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Research Article

Common Fixed Point Theorems for Hybrid Pairs of Occasionally Weakly Compatible Mappings Satisfying Generalized Contractive Condition of Integral Type

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We obtain several fixed point theorems for hybrid pairs of single-valued and multivalued occasionally weakly compatible maps defined on a symmetric space satisfying a contractive condition of integral type. The results of this paper essentially contain every theorem on hybrid and multivalued self-maps of a metric space as a special case.

1. Introduction and preliminaries

The study of fixed point theorems, involving four single-valued maps, began with the assumption that all of the maps commuted. Sessa [1] weakened the condition of commutativity to that of pairwise weakly commuting. Jungck generalized the notion of weak commutativity to that of pairwise compatible [2] and then pairwise weakly compatible maps [3]. In the recent paper of Jungck and Rhoades [4], the concept of occasionally weakly commuting maps (owc) was introduced. In that paper, it was shown that essentially every theorem involving four maps becomes a special case of one of the results on owc maps. In this paper, we show that the same is true for the theorems involving four maps, in which two of them are multivalued and for which the contractive condition is of integral type. Branciari [5] obtained a fixed point theorem for a single valued mapping satisfying an analogue of Banach's contraction principle for an integral-type inequality. Rhoades [6] proved two fixed point theorems involving more general contractive conditions (see also [7–9]). The aim of this paper is to extend the concept of occasionally weakly compatible maps to hybrid pairs of single-valued and multivalued maps in the setting of symmetric space satisfying a contractive condition of integral type. Our results complement, extend, and unify comparable results in the literature.

Consistent with [10–12], we will use the following notations, where (X,d) is a metric space, for $x \in X$ and $A \subseteq X$, $d(x,A) = \inf\{d(y,A) : y \in A\}$, and $CB(X)$ is the class of all nonempty bounded and closed subsets of X. Let H be a Hausdorff metric induced by the metric d of X, given by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\right\} \tag{1.1}$$

for every $A,B \in CB(X)$.

Definition 1.1. Let X be a set. A symmetric on X is a mapping $d : X \times X \to [0,\infty)$ such that

$$\begin{aligned} &d(x,y) = 0 \quad \text{iff } x = y,\\ &d(x,y) = d(y,x). \end{aligned} \tag{1.2}$$

A set X together with a symmetric d is called a *symmetric space.*

Definition 1.2. Maps $f : X \to X$ and $T : X \to CB(X)$ are said to be occasionally weakly compatible (owc) if and only if there exists some point x in X such that $fx \in Tx$ and $fTx \subseteq Tfx$.

The following lemma due to Dube [13] will be used.

Lemma 1.3. *Let $A,B \in CB(X)$, then for any $a \in A$,*

$$d(a,B) \le H(A,B). \tag{1.3}$$

Example 1.4. Let $X = [0,\infty)$ with usual metric. Define $f : X \to X$, $T : X \to CB(X)$ by

$$\begin{aligned} fx &= \begin{cases} 0, & 0 \le x < 1,\\ 2x, & 1 \le x < \infty, \end{cases}\\ Tx &= \begin{cases} \{x\}, & 0 \le x < 1,\\ [1,1+4x], & 1 \le x < \infty. \end{cases} \end{aligned} \tag{1.4}$$

It can be easily verified that $x = 1$ is coincidence point of f and T, but f and T are not weakly compatible there. However, the pair $\{f,T\}$ is occasionally weakly compatible.

2. Common fixed point theorems

In this section, we establish several common fixed point theorems for hybrid pairs of single-valued and multivalued maps defined on a symmetric space, which is more general than a metric space. Define $\mathcal{F} = \{\varphi : \mathbb{R}^+ \to \mathbb{R}^+ : \varphi$ is a Lebesgue integral mapping which is summable, nonnegative, and satisfies $\int_0^{\epsilon} \varphi(t)dt > 0$, for each $\epsilon > 0\}$.

Theorem 2.1. *Let f, g be self-maps of a metric space (X,d) and let T, S be maps from X into $CB(X)$ such that the pairs of $\{f,T\}$ and $\{g,S\}$ are owc. If*

$$\int_0^{H(Tx,Sy)} \varphi(t)dt < \int_0^{M(x,y)} \varphi(t)dt, \tag{2.1}$$

where $\varphi \in F$ *and*

$$M(x,y) = \max\{d(fx,gy),d(fx,Tx),d(gy,Sy),d(fx,Sy),d(gy,Tx)\} \tag{2.2}$$

for all $x,y \in X$ *for which (2.2) is positive. Then* f, g, T *and* S *have a common fixed point.*

Proof. By hypothesis, there exist points x, y in X such that $fx \in Tx, gy \in Sy$, $fTx \subseteq Tfx$, and $gSy \subseteq Sgy$. Using the triangle inequality and Lemma 1.3, we obtain $d(f^2x,g^2y) \leq H(Tfx,Sgy)$. We first show that $gy = fx$. Suppose not. Then consider

$$\begin{aligned} M(fx,gy) &= \max\{d(f^2x,g^2y),d(f^2x,Tfx),d(g^2y,Sgy),d(f^2x,Sgy),d(g^2y,Tfx)\} \\ &\leq H(Tfx,Sgy). \end{aligned} \tag{2.3}$$

Condition (2.1) then implies that

$$\int_0^{H(Tfx,Sgy)} \varphi(t)dt < \int_0^{M(fx,gy)} \varphi(t)dt \leq \int_0^{H(Tfx,Sgy)} \varphi(t)dt, \tag{2.4}$$

which is a contradiction and hence $gy = fx$. Using the triangle inequality, we obtain $d(fx,g^2y) \leq H(Tx,Sfx)$. Next, we claim that $x = fx$. If not, then consider

$$\begin{aligned} M(x,fx) &= \max\{d(fx,g^2y),d(fx,Tx),d(g^2y,Sgy),d(gy,Sgy),d(g^2y,Tx)\} \\ &\leq H(Tx,Sfx). \end{aligned} \tag{2.5}$$

Condition (2.1) implies

$$\int_0^{H(Tx,Sgy)} \varphi(t)dt < \int_0^{M(x,fx)} \varphi(t)dt = \int_0^{H(Tx,Sgy)} \varphi(t)dt, \tag{2.6}$$

which is again a contradiction and the claim follows. Similarly, we obtain $y = gy$. Thus f, g, T, and S have a common fixed point. □

Theorem 2.2. *Let* f, g *be self-maps of the symmetric space* (X,d) *and let* T, S *be maps from* X *into* $CB(X)$ *such that the pairs of* $\{f,T\}$ *and* $\{g,S\}$ *are owc. If*

$$\int_0^{(H(Tx,Sy))^p} \varphi(t)dt < \int_0^{M_p(x,y)} \varphi(t)dt, \tag{2.7}$$

where $\varphi \in F$ *and*

$$\begin{aligned} &M_p(x,y) \\ &\quad = \alpha(d(gy,Tx))^p + (1-\alpha)\max\{(d(fx,Tx))^p,(d(gy,Sy))^p,(d(fx,Tx))^{p/2}(d(gy,Tx))^{p/2}, \\ &\qquad (d(gy,Tx))^{p/2}(d(fx,Sy))^{p/2}\}, \end{aligned} \tag{2.8}$$

for all $x,y \in X$ *for which (2.8) is not zero,* $\alpha,\beta \in (0,1]$, *and* $p \geq 1$. *Then* f, g, T *and* S *have a common fixed point.*

Proof. By hypothesis, there exist points x, y in X such that $fx \in Tx, gy \in Sy, fTx \subseteq Tfx$, and $gSy \subseteq Sgy$. We first show that $gy = fx$. Suppose not. Then consider

$$\begin{aligned} M_p(fx,gy) = {} & \alpha(d(g^2y,Tfx))^p \\ & + (1-\alpha)\max\{(d(f^2x,Tfx))^p,(d(g^2y,Sgy))^p, \\ & \qquad (d(f^2x,Tfx))^{p/2}(d(g^2y,Tfx))^{p/2}, \\ & \qquad (d(g^2y,Tfx))^{p/2}(d(f^2x,Sgy))^{p/2}\} \\ = {} & \alpha(d(g^2y,Tfx))^p + (1-\alpha)(d(g^2y,Tfx))^{p/2}(d(f^2x,Sgy))^{p/2} \\ \le {} & \alpha(H(Tfx,Sgy))^p + (1-\alpha)(H(Tfx,Sgy))^p = (H(Tfx,Sgy))^p. \end{aligned} \tag{2.9}$$

Condition (2.7) then implies that

$$\int_0^{(H(Tfx,Sgy))^p}\varphi(t)dt < \int_0^{M_p(fx,gy)}\varphi(t)dt \le \int_0^{(H(Tfx,Sgy))^p}\varphi(t)dt, \tag{2.10}$$

which is a contradiction, and hence $gy = fx$. Now, we claim that $x = fx$. If not, then since $fx = gy$,

$$\begin{aligned} M_p(x,fx) = {} & \alpha(d(gfx,Tx))^p \\ & + (1-\alpha)\max\{(d(fx,Tx))^p,(d(gfx,Sfx))^p, \\ & \qquad (d(fx,Tx))^{p/2}(d(gfx,Tx))^{p/2}, \\ & \qquad (d(gfx,Tx))^{p/2}(d(fx,Sfx))^{p/2}\} \\ = {} & \alpha(d(gfx,Tx))^p + (1-\alpha)(d(g^2y,Tx))^{p/2}(d(fx,Sgy))^{p/2} \\ \le {} & \alpha(H(Tx,Sgy))^p + (1-\alpha)(H(Tx,Sgy))^p = (H(Tx,Sgy))^p. \end{aligned} \tag{2.11}$$

Condition (2.7) then implies that

$$\int_0^{(H(Tx,Sgy))^p}\varphi(t)dt < \int_0^{M_p(x,gy)}\varphi(t)dt \le \int_0^{(H(Tx,Sgy))^p}\varphi(t)dt, \tag{2.12}$$

which is again a contradiction, and the claim follows. Similarly, we obtain $y = gy$. Thus, f, g, T, and S have a common fixed point. □

Corollary 2.3. *Let f, g be self-maps of a metric space (X,d) and let T, S be maps from X into $CB(X)$ such that the pairs of $\{f,T\}$ and $\{g,S\}$ are owc. If*

$$\int_0^{H(Tx,Sy)}\varphi(t)dt < \int_0^{M(x,y)}\varphi(t)dt, \tag{2.13}$$

where $\varphi \in F$ and

$$M(x,y) = h\max\left\{d(fx,gy),d(fx,Tx),d(gy,Sy),\frac{1}{2}[d(fx,Sy)+d(gy,Tx)]\right\} \tag{2.14}$$

for all $x, y \in X$ for which (2.14) is not zero and $h \in [0,1)$. Then f, g, T, and S have a common fixed point.

Proof. Since (2.14) is a special case of (2.2), the result follows immediately from Theorem 2.1. □

Every contractive condition of integral type automatically includes a corresponding contractive condition, not involving integrals, by setting $\varphi(t) = 1$ over $\mathbb{R}^+$. Theorem 1 of [14], [15, Theorem 2.3], and [16, Theorem 2] are special cases of Corollary 2.3. Also [17, Theorem 2] and [18, Theorem 1] become special cases of the corollary if we take $S = T$ and $f = g$.

Corollary 2.4. *Let f be a self-map of the symmetric space (X,d) and let T be a map from X into $CB(X)$ such that f and T are owc and for all $x, y \in X$ for which (2.16) is not zero,*

$$\int_0^{H(Tx,Ty)} \varphi(t)dt < \int_0^{M(x,y)} \varphi(t)dt, \tag{2.15}$$

where $\varphi \in F$ and

$$M(x,y) = \max\left\{d(fx,Ty), \frac{1}{2}[d(fx,Tx)+d(fy,Ty)], \frac{1}{2}[d(fy,Tx)+d(fx,Ty)]\right\}. \tag{2.16}$$

Then f and T have a common fixed point.

Proof. Since (2.16) is the special case of (2.2) with $S = T$ and $f = g$, the result follows immediately from Theorem 2.1. □

Corollary 2.5. *Let f, g be self-maps of a metric space (X,d) and T, S be maps from X into $CB(X)$ such that the pairs of $\{f,T\}$ and $\{g,S\}$ are owc and for all $x \neq y \in X$,*

$$\int_0^{H(Tx,Sy)} \varphi(t)dt < \int_0^{M(x,y)} \varphi(t)dt, \tag{2.17}$$

where $\varphi \in F$ and

$$M(x,y) = \alpha d(fx,gy) + \beta \max\{d(fx,Tx), d(gy,Sy)\} + \gamma \max\{d(fx,gy), d(fx,Sy), d(gy,Tx)\}, \tag{2.18}$$

with $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$. Then f, g, T, and S have a common fixed point.

Proof. Since (2.18) is a special case of (2.2), the result follows immediately from Theorem 2.1. □

Define $G = \{g : \mathbb{R}^5 \to \mathbb{R}^+\}$ such that

(g_1) g is nondecreasing in the 4th and 5th variables,

(g_2) if $u, v \in \mathbb{R}^+$ are such that $u \leq g(v,v,u,u+v,0)$, $u \leq g(v,u,v,u+v,0)$, $v \leq g(u,u,v,u+v,0)$, or $u \leq g(v,u,v,u,u+v)$, then $u \leq hv$, where $0 < h < 1$ is constant,

(g_3) if $u \in \mathbb{R}^+$ is such that $u \le g(u,0,0,u,u)$, $u \le g(0,u,0,u,u)$ or $u \le g(0,0,u,u,u)$, then $u = 0$.

Theorem 2.6. *Let f, g be self-maps of the metric space (X,d) and let T, S be maps from X into $CB(X)$ such that the pairs of $\{f,T\}$ and $\{g,S\}$ are owc. If*

$$\int_0^{H(Tx,Sy)} \varphi(t)dt < g\left(\int_0^{d(fx,gy)} \varphi(t)dt, \int_0^{d(fx,Tx)} \varphi(t)dt, \int_0^{d(gy,Sy)} \varphi(t)dt, \int_0^{d(fx,Sy)} \varphi(t)dt, \int_0^{d(gy,Tx)} \varphi(t)dt\right), \tag{2.19}$$

where $\varphi \in F$ and for all $x, y \in X$ for which the right-hand side of (2.19) is not zero, where $g \in G$, then f, g, T, and S have a common fixed point.

Proof. By hypothesis, there exist points x, y in X such that $fx \in Tx$, $gy \in Sy$, $fTx \subseteq Tfx$, and $gSy \subseteq Sgy$. Also, using the triangle inequality and Lemma 1.3, we obtain $d(fx,gy) \le H(Tx,Sy)$. First, we show that $gy = fx$. Suppose not. Then condition (2.19) implies that

$$\begin{aligned}\int_0^{H(Tx,Sy)} \varphi(t)dt &< g\left(\int_0^{d(fx,gy)} \varphi(t)dt, 0, 0, \int_0^{d(fx,Sy)} \varphi(t)dt, \int_0^{d(gy,Tx)} \varphi(t)dt\right) \\ &\le g\left(\int_0^{H(Tx,Sy)} \varphi(t)dt, 0, 0, \int_0^{H(Tx,Sy)} \varphi(t)dt, \int_0^{H(Tx,Sy)} \varphi(t)dt\right)\end{aligned} \tag{2.20}$$

which, from (g_3), gives $\int_0^{H(Tx,Sy)} \varphi(t)dt = 0$, and hence $H(Tx,Sy) = 0$, which implies that $d(fx,gy) = 0$. Hence the claim follows. Using the triangle inequality, we obtain $d(fx, f^2x) \le H(Tfx,Sy)$. Next, we claim that $fx = f^2x$. If not, then condition (2.19) implies that

$$\begin{aligned}\int_0^{H(Tfx,Sy)} \varphi(t)dt &< g\left(\int_0^{d(f^2x,gy)} \varphi(t)dt, 0, 0, \int_0^{d(f^2x,Sy)} \varphi(t)dt, \int_0^{d(gy,Tfx)} \varphi(t)dt\right) \\ &\le g\left(\int_0^{H(Tfx,Sy)} \varphi(t)dt, 0, 0, \int_0^{H(Tfx,Sy)} \varphi(t)dt, \int_0^{H(Tfx,Sy)} \varphi(t)dt\right)\end{aligned} \tag{2.21}$$

which, from (g_3), gives $H(Tfx,Sy) = 0$, which implies that $d(fx, f^2x) = 0$. Hence the claim follows. Similarly, it can be shown that $gy = g^2y$ which proves the result. □

A control function Φ is defined by $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ which is continuous monotonically increasing, $\Phi(2t) \le 2\Phi(t)$ and $\Phi(0) = 0$ if and only if $t = 0$. Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $\Psi(t) < t$ for each $t > 0$.

Theorem 2.7. *Let f, g be self-maps of the metric space (X,d) and let T, S be maps from X into $CB(X)$ such that the pairs of $\{f,T\}$ and $\{g,S\}$ are owc. If*

$$\int_0^{\Phi(H(Tx,Sy))} \varphi(t)dt < \Psi\left(\int_0^{M(x,y)} \varphi(t)dt\right), \tag{2.22}$$

where $\varphi \in F$ and

$$M(x,y) = \max\left\{\Phi(d(fx,gy)),\Phi(d(fx,Tx)),\Phi(d(gy,Sy)),\frac{1}{2}[\Phi(d(fx,Sy))+\Phi(d(gy,Tx))]\right\} \tag{2.23}$$

for all $x,y \in X$ for which (2.23) is not zero. Then f, g, T and let S have a common fixed point.

Proof. By hypothesis, there exist points x, y in X such that $fx \in Tx, gy \in Sy, fTx \subseteq Tfx$, and $gSy \subseteq Sgy$. Also, using the triangle inequality, we obtain $d(fx,gy) \le H(Tx,Sy)$. First, we show that $H(Tx,Sy) = 0$. Suppose not. Then consider

$$M(x,y) = \max\left\{\Phi(d(fx,gy)),0,0,\frac{1}{2}\Phi(2H(Tx,Sy))\right\} = \Phi(H(Tx,Sy)). \tag{2.24}$$

Condition (2.22) implies that

$$0 < \int_0^{\Phi(H(Tx,Sy))}\varphi(t)dt < \Psi\left(\int_0^{M(x,y)}\varphi(t)dt\right) < \int_0^{\Phi(H(Tx,Sy))}\varphi(t)dt, \tag{2.25}$$

which is a contradiction. Therefore $H(Tx,Sy) = 0$, which implies that $d(fx,gy) = 0$. Hence the claim follows. Using the triangle inequality, we obtain $d(fx,f^2x) \le H(Tfx,Sy)$. Next, we claim that $H(Tfx,Sy) = 0$. If not, then consider

$$M(fx,y) = \max\left\{\Phi(d(f^2x,gy)),0,0,\frac{1}{2}\Phi(2H(Tfx,Sy))\right\} = \Phi(H(Tfx,Sy)). \tag{2.26}$$

Then condition (2.22) implies that

$$0 < \int_0^{\Phi(H(Tfx,Sy))}\varphi(t)dt < \Psi\left(\int_0^{M(fx,y)}\varphi(t)dt\right) < \int_0^{\Phi(H(Tfx,Sy))}\varphi(t)dt, \tag{2.27}$$

which is a contradiction. Therefore ,$H(Tfx,Sy) = 0$, which implies that $d(fx,f^2x) = 0$. Hence the claim follows. Similarly, it can be shown that $gy = g^2y$, which proves the result. □

Theorem 1 of [19] and [20, Theorem 1] become special cases of Theorem 2.7 with $\Phi(x) = 1$.

Remark 2.8. It is natural to ask if integral contractive conditions are indeed generalizations of corresponding contractive conditions not involving integrals. We illustrate this fact with an example. In [6, Theorem 4], a unique fixed point was established for a self-map of complete metric space X satisfying the integral condition

$$\int_0^{d(Tx,Ty)}\varphi(t)dt \le h\int_0^{M(x,y)}\varphi(t)dt, \tag{2.28}$$

for all $x, y \in X$, where $0 \leq h < 1$ and

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \tag{2.29}$$

It was also assumed that there was a point in X with bounded orbit.

If there exists points x, y in X for which $d(Tx,Ty) \geq M(x,y)$, then one obtains a contradiction to (2.28). Therefore for all x, y in X,

$$d(Tx,Ty) < M(x,y). \tag{2.30}$$

Even if one assumes the continuity of T, Taylor [21] has shown that there exists a map as T satisfying (2.30), with bounded orbit, but which does not possess a fixed point.

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