Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2007, Article ID 64306, 7 pages
doi:10.1155/2007/64306

Research Article

Hybrid Iteration Method for Fixed Points of Nonexpansive Mappings in Arbitrary Banach Spaces

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Received 20 June 2007; Accepted 23 November 2007

Recommended by Nanjing Huang

We prove that recent results of Wang (2007) concerning the iterative approximation of fixed points of nonexpansive mappings using a hybrid iteration method in Hilbert spaces can be extended to arbitrary Banach spaces without the strong monotonicity assumption imposed on the hybrid operator.

1. Introduction

Let E be a real Banach space. A mapping $T : E \to E$ is said to be L-*Lipschitzian* if there exists $L > 0$ such that

$$||Tx - Ty|| \le L\|x - y\|, \quad \forall x, y \in E. \tag{1.1}$$

T is said to be *nonexpansive* if $L = 1$ in (1.1).

Several authors have studied various methods for the iterative approximation of fixed points of nonexpansive mappings. Recently, Wang [1] studied the following iteration method in Hilbert spaces.

The hybrid iteration method. Let H be a Hilbert space, $T : H \to H$ a nonexpansive mapping with $F(T) = \{x \in H : Tx = x\} \neq \emptyset$, and $F : H \to H$ an L-Lipschitzian mapping which is also η-*strongly monotone,* where T is η-strongly monotone if there exists $\eta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \eta\|x - y\|^2, \quad \forall x, y \in H. \tag{1.2}$$

Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ be real sequences in $[0,1)$, and $\mu > 0$, then the sequence $\{x_n\}_{n=1}^{\infty}$ is generated from an arbitrary $x_1 \in H$ by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T^{\lambda_{n+1}} x_n, \quad n \geq 1, \tag{1.3}$$

where $T^{\lambda_{n+1}} x_n := Tx_n - \lambda_{n+1}\mu F(Tx_n)$, $\mu > 0$. Wang's work was motivated by earlier results of Xu and Kim [2] and Yamada [3], in addition to several other related results. Using this iteration method, Wang proved the following main results.

Lemma 1.1 (see [1, page 3]). *Let H be a Hilbert space, $T : H \to H$ a nonexpansive mapping with $F(T) = \{x \in H : Tx = x\} \neq \varnothing$, and $F : H \to H$ an η-strongly monotone and L-Lipschitzian mapping. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated from an arbitrary $x_1 \in H$ by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T^{\lambda_{n+1}} x_n, \quad n \geq 1, \tag{1.4}$$

where $T^{\lambda_{n+1}} x_n := Tx_n - \lambda_{n+1}\mu F(Tx_n)$, $\mu > 0$, and let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ be real sequences in $[0,1)$ satisfying the following conditions:

(i) $0 < \alpha \leq \alpha_n \leq \beta < 1$, *for some* $\alpha, \beta \in (0,1)$,

(ii) $\sum_{n=1}^{\infty} \lambda_n < \infty$,

(iii) $0 < \mu < 2\eta/L^2$.

Then,

(a) $\lim_{n\to\infty} \|x_n - x^*\|$ *exists for each* $x^* \in F(T)$,

(b) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Theorem 1.2 (see [1, page 5]). *Let H, T, $F(T)$, F, $\{T^{\lambda_{n+1}}\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\lambda_n\}_{n=1}^{\infty}$, μ, α, and β be as in Lemma 1.1. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated from an arbitrary $x_1 \in H$ by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T^{\lambda_{n+1}} x_n, \quad n \geq 1. \tag{1.5}$$

Then,

(a) $\{x_n\}_{n=1}^{\infty}$ *converges weakly to a fixed point of T,*

(b) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a fixed point of T if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, *where* $d(x, F(T)) := \inf\{\|x-p\| : p \in F(T)\}$.

It is our purpose in this paper to extend Lemma 1.1 and Theorem 1.2 from Hilbert spaces to arbitrary Banach spaces. Our results are much more general and applicable than the results of Wang [1] because the strong monotonicity condition imposed on F by Wang is not required in our results.

2. Preliminaries

In the sequel, we will need what follows.

A Banach space E is said to satisfy *Opial's condition* (see, e.g., [4]) if for each sequence $\{x_n\}_{n=1}^{\infty}$ in E which converges weakly to a point $x \in E$, we have

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||, \quad \forall y \in E. \tag{2.1}$$

Let E be a Banach space. A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *demiclosed at a point* $p \in D(T)$ if, whenever, $\{x_n\}_{n=1}^{\infty}$ is a sequence in E which converges weakly to a point $x \in E$ and $\{Tx_n\}_{n=1}^{\infty}$ converges strongly to p, then $Tx = p$. Furthermore, T is said to be *demicompact* if, whenever, $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in $D(T)$ such that $\{x_n - Tx_n\}_{n=1}^{\infty}$ converges strongly, then $\{x_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly. T is said to satisfy condition (A) if $F(T) \neq \emptyset$ and there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x,F(T)))$ for all $x \in D(T)$, where $d(x,F(T)) := \inf\{\|x - p\| : p \in F(T)\}$.

Lemma 2.1 (see [5]). *Let E be a reflexive Banach space satisfying Opial's condition and let K be a nonempty closed convex subset of E. Let $T : K \to E$ be a nonexpansive mapping. Then, $(I - T)$ is demiclosed on K, where I is the identity mapping.*

Lemma 2.2 (see [6, page 1184], [7]). *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 1. \tag{2.2}$$

If $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty}b_n < \infty$, then $\lim_{n\to\infty}a_n$ exists. In particular, if $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty}a_n = 0$.

Lemma 2.3 (see [8, page 770]). *Let E be an arbitrary normed space and let $\{t_n\}_{n=1}^{\infty}$ be a real sequence satisfying the following conditions:*

(i) $0 \leq t_n \leq t < 1$ *for all* $n \geq 1$ *and for some* $t \in (0,1)$,
(ii) $\sum_{n=1}^{\infty}t_n = \infty$.

Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be two sequences in E such that

(iii) $u_{n+1} = (1 - t_n)u_n + t_n v_n$, $n \geq 1$,
(iv) $\lim_{n\to\infty}\|u_n\| = d$ *for some* $d \in [0,\infty)$,
(v) $\limsup_{n\to\infty}\|v_n\| \leq d$,
(vi) $\{\sum_{j=1}^{n}t_j v_j\}_{n=1}^{\infty}$ *is bounded.*

Then, $d = 0$.

3. Main results

Theorem 3.1. *Let E be an arbitrary real Banach space, $T : E \to E$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $F : E \to E$ an L-Lipschitzian mapping. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated from an arbitrary $x_1 \in E$ by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^{\lambda_{n+1}}x_n, \quad n \geq 1, \tag{3.1}$$

where $T^{\lambda_{n+1}}x_n := Tx_n - \lambda_{n+1}\mu F(Tx_n)$, $\mu > 0$, and $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=2}^{\infty}$ are real sequences in $[0,1)$ satisfying the following conditions:

(i) $0 < \alpha \leq \alpha_n < 1$ *for all* $n \geq 1$ *and for some* $\alpha \in (0,1)$,
(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$,
(iii) $\sum_{n=2}^{\infty}\lambda_n < \infty$.

Then,

(a) $\lim_{n\to\infty}\|x_n - x^*\|$ *exists for each* $x^* \in F(T)$,

(b) $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$,

(c) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a fixed point of* T *if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Proof. Let $x^* \in F(T)$ be arbitrary, then,

$$\begin{aligned}
||x_{n+1} - x^*|| &= ||\alpha_n(x_n - x^*) + (1-\alpha_n)(Tx_n - x^*) - (1-\alpha_n)\lambda_{n+1}\mu F(Tx_n)|| \\
&\le ||\alpha_n(x_n - x^*) + (1-\alpha_n)(Tx_n - x^*)|| + (1-\alpha_n)\lambda_{n+1}\mu||F(Tx_n)|| \\
&\le \alpha_n||x_n - x^*|| + (1-\alpha_n)||Tx_n - x^*|| + (1-\alpha_n)\lambda_{n+1}\mu||F(Tx_n) - F(x^*)|| \\
&\quad + (1-\alpha_n)\lambda_{n+1}\mu||F(x^*)|| \\
&\le ||x_n - x^*|| + (1-\alpha_n)\lambda_{n+1}\mu L||x_n - x^*|| + (1-\alpha_n)\lambda_{n+1}\mu||F(x^*)|| \\
&= [1+\delta_n]||x_n - x^*|| + \sigma_n,
\end{aligned} \tag{3.2}$$

where $\delta_n = (1-\alpha_n)\lambda_{n+1}\mu L$ and $\sigma_n = (1-\alpha_n)\lambda_{n+1}\mu\|F(x^*)\|$.

Since $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty}\sigma_n < \infty$, it follows from Lemma 2.2 that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. This completes the proof of (a).

Since $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ is bounded, there exists $M > 0$ such that

$$||x_n - x^*|| \le M, \quad \forall n \ge 1. \tag{3.3}$$

Observe that

$$\begin{aligned}
||x_{n+1} - Tx_{n+1}|| &= ||\alpha_n x_n + (1-\alpha_n)T^{\lambda_{n+1}}x_n - Tx_{n+1}|| \\
&\le ||\alpha_n(x_n - Tx_{n+1}) + (1-\alpha_n)(Tx_n - Tx_{n+1})|| \\
&\quad + (1-\alpha_n)\lambda_{n+1}\mu||F(Tx_n)|| \\
&\le \alpha_n||(x_n - Tx_{n+1})|| + (1-\alpha_n)||Tx_n - Tx_{n+1}|| \\
&\quad + (1-\alpha_n)\lambda_{n+1}\mu||F(Tx_n)|| \\
&\le \alpha_n||x_n - x_{n+1}|| + \alpha_n||x_{n+1} - Tx_{n+1}|| + (1-\alpha_n)||x_n - x_{n+1}|| \\
&\quad + (1-\alpha_n)\lambda_{n+1}\mu||F(Tx_n)||.
\end{aligned} \tag{3.4}$$

Thus,

$$\begin{aligned}
||x_{n+1} - Tx_{n+1}|| &\le \frac{1}{1-\alpha_n}||x_n - x_{n+1}|| + \lambda_{n+1}\mu||F(Tx_n)|| \\
&\le ||x_n - Tx_n|| + 2\lambda_{n+1}\mu||F(Tx_n)||.
\end{aligned} \tag{3.5}$$

It follows from (3.3) that

$$||F(Tx_n)|| \le L||x_n - x^*|| + ||F(x^*)|| \le LM + ||F(x^*)|| \le D \tag{3.6}$$

for all $n \geq 1$ and for some $D > 0$. Using (3.6) in (3.5), we obtain

$$||x_{n+1} - Tx_{n+1}|| \leq ||x_n - Tx_n|| + 2\lambda_{n+1}\mu D = ||x_n - Tx_n|| + \gamma_n, \tag{3.7}$$

where $\gamma_n = 2\lambda_{n+1}\mu D$. Since $\sum_{n=1}^{\infty}\gamma_n < \infty$, it follows from Lemma 2.2 that $\lim_{n\to\infty}\|x_n - Tx_n\|$ exists. Let $\lim_{n\to\infty}\|x_n - Tx_n\| = d$, and set $u_n = x_n - Tx_n$ so that

$$u_{n+1} = (1 - t_n)u_n + t_n v_n, \tag{3.8}$$

where $t_n = 1 - \alpha_n$ and $v_n = (1/(1-\alpha_n))(Tx_n - Tx_{n+1}) - \lambda_{n+1}\mu F(Tx_n)$. Observe that $0 < t_n \leq 1 - \alpha = t \in (0,1)$ and $\sum_{n=1}^{\infty} t_n = \infty$. Furthermore,

$$\begin{aligned}
||v_n|| &\leq \frac{1}{1-\alpha_n}||Tx_n - Tx_{n+1}|| + \lambda_{n+1}\mu||F(Tx_n)|| \\
&\leq \frac{1}{1-\alpha_n}||x_n - x_{n+1}|| + \lambda_{n+1}\mu||F(Tx_n)|| \\
&\leq ||x_n - Tx_n|| + 2\lambda_{n+1}\mu||F(Tx_n)|| \\
&\leq ||x_n - Tx_n|| + 2\lambda_{n+1}\mu D.
\end{aligned} \tag{3.9}$$

Thus, $\limsup_{n\to\infty}\|v_n\| \leq d$. Also,

$$\begin{aligned}
\left\|\sum_{j=1}^{n} t_j v_j\right\| &= \left\|\sum_{j=1}^{n}(1-\alpha_j)\left[\frac{1}{(1-\alpha_j)}(Tx_j - Tx_{j+1}) - \lambda_{j+1}\mu F(Tx_j)\right]\right\| \\
&\leq \left\|\sum_{j=1}^{n}(Tx_j - Tx_{j+1})\right\| + \sum_{j=1}^{n}(1-\alpha_j)\lambda_{j+1}\mu||F(Tx_j)|| \\
&< ||Tx_1 - Tx_{n+1}|| + \mu D\sum_{j=1}^{n}\lambda_{j+1} \\
&\leq ||x_1 - x_{n+1}|| + \mu D\sum_{j=1}^{n}\lambda_{j+1} \\
&\leq ||x_1 - x^*|| + ||x_{n+1} - x^*|| + \mu D\sum_{j=1}^{\infty}\lambda_{j+1} \leq K
\end{aligned} \tag{3.10}$$

for all $n \geq 1$ and for some $K > 0$. Hence, $\{\sum_{j=1}^{n} t_j v_j\}_{n=1}^{\infty}$ is bounded. It now follows from Lemma 2.3 that $\lim_{n\to\infty}\|v_n\| = \lim_{n\to\infty}\|x_n - Tx_n\| = 0$. This completes the proof of (b).

From (3.2), we obtain

$$||x_{n+1} - x^*|| \leq [1 + \delta_n]||x_n - x^*|| + \sigma_n \leq ||x_n - x^*|| + M\delta_n + \sigma_n = ||x_n - x^*|| + \beta_n, \tag{3.11}$$

where $\beta_n = M\delta_n + \sigma_n$. Hence, $d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + \beta_n$. Since $\sum_{n=1}^{\infty}\beta_n < \infty$, it follows from Lemma 2.2 that $\lim_{n\to\infty} d(x_n, F(T))$ exists.

If $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point p of T, then $\lim_{n\to\infty}\|x_n - p\| = 0$. Since

$$0 \le d(x_n, F(T)) \le ||x_n - p||, \tag{3.12}$$

we have $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Conversely, suppose $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, then we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus for arbitrary $\epsilon > 0$, there exists a positive integer N_1 such that $d(x_n, F(T)) < \epsilon/4$ for all $n \ge N_1$. Furthermore, $\sum_{n=1}^{\infty}\beta_n < \infty$ implies that there exists a positive integer N_2 such that $\sum_{j=n}^{\infty}\beta_j < \epsilon/4$ for all $n \ge N_2$. Choose $N = \max\{N_1, N_2\}$, then $d(x_N, F(T)) < \epsilon/4$ and $\sum_{j=N}^{\infty}\beta_j < \epsilon/4$. It follows from (3.11) that for all $n, m \ge N$ and for all $p \in F(T)$, we have

$$\begin{aligned} ||x_n - x_m|| &\le ||x_n - p|| + ||x_m - p|| \\ &\le ||x_N - p|| + \sum_{j=N+1}^{n} \beta_j + ||x_N - p|| + \sum_{j=N+1}^{m} \beta_j \\ &\le 2||x_N - p|| + 2\sum_{j=N}^{\infty} \beta_j. \end{aligned} \tag{3.13}$$

Taking infimum over all $p \in F(T)$, we obtain

$$||x_n - x_m|| \le 2d(x_N, F(T)) + 2\sum_{j=N}^{\infty}\beta_j < \epsilon, \quad \forall n, m \ge N. \tag{3.14}$$

Thus, $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Suppose $\lim_{n\to\infty} x_n = u$, then since $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, we have $u \in F(T)$. This completes the proof of (c). □

Theorem 3.2. *Let E be a real reflexive Banach space satisfying Opial's condition, $T : E \to E$ a nonexpansive mapping with $F(T) \ne \emptyset$, and $F : E \to E$ an L-Lipschitzian mapping. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated from an arbitrary $x_1 \in H$ by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^{\lambda_{n+1}} x_n, \quad n \ge 1, \tag{3.15}$$

where $T^{\lambda_{n+1}} x_n := Tx_n - \lambda_{n+1}\mu F(Tx_n)$, $\mu > 0$, and $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=2}^{\infty}$ are real sequences in $[0,1)$ satisfying the following conditions:

(i) $0 < \alpha \le \alpha_n < 1$ *for all* $n \ge 1$ *and for some* $\alpha \in (0,1)$,

(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$,

(iii) $\sum_{n=2}^{\infty}\lambda_n < \infty$.

Then, $\{x_n\}_{n=1}^{\infty}$ converges weakly to a fixed point of T.

Proof. From Lemma 2.1, $(I - T)$ is demiclosed at zero, and since $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and E satisfies Opial's condition, it follows from standard argument that $\{x_n\}_{n=1}^{\infty}$ converges weakly to a fixed point of T. □

Remark 3.3. It follows from Lemma 2.2 and Theorem 3.1 that under the hypothesis of Theorem 3.1, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point p of T if and only if $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ which converges strongly to p. Thus, under the hypothesis of Theorem 3.1, if T is in addition completely continuous or demicompact, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

Furthermore, if T satisfies condition (A), then $\liminf_{n\to\infty} d(x_n, F(T)) = 0$; so under the conditions of Theorem 3.1, if T satisfies condition (A), then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

Remark 3.4. Theorems 3.1 and 3.2 and Remark 3.3 extend the results of [1] from Hilbert spaces to much more general Banach spaces as considered here. Furthermore, the strong monotonicity condition imposed on F in [1] is not required in our results.

Prototypes of our real sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ are $\alpha_n = n/(n+1), n \geq 1$ and $\lambda_n = 1/(n+1)^2, n \geq 1$.

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