \emptyset$, *then the sequence* $\{x_n\}$ *defined by (1.2) converges strongly to* Pu, *where* P *is the metric projection of* H *onto* $T^{-1}0$.

Motivated and inspired by the above result, in this paper, we suggest and analyze an iterative algorithm which has strong convergence. Further, using this result, we consider the convex minimization problem of finding a minimizer of a proper lower-semicontinuous convex function and the variational problem of finding a solution of a variational inequality.

2. Preliminaries

Recall that a mapping $U : H \to H$ is said to be nonexpansive if $\|Ux - Uy\| \leq \|x - y\|$ for all $x, y \in H$. We denote the set of all fixed points of U by $F(U)$. A multivalued operator $T : H \to 2^H$ with domain $D(T)$ and range $R(T)$ is said to be monotone if for each $x_i \in D(T)$ and $y_i \in Tx_i$, $i = 1,2$, we have $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$.

A monotone operator T is said to be maximal if its graph $G(T) = \{(x,y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. Let I denote the identity operator on H and let $T : H \to 2^H$ be a maximal monotone operator. Then we can define, for each $r > 0$, a nonexpansive single-valued mapping $J_r : H \to H$ by $J_r = (I + rT)^{-1}$. It is called the resolvent (or the proximal mapping) of T. We also define the Yosida approximation A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in TJ_r x$ and $\|A_r x\| \leq \inf\{\|y\| : y \in Tx\}$ for all $x \in H$.

Before starting the main result of this paper, we include some lemmas.

Lemma 2.1 (see [8]). *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. *Suppose* $x_{n+1} = \alpha_n x_n + (1-\alpha_n) z_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,* $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.2 (the resolvent identity). *For* $\lambda, \mu > 0$, *there holds the identity*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda x\right), \quad x \in X. \tag{2.1}$$

Lemma 2.3 (see [9]). *Let* E *be a real Banach space. Then for all* $x, y \in E$

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle \quad \forall j(x+y) \in J(x+y). \tag{2.2}$$

Lemma 2.4 (see[10]). *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the property* $a_{n+1} \leq (1-s_n)a_n + s_n t_n$, $n \geq 0$, *where* $\{s_n\} \subset (0,1)$ *and* $\{t_n\}$ *are such that*

(i) $\sum_{n=0}^{\infty} s_n = \infty$,

(ii) *either* $\limsup_{n\to\infty} t_n \leq 0$ *or* $\sum_{n=0}^{\infty} |s_n t_n| < \infty$.

Then $\{a_n\}$ *converges to zero.*

3. Main result

Let $T : H \to 2^H$ be a maximal monotone operator and let $J_r : H \to H$ be the resolvent of T for each $r > 0$. Then we consider the following algorithm: for fixed $u \in H$ and given $x_0 \in H$ arbitrarily, let the sequence $\{x_n\}$ is generated by

$$\begin{aligned} y_n &\approx J_{r_n} x_n, \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \delta_n y_n, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$ are three real numbers in $[0,1]$ and $\{r_n\} \subset (0,\infty)$. Here the criterion for the approximate computation of y_n in (3.1) will be

$$||y_n - J_{r_n} x_n|| \le \sigma_n, \tag{3.2}$$

where $\sum_{n=0}^{\infty} \sigma_n < \infty$.

Theorem 3.1. *Let $T : H \to 2^H$ be a maximal monotone operator. Assume $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, and $\{r_n\}$ satisfy the following control conditions:*
(i) $\alpha_n + \beta_n + \delta_n = 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(iv) $r_n \ge \epsilon > 0$ *for all n and* $r_{n+1} - r_n \to 0 (n \to \infty)$.
If $T^{-1}0 \neq \emptyset$, then $\{x_n\}$ defined by (3.1) under criterion (3.2) converges strongly to Pu, where P is the metric projection of H onto $T^{-1}0$.

Proof. From $T^{-1}0 \neq \emptyset$, there exists $p \in T^{-1}0$ such that $J_s p = p$ for all $s > 0$. Then we have

$$\begin{aligned} ||x_{n+1} - p|| &\le \alpha_n \|u - p\| + \beta_n ||x_n - p|| + \delta_n ||y_n - p|| \\ &\le \alpha_n \|u - p\| + \beta_n ||x_n - p|| + \delta_n (\sigma_n + ||J_{r_n} x_n - p||) \\ &\le \alpha_n \|u - p\| + \beta_n ||x_n - p|| + \delta_n ||x_n - p|| + \delta_n \sigma_n \\ &= \alpha_n \|u - p\| + (1 - \alpha_n) ||x_n - p|| + \delta_n \sigma_n. \end{aligned} \tag{3.3}$$

An induction gives that

$$||x_n - p|| \le \max\{\|u - p\|, ||x_0 - p||\} + \sum_{k=0}^{n} \sigma_k \tag{3.4}$$

for all $n \ge 0$. This implies that $\{x_n\}$ is bounded. Hence $\{J_{r_n} x_n\}$ and $\{y_n\}$ are also bounded. Define a sequence $\{z_n\}$ by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n, \quad n \ge 0. \tag{3.5}$$

Then we observe that

$$\begin{aligned} z_{n+1}-z_n &= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1}-\beta_n x_n}{1-\beta_n} \\ &= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)u + \frac{\delta_{n+1}}{1-\beta_{n+1}}(y_{n+1}-y_n) \\ &\quad + \left(\frac{\delta_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n}{1-\beta_n}\right)y_n. \end{aligned} \tag{3.6}$$

If $r_{n-1} \le r_n$, from Lemma 2.2, using the resolvent identity

$$J_{r_n}x_n = J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}x_n + \left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}x_n\right), \tag{3.7}$$

we obtain

$$\begin{aligned} \|J_{r_n}x_n - J_{r_{n-1}}x_{n-1}\| &\le \frac{r_{n-1}}{r_n}\|x_n-x_{n-1}\| + \left(\frac{r_n-r_{n-1}}{r_n}\right)\|J_{r_n}x_n - x_{n-1}\| \\ &\le \|x_n-x_{n-1}\| + \frac{1}{\epsilon}|r_{n-1}-r_n|\,\|J_{r_n}x_n-x_{n-1}\|. \end{aligned} \tag{3.8}$$

Similarly, we can prove that the last inequality holds if $r_{n-1} \ge r_n$.

On the other hand, from (3.2), we have

$$\begin{aligned} \|y_{n+1}-y_n\| &\le \|y_{n+1}-J_{r_{n+1}}x_{n+1}\| + \|y_n - J_{r_n}x_n\| + \|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\| \\ &\le \sigma_{n+1}+\sigma_n + \|J_{r_{n+1}}x_{n+1}-J_{r_n}x_n\|. \end{aligned} \tag{3.9}$$

Thus it follows from (3.5) that

$$\begin{aligned} &\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \\ &\quad\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(\|u\|+\|y_n\|) + \frac{\delta_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| \\ &\qquad + \frac{\delta_{n+1}}{1-\beta_{n+1}}\frac{1}{\epsilon}|r_{n+1}-r_n| \times \|J_{r_{n+1}}x_{n+1}-x_n\| + \sigma_{n+1}+\sigma_n - \|x_{n+1}-x_n\| \\ &\quad\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(\|u\|+\|y_n\|) + \sigma_{n+1}+\sigma_n \\ &\qquad + \frac{\delta_{n+1}}{1-\beta_{n+1}}\frac{1}{\epsilon}|r_{n+1}-r_n| \times \|J_{r_{n+1}}x_{n+1}-x_n\|, \end{aligned} \tag{3.10}$$

which implies that $\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \le 0$. Hence, by Lemma 2.1, we have $\lim_{n\to\infty}\|z_n-x_n\| = 0$. Consequently, it follows from (3.5) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\| = 0. \tag{3.11}$$

On the other hand,

$$\begin{aligned} \|x_n-y_n\| &\le \|x_{n+1}-x_n\| + \|x_{n+1}-y_n\| \\ &\le \|x_{n+1}-x_n\| + \alpha_n\|u-y_n\| + \beta_n\|x_n-y_n\|, \end{aligned} \tag{3.12}$$

and so, by (ii), (iii), (3.11), and (3.12), we have $\lim_{n\to\infty}\|x_n - y_n\| = 0$. It follows that

$$||A_{r_n}x_n|| \le \frac{1}{r_n}[||x_n - y_n|| + ||y_n - J_{r_n}x_n||] \le \frac{1}{\epsilon}[||x_n - y_n|| + \sigma_n] \longrightarrow 0. \tag{3.13}$$

We next prove that

$$\limsup_{n\to\infty} \langle u - Pu, x_{n+1} - Pu\rangle \le 0, \tag{3.14}$$

where P is the metric projection of H onto $T^{-1}0$. To prove this, it is sufficient to show $\limsup_{n\to\infty}\langle u - Pu, J_{r_n}x_n - Pu\rangle \le 0$, because $x_{n+1} - J_{r_n}x_n \to 0$. Now there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that

$$\lim_{i\to\infty} \langle u - Pu, J_{r_{n_i}}x_{n_i} - Pu\rangle = \limsup_{n\to\infty} \langle u - Pu, J_{r_n}x_n - Pu\rangle. \tag{3.15}$$

Since $\{J_{r_n}x_n\}$ is bounded, we may assume that $\{J_{r_{n_i}}x_{n_i}\}$ converges weakly to some $v \in H$. Then it follows that $v \in T^{-1}0$. Indeed, since $A_{r_n}x_n \in TJ_{r_n}x_n$ and T is monotone, we have $\langle s - J_{r_{n_i}}x_{n_i}, s' - A_{r_{n_i}}x_{n_i}\rangle \ge 0$, where $s' \in Ts$. From $A_{r_n}x_n \to 0$, we obtain $\langle s - v, s'\rangle \ge 0$ whenever $s' \in Ts$. Hence, from the maximality of T, we have $v \in T^{-1}0$. Since P is the metric projection of H onto $T^{-1}0$, we obtain

$$\limsup_{n\to\infty} \langle u - Pu, J_{r_n}x_n - Pu\rangle = \lim_{i\to\infty} \langle u - Pu, J_{r_{n_i}}x_{n_i} - Pu\rangle = \langle u - Pu, v - Pu\rangle \le 0. \tag{3.16}$$

That is, (3.14) holds.

Finally, to prove that $x_n \to p$, we apply Lemma 2.3 to get

$$\begin{aligned} ||x_{n+1} - Pu||^2 &\le (||\beta_n(x_n - Pu) + \delta_n(y_n - Pu)||)^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle \\ &\le (\beta_n||x_n - Pu|| + \delta_n||x_n - Pu|| + \delta_n\sigma_n)^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle \\ &= ((1-\alpha_n)||x_n - Pu|| + \delta_n\sigma_n)^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle \\ &\le (1-\alpha_n)||x_n - Pu||^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle + M\sigma_n, \end{aligned} \tag{3.17}$$

where $M > 0$ is some constant such that $2(1-\alpha_n)\delta_n\|x_n - Pu\| + \delta_n^2\sigma_n \le M$. An application of Lemma 2.4 yields that $\|x_n - Pu\| \to 0$. This completes the proof. □

Remark 3.2. It is clear that the algorithm (3.1) includes the algorithm (1.2) as a special case. Our result can be considered as a complement of Kamimura and Takahashi [7] and others.

4. Applications

Let $f : H \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Then we can define the subdifferential ∂f of f by

$$\partial f(x) = \{z \in H : f(y) \ge f(x) + \langle y - x, z\rangle \ \forall y \in H\} \tag{4.1}$$

for all $x \in H$. It is well known that ∂f is a maximal monotone operator of H into itself; see Minty [11] and Rockafellar [12, 13].

In this section, we investigate our algorithm in the case of $T = \partial f$. Our discussion follows Rockafellar [14, Section 4]. If $T = \partial f$, the algorithm (3.1) is reduced to the following algorithm:

$$\begin{aligned} y_n &\approx \arg\min_{z\in H}\left\{f(z)+\frac{1}{2r_n}\|z-x_n\|^2\right\},\\ x_{n+1} &= \alpha_n u+\beta_n x_n+\delta_n y_n, \quad n\in N, \end{aligned} \tag{4.2}$$

with the following criterion:

$$d(0,S_n(y_n)) \le \frac{\sigma_n}{r_n}, \tag{4.3}$$

where $\sum_{n=0}^{\infty}\sigma_n < \infty$, $S_n(z) = \partial f(z) + (z - x_n)/r_n$, and $d(0,A) = \inf\{\|x\| : x \in A\}$. About (4.3), the following lemma was proved in Rockafellar [2, Proposition 3].

Lemma 4.1. *If y_n is chosen according to criterion (4.3), then $\|y_n - J_{r_n}x_n\| \le \sigma_n$ holds, where $J_{r_n} = (I + r_n\partial f)^{-1}$.*

Theorem 4.2. *Let $f : H \to (-\infty,\infty]$ be a proper lower semicontinuous convex function. Assume $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, and $\{r_n\}$ satisfy the same conditions (i)–(iv) as in Theorem 3.1. If $(\partial f)^{-1}0 \ne \emptyset$, then $\{x_n\}$ defined by (4.2) with criterion (4.3) converges strongly to $v \in H$, which is the minimizer of f nearest to u.*

Proof. Putting $g_n(z) = f(z) + \|z - x_n\|^2/2r_n$, we obtain

$$\partial g_n(z) = \partial f(z) + \frac{1}{r_n}(z - x_n) = S_n(z) \tag{4.4}$$

for all $z \in H$ and $J_{r_n}x_n = (I + r_n\partial f)^{-1}x_n = \arg\min_{z\in H} g_n(z)$. It follows from Theorem 3.1 and Lemma 4.1 that $\{x_n\}$ converges strongly to $v \in H$ and $f(v) = \min_{z\in H} f(z)$. This completes the proof. □

Next we consider a variational inequality. Let C be a nonempty closed convex subset of H and let T be a single-valued operator of C into H. We denote by $\mathrm{VI}(C,T)$ the set of solutions of the variational inequality, that is,

$$\mathrm{VI}(C,T) = \{w \in X : \langle s - w, Tw\rangle \ge 0,\ \forall s \in C\}. \tag{4.5}$$

A single-valued operator T is called semicontinuous if T is continuous from each line segment of C to H with the weak topology. Let F be a single-valued monotone and semicontinuous operator of C into H and let N_Cz be the normal cone to C at $z \in C$, that is, $N_Cz = \{w \in H : \langle z - s, w\rangle \ge 0,\ \forall s \in C\}$. Letting

$$Az = \begin{cases} Fz + N_Cz, & z \in C,\\ \emptyset, & z \in H \setminus C, \end{cases} \tag{4.6}$$

we have that A is a maximal monotone operator (see Rockafellar [14, Theorem 3]). We can also check that $0 \in Av$ if and only if $v \in \mathrm{VI}(C,F)$ and that $J_r x = \mathrm{VI}(C,F_{r,x})$ for all $r > 0$ and $x \in H$, where $F_{r,x}z = Fz + (z-x)/r$ for all $z \in C$. Then we have the following result.

Corollary 4.3. *Let F be a single-valued monotone and semicontinuous operator of C into H. For fixed $u \in H$, let the sequence $\{x_n\}$ be generated by*

$$\begin{aligned} y_n &\approx \mathrm{VI}\left(C, F_{r_n,x_n}\right), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \delta_n y_n. \end{aligned} \tag{4.7}$$

Here the criterion for the approximate computation of y_n in (4.7) will be

$$\left|\left|y_n - \mathrm{VI}\left(C, F_{r_n,x_n}\right)\right|\right| \le \sigma_n, \tag{4.8}$$

where $\sum_{n=0}^{\infty} \sigma_n < \infty$. Assume $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, and $\{r_n\}$ satisfy the same conditions (i)–(iv) as in Theorem 3.1. If $\mathrm{VI}(C,F) \neq \emptyset$, then $\{x_n\}$ defined by (4.7) with criterion (4.8) converges strongly to the point of $\mathrm{VI}(C,F)$ nearest to u.

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