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Research Article

The Equivalence between T-Stabilities of The Krasnoselskij and The Mann Iterations

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We prove the equivalence between the T-stabilities of the Krasnoselskij and the Mann iterations; a consequence is the equivalence with the T-stability of the Picard-Banach iteration.

1. Introduction

Let X be a normed space and T a selfmap of X. Let x_0 be a point of X, and assume that $x_{n+1} = f(T,x_n)$ is an iteration procedure, involving T, which yields a sequence $\{x_n\}$ of points from X. Suppose $\{x_n\}$ converges to a fixed point x^* of T. Let $\{\xi_n\}$ be an arbitrary sequence in X, and set $\epsilon_n = \|\xi_{n+1} - f(T,\xi_n)\|$ for all $n \in \mathbb{N}$.

Definition 1.1 [1]. If $(\lim_{n\to\infty} \epsilon_n = 0) \Rightarrow (\lim_{n\to\infty} \xi_n = p)$, then the iteration procedure $x_{n+1} = f(T,x_n)$ is said to be T-stable with respect to T.

Remark 1.2 [1]. In practice, such a sequence $\{\xi_n\}$ could arise in the following way. Let x_0 be a point in X. Set $x_{n+1} = f(T,x_n)$. Let $\xi_0 = x_0$. Now $x_1 = f(T,x_0)$. Because of rounding or discretization in the function T, a new value ξ_1 approximately equal to x_1 might be obtained instead of the true value of $f(T,x_0)$. Then to approximate x_2, the value $f(T,\xi_1)$ is computed to yield ξ_2, an approximation of $f(T,\xi_1)$. This computation is continued to obtain $\{\xi_n\}$ an approximate sequence of $\{x_n\}$.

Let X be a normed space, D a nonempty, convex subset of X, and T a selfmap of D, let $p_0 = e_0 \in D$. The Mann iteration (see [2]) is defined by

$$e_{n+1} = (1-\alpha_n)e_n + \alpha_n Te_n, \tag{1.1}$$

where $\{\alpha_n\} \subset (0,1)$. The Ishikawa iteration is defined (see [3]) by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$. The Krasnoselskij iteration (see [4]) is defined by

$$p_{n+1} = (1-\lambda)p_n + \lambda T p_n, \tag{1.3}$$

where $\lambda \in (0,1)$. Recently, the equivalence between the T-stabilities of Mann and Ishikawa iterations, respectively, for modified Mann-Ishikawa iterations was shown in [5]. In the present paper, we shall prove the equivalence between the T-stabilities of the Krasnoselskij and the Mann iterations. Next, $\{u_n\}, \{v_n\} \subset X$ are arbitrary.

Definition 1.3.

(i) The Mann iteration (1.1) is said to be T-stable if and only if for all $\{\alpha_n\} \subset (0,1)$ and for every sequence $\{u_n\} \subset X$,

$$\lim_{n\to\infty} \varepsilon_n = 0 \Longrightarrow \lim_{n\to\infty} u_n = x^*, \tag{1.4}$$

where $\varepsilon_n := \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T u_n\|$.

(ii) The Krasnoselskij iteration (1.3) is said to be T-stable if and only if for all $\lambda \in (0,1)$, and for every sequence $\{v_n\} \subset X$,

$$\lim_{n\to\infty} \delta_n = 0 \Longrightarrow \lim_{n\to\infty} v_n = x^*, \tag{1.5}$$

where $\delta_n := \|v_{n+1} - (1-\lambda)v_n - \lambda T v_n\|$.

2. Main results

Theorem 2.1. *Let X be a normed space and $T : X \to X$ a map with bounded range and $\{\alpha_n\} \subset (0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lambda$, $\lambda \in (0,1)$. Then the following are equivalent:*

(i) *the Mann iteration is T-stable,*

(ii) *the Krasnoselskij iteration is T-stable.*

Proof. We prove that (i)⇒(ii). If $\lim_{n\to\infty} \delta_n = 0$, then $\{v_n\}$ is bounded. Set

$$M_1 := \max\left\{\sup_{x\in X}\{\|T(x)\|\}, \|v_0\|, \|u_0\|\right\}. \tag{2.1}$$

Observe that $\|v_1\| \leq \delta_0 + (1-\lambda)\|v_0\| + \lambda\|Tv_0\| \leq \delta_0 + M_1$. Set $M := M_1 + 1/\lambda$. Suppose that $\|v_n\| \leq M$ to prove that $\|v_{n+1}\| \leq M$. Remark that

$$\begin{aligned} ||v_{n+1}|| &\leq \delta_n + (1-\lambda)\delta_{n-1} + \cdots + (1-\lambda)^n\delta_0 + M_1 \\ &\leq 1 + (1-\lambda) + \cdots + (1-\lambda)^n + M_1 \\ &\leq \frac{1}{1-(1-\lambda)} + M_1 = M. \end{aligned} \tag{2.2}$$

Suppose that $\lim_{n\to\infty}\delta_n = 0$ to note that

$$\begin{aligned}
\varepsilon_n &= \|v_{n+1} - (1-\alpha_n)v_n - \alpha_n T v_n\| \\
&= \|v_{n+1} - v_n + \lambda v_n - \lambda v_n + \alpha_n v_n - \lambda T v_n + \lambda T v_n - \alpha_n T v_n\| \\
&\le \|v_{n+1} - (1-\lambda)v_n - \lambda T v_n\| + |\lambda - \alpha_n|\,\|v_n - T v_n\| \\
&\le \|v_{n+1} - (1-\lambda)v_n - \lambda T v_n\| + 2M|\lambda - \alpha_n| \\
&= \delta_n + 2M|\lambda - \alpha_n| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{2.3}$$

Condition (i) assures that if $\lim_{n\to\infty}\varepsilon_n = 0$, then $\lim_{n\to\infty} v_n = x^*$. Thus, for a $\{v_n\}$ satisfying

$$\lim_{n\to\infty}\delta_n = \lim_{n\to\infty}\|v_{n+1} - (1-\lambda)v_n - \lambda T v_n\| = 0, \tag{2.4}$$

we have shown that $\lim_{n\to\infty} v_n = x^*$.

Conversely, we prove (ii)⇒(i). First, we prove that $\{u_n\}$ is bounded. Since $\lim_{n\to\infty}\alpha_n = \lambda$, for $\beta \in (0,1)$ given, there exists $n_0 \in N$, such that $1-\alpha_n \le \beta$, for all $n \ge n_0$. Set $M_1 := \max\{\sup_{x\in X}\|Tx\|, \|u_0\|\}$ and $M := n_0 + 1 + \beta/(1-\beta) + M_1$ to obtain

$$\begin{aligned}
\|u_{n+1}\| &\le [\varepsilon_n + (1-\alpha_1)\varepsilon_{n-1} + (1-\alpha_1)(1-\alpha_2)\varepsilon_{n-2} \\
&\quad + \cdots + (1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{n_0})\varepsilon_{n-n_0}] \\
&\quad + (1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{n_0})(1-\alpha_{n_0+1})\varepsilon_{n-n_0-1} \\
&\quad + \cdots + (1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_n)\varepsilon_0 + M_1 \\
&\le (n_0+1) + (1-\alpha_{n_0+1}) + (1-\alpha_{n_0+1})(1-\alpha_{n_0+2})\cdots \\
&\quad + (1-\alpha_{n_0+1})\cdots(1-\alpha_n)\varepsilon_0 + M_1 \\
&\le n_0 + 1 + \beta + \beta^2 + \cdots + \beta^{n-n_0} + M_1 < M.
\end{aligned} \tag{2.5}$$

Suppose $\lim_{n\to\infty}\varepsilon_n = 0$. Observe that

$$\begin{aligned}
\delta_n &= \|u_{n+1} - (1-\lambda)u_n - \lambda T u_n\| \\
&= \|u_{n+1} - u_n + \lambda u_n - \lambda T u_n + \alpha_n u_n - \alpha_n u_n - \alpha_n T u_n + \alpha_n T u_n\| \\
&\le \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T u_n\| + |\lambda - \alpha_n|\,\|u_n - T u_n\| \\
&\le \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T u_n\| + 2M|\lambda - \alpha_n| \\
&= \varepsilon_n + 2M|\lambda - \alpha_n| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{2.6}$$

Condition (ii) assures that if $\lim_{n\to\infty}\delta_n = 0$, then $\lim_{n\to\infty} v_n = x^*$. Thus, for a $\{u_n\}$ satisfying

$$\lim_{n\to\infty}\varepsilon_n = \lim_{n\to\infty}\left\|u_{n+1}-(1-\alpha_n)u_n-\alpha_n T u_n\right\| = 0, \tag{2.7}$$

we have shown that $\lim_{n\to\infty} u_n = x^*$. □

Remark 2.2. Let X be a normed space and $T: X\to X$ a map with bounded range and $\{\alpha_n\}\subset(0,1)$ satisfy $\lim_{n\to\infty}\alpha_n=\lambda$, $\lambda\in(0,1)$. If the Mann iteration is not T-stable, then the Krasnoselskij iteration is not T-stable, and conversely.

Example 2.3. Let $T:[0,1)\to[0,1)$ be given by $Tx = x^2$, and $\lambda = 1/2$. Then the Krasnoselskij iteration converges to the unique fixed point $x^*=0$, and it is not T-stable.

The Krasnoselskij iteration converges because, supposing $F := \sup_n p_n < 1$, the sequence $p_n\to 0$, as we can see from

$$\begin{aligned} p_{n+1} &= \left(1-\frac{1}{2}\right)p_n + \frac{1}{2}p_n^2 = \frac{1}{2}p_n + \frac{1}{2}p_n^2 \\ &= \frac{1}{2}p_n(1+p_n) \le \frac{1+F}{2}p_n = \left(\frac{1+F}{2}\right)^n p_0 \longrightarrow 0; \end{aligned} \tag{2.8}$$

set $v_n = n/(n+1)$ and note that v_n does not converge to zero, while δ_n does:

$$\delta_n = \left|\frac{n+1}{n+2} - \frac{1}{2}\frac{n}{n+1} - \frac{1}{2}\frac{n^2}{(n+1)^2}\right| = \frac{n^2+4n+2}{2(n+1)^2(n+2)} \longrightarrow 0. \tag{2.9}$$

The Mann iteration also converges because (supposing $E := \sup_n e_n < 1$) one has

$$\begin{aligned} e_{n+1} &= (1-\alpha_n)e_n + \alpha_n e_n^2 = (1-(1-E)\alpha_n)e_n \\ &\le \prod_{k=1}^{n}(1-(1-E)\alpha_k)e_0 \le \exp\left(-(1-E)\sum_{k=1}^{n}\alpha_k\right)e_0 \longrightarrow 0; \end{aligned} \tag{2.10}$$

the last inequality is true because $1-x\le\exp(-x)$, $\forall x\ge 0$, and $\sum\alpha_n = +\infty$.

Take $u_n = n/(n+1)\to 1$, and note that $\varepsilon_n\to 0$ because

$$\varepsilon_n = \left|\frac{n+1}{n+2} - (1-\alpha_n)\frac{n}{n+1} - \alpha_n\frac{n^2}{(n+1)^2}\right| = \frac{\alpha_n n^2 + (2\alpha_n+1)n+1}{(n+1)^2(n+2)}. \tag{2.11}$$

So the Mann iteration is not T-stable. Actually, by use of Theorem 2.1, one can easily obtain the non-T-stability of the other iteration, provided that the previous one is not stable.

The following result takes in consideration the case in which no condition on $\{\alpha_n\}$ are imposed.

Theorem 2.4. *Let X be a normed space and $T: X\to X$ a map, and $\{\alpha_n\}\subset(0,1)$. If*

$$\lim_{n\to\infty}\left\|v_n - Tv_n\right\| = 0, \qquad \lim_{n\to\infty}\left\|u_n - Tu_n\right\| = 0, \tag{2.12}$$

then the following are equivalent:

(i) *the Mann iteration is T-stable,*

(ii) *the Krasnoselskij iteration is T-stable.*

Proof. We prove that (i)⇒(ii). Suppose $\lim_{n\to\infty}\delta_n = 0$, to note that,

$$\begin{aligned}\varepsilon_n &= \|v_{n+1} - (1-\alpha_n)v_n - \alpha_n T v_n\| \\ &= \|v_{n+1} - v_n + \lambda v_n - \lambda v_n + \alpha_n v_n - \lambda T v_n + \lambda T v_n - \alpha_n T v_n\| \\ &\le \|v_{n+1} - (1-\lambda)v_n - \lambda T v_n\| + |\lambda - \alpha_n|\,\|v_n - T v_n\| \\ &\le \delta_n + 2\|v_n - T v_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{2.13}$$

Condition (i) assures that if $\lim_{n\to\infty}\varepsilon_n = 0$, then $\lim_{n\to\infty} v_n = x^*$. Thus, for a $\{v_n\}$ satisfying

$$\lim_{n\to\infty}\delta_n = \lim_{n\to\infty}\|v_{n+1} - (1-\lambda)v_n - \lambda T v_n\| = 0, \tag{2.14}$$

we have shown that $\lim_{n\to\infty} v_n = x^*$.

Conversely, we prove (ii)⇒(i). Suppose $\lim_{n\to\infty}\varepsilon_n = 0$. Observe that

$$\begin{aligned}\delta_n &= \|u_{n+1} - (1-\lambda)u_n - \lambda T u_n\| \\ &= \|u_{n+1} - u_n + \lambda u_n - \lambda T u_n + \alpha_n u_n - \alpha_n u_n - \alpha_n T u_n + \alpha_n T u_n\| \\ &\le \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T u_n\| + |\lambda - \alpha_n|\,\|u_n - T u_n\| \\ &\le \varepsilon_n + 2\|u_n - T u_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{2.15}$$

Condition (ii) assures that if $\lim_{n\to\infty}\delta_n = 0$, then $\lim_{n\to\infty} v_n = x^*$. Thus, for a $\{u_n\}$ satisfying

$$\lim_{n\to\infty}\varepsilon_n = \lim_{n\to\infty}\|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T u_n\| = 0, \tag{2.16}$$

we have shown that $\lim_{n\to\infty} u_n = x^*$. □

Remark 2.5. Let X be a normed space and $T : X \to X$ a map, $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty}\|v_n - Tv_n\| = 0$, $\lim_{n\to\infty}\|u_n - Tu_n\| = 0$. If the Mann iteration is not T-stable, then the Krasnoselskij iteration is not T-stable, and conversely.

Note that one can consider the usual conditions $\lambda = 1/2$, $\lim\alpha_n = 0$, and $\sum\alpha_n = \infty$ in Theorem 2.4 and Remark 2.5.

Example 2.6. Again, let $T : [0,1) \to [0,1)$ be given by $Tx = x^2$, and $\lambda = 1/2$, $\alpha_n = 1/n$. Set $v_n = u_n = n/(n+1)$, to note that $\lim_{n\to\infty} u_n = 1$, and

$$\lim_{n\to\infty}\|v_n - T v_n\| = \lim_{n\to\infty}\frac{n}{(n+1)^2} = 0. \tag{2.17}$$

Hence, neither the Mann nor the Krasnoselskij iteration is T-stable, as we can see from Example 2.3.

3. Further results

Let $q_0 \in X$ be fixed, and let $q_{n+1} = Tq_n$ be the Picard-Banach iteration.

Definition 3.1. The Picard iteration is said to be T-stable if and only if for every sequence $\{q_n\} \subset X$ given,

$$\lim_{n\to\infty} \Delta_n = 0 \Longrightarrow \lim_{n\to\infty} q_n = x^*, \tag{3.1}$$

where $\Delta_n := \|q_{n+1} - Tq_n\|$.

In [6], the equivalence between the T-stabilities of Picard-Banach iteration and Mann iteration is given, that is, the following holds.

THEOREM 3.2 [6]. *Let X be a normed space and $T : X \to X$ a map. If*

$$\lim_{n\to\infty} ||q_n - Tq_n|| = 0, \qquad \lim_{n\to\infty} ||u_n - Tu_n|| = 0, \tag{3.2}$$

then the following are equivalent:
(i) *for all $\{\alpha_n\} \subset (0,1)$, the Mann iteration is T- stable,*
(ii) *the Picard iteration is T-stable.*

Theorems 2.4 and 3.2 lead to the following conclusion.

COROLLARY 3.3. *Let X be a normed space and $T : X \to X$ a map. If*

$$\lim_{n\to\infty} ||q_n - Tq_n|| = 0, \qquad \lim_{n\to\infty} ||v_n - Tv_n|| = 0, \qquad \lim_{n\to\infty} ||u_n - Tu_n|| = 0, \tag{3.3}$$

then the following are equivalent:
(i) *for all $\{\alpha_n\} \subset (0,1)$, the Mann iteration is T-stable,*
(ii) *the Picard-Banach iteration is T-stable,*
(iii) *the Krasnoselskij iteration is T-stable.*

Remark 3.4. Let X be a normed space and $T : X \to X$ a map, $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty} \|q_n - Tq_n\| = 0$, $\lim_{n\to\infty} \|v_n - Tv_n\| = 0$, $\lim_{n\to\infty} \|u_n - Tu_n\| = 0$. If the Mann or Krasnoselskij iteration is not T-stable, then the Picard-Banach iteration is not T-stable, and conversely.

Example 3.5. To see that the Picard-Banach iteration is also not T-stable, consider $T : [0,1) \to [0,1)$, by $Tx = x^2$.

Indeed, setting $q_n = n/(n+1)$, we have

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} \frac{n}{n+1} = 1,$$
$$\lim_{n\to\infty} \left| \frac{n}{n+1} - \left(\frac{n}{n+1}\right)^2 \right| = \frac{n}{(n+1)^2} = 0. \tag{3.4}$$

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