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Research Article

A Fixed Point Theorem Based on Miranda

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A new fixed point theorem is proved by using the theorem of Miranda.

1. Introduction

In 1940, Miranda published the following theorem ([1]).

Theorem 1.1. *Let* $\Omega = \{x \in \mathbb{R}^n : |x_i| \leq L,\ i = 1,\dots,n\}$ *and let* $f : \Omega \to \mathbb{R}^n$ *be continuous satisfying*

$$\begin{aligned} &f_i(x_1,x_2,\dots,x_{i-1},-L,x_{i+1},\dots,x_n) \geq 0, \\ &f_i(x_1,x_2,\dots,x_{i-1},+L,x_{i+1},\dots,x_n) \leq 0, \end{aligned} \qquad \forall\, i\{1,\dots,n\}. \tag{1.1}$$

Then, $f(x) = 0$ *has a solution in* Ω.

For $n = 1$, Theorem 1.1 reduces to the well-known intermediate-value theorem. Miranda proved his theorem using the Brouwer fixed point theorem. Using the Brouwer degree of a mapping, Vrahatis gave another short proof of Theorem 1.1 (see [2]). Following this proof it is easy to see that Theorem 1.1 is also true, if L is dependent of i; that is, Ω can also be a rectangle and need not to be a cube. Even some L_i can be zero. Very often, the theorem of Miranda is stated as in the following corollary (see also [3, 4]), which is not the theorem of Miranda in its original form, but a consequence of it.

Corollary 1.2. *Let* $\hat{x} \in \mathbb{R}^n$, $L = (l_i) \in \mathbb{R}^n$, $l_i \geq 0$, *for* $i = 1,\dots,n$, *let* Ω *be the rectangle* $\Omega := \{x \in \mathbb{R}^n : |x_i - \hat{x}_i| \leq l_i, i = 1,\dots,n\}$ *and let* $f : \Omega \to \mathbb{R}^n$ *be a continuous function on* Ω.

Also let

$$F_i^+ := \{x \in \Omega : x_i = \hat{x}_i + l_i\}, \quad F_i^- := \{x \in \Omega : x_i = \hat{x}_i - l_i\}, \quad i = 1,\dots,n, \tag{1.2}$$

be the pairs of parallel opposite faces of the rectangle Ω. *If for all* $i = 1,\dots,n$

$$f_i(x)\cdot f_i(y) \le 0, \quad \forall\, x \in F_i^+,\ \forall\, y \in F_i^-, \tag{1.3}$$

then there exists some $x^* \in \Omega$ *satisfying* $f(x^*) = 0$.

In principle, Corollary 1.2 says that Theorem 1.1 is also true if the $\le$ -sign and the $\ge$ -sign are exchanged with each other in (1.1). Corollary 1.2 also says that Theorem 1.1 is not restricted to a rectangle with 0 as its center.

Many generalizations have been given (see, e.g., [2, 4–6] for the finite-dimensional case and see [7, 8] for the infinite-dimensional case). In the presented paper we give a generalization of Corollary 1.2 in the infinite-dimensional Hilbert space l^2. Finally, we prove a fixed point version of Theorem 1.1 in l^2.

2. The infinite-dimensional case

Let l^2 be the infinite-dimensional Hilbert space of all square summable sequences of real numbers equipped with the natural order

$$x \le y :\Longleftrightarrow x_i \le y_i, \quad \forall_i \in \mathbb{N}, \tag{2.1}$$

and equipped with the norm $\|x\| := \sqrt{\sum_{i=1}^{\infty} x_i^2}$.

Theorem 2.1. *Let* $\hat{x} = \{\hat{x}_i\}_{i=1}^{\infty} \in l^2$, $L = \{l_i\}_{i=1}^{\infty} \in l^2$, $l_i \ge 0$, *for all* $i \in \mathbb{N}$, $\Omega := \{x \in l^2 : |x_i - \hat{x}_i| \le l_i, \text{for all}\, i \in \mathbb{N}\}$ *and let* $f : \Omega \to l^2$ *be a continuous function on* Ω. *Also let*

$$F_i^+ := \{x \in \Omega : x_i = \hat{x}_i + l_i\}, \quad F_i^- := \{x \in \Omega : x_i = \hat{x}_i - l_i\}, \quad \forall\, i \in \mathbb{N}. \tag{2.2}$$

If for all $i \in \mathbb{N}$ *it holds that*

$$f_i(x)\cdot f_i(y) \le 0, \quad \forall\, x \in F_i^+,\ \forall\, y \in F_i^-, \tag{2.3}$$

then there exists some $x^* \in \Omega$ *satisfying* $f(x^*) = 0$.

Proof. For fixed $n \in \mathbb{N}$, we consider the function $\tilde{h}^{(n)} : \Omega \to l^2$ defined by

$$\tilde{h}^{(n)}(x) := \begin{pmatrix} f_1(x_1, x_2, \dots, x_{n-1}, x_n, x_{n+1}, \dots) \\ \vdots \\ f_n(x_1, x_2, \dots, x_{n-1}, x_n, x_{n+1}, \dots) \\ 0 \\ \vdots \end{pmatrix}. \tag{2.4}$$

Since Ω is compact and since f is continuous, the set $f(\Omega)$ is compact. Therefore, for given $\varepsilon > 0$ there is a finite set of elements $v^{(1)}, \dots, v^{(p)} \in f(\Omega)$ such that if $f(x) \in f(\Omega)$, then there is a $v \in \{v^{(1)},\dots,v^{(p)}\}$ such that

$$\|f(x)-v\| \le \varepsilon \tag{2.5}$$

and there exists $n_1 = n_1(\varepsilon) \in \mathbb{N}$ such that for all $n > n_1$ it holds that

$$\sqrt{\sum_{j=n+1}^{\infty}(v_j)^2} \le \varepsilon, \quad \forall v \in \{v^{(1)},\dots,v^{(p)}\}. \tag{2.6}$$

So, if $n > n_1$ is valid, then for all $f(x) \in f(\Omega)$ we have some $v \in \{v^{(1)},\dots,v^{(p)}\}$ such that

$$\|f(x)-\tilde{h}^{(n)}(x)\| = \left\| \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_{n+1}(x) \\ f_{n+2}(x) \\ \vdots \end{pmatrix} \right\| \le \|f(x)-v\| + \left\| \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_{n+1} \\ v_{n+2} \\ \vdots \end{pmatrix} \right\| \le 2\varepsilon \tag{2.7}$$

for all $x \in \Omega$. Now, for fixed $n \in \mathbb{N}$ we define

$$\Omega_n := \begin{pmatrix} [\hat{x}_1 - l_1, \hat{x}_1 + l_1] \\ \vdots \\ [\hat{x}_n - l_n, \hat{x}_n + l_n] \end{pmatrix} \tag{2.8}$$

and $h^{(n)} : \Omega_n \to \mathbb{R}^n$ by

$$h^{(n)}(x) := \begin{pmatrix} f_1(x_1, x_2, \dots, x_{n-1}, x_n, \hat{x}_{n+1}, \hat{x}_{n+2}, \dots) \\ \vdots \\ f_n(x_1, x_2, \dots, x_{n-1}, x_n, \hat{x}_{n+1}, \hat{x}_{n+2}, \dots) \end{pmatrix}. \tag{2.9}$$

Due to (2.3) and Corollary 1.2 there exists $x^{(n)} \in \Omega_n$ with

$$h^{(n)}(x^{(n)}) = 0. \tag{2.10}$$

Setting

$$\tilde{x}^{(n)} := \begin{pmatrix} x^{(n)} \\ \hat{x}_{n+1} \\ \hat{x}_{n+2} \\ \vdots \end{pmatrix}, \tag{2.11}$$

it holds that

$$\tilde{x}^{(n)} \in \Omega, \quad \tilde{h}^{(n)}(\tilde{x}^{(n)}) = 0. \tag{2.12}$$

Now, let $n > n_1$. Then,

$$||f(\widetilde{x}^{(n)})|| = ||f(\widetilde{x}^{(n)}) - \widetilde{h}^{(n)}(\widetilde{x}^{(n)})|| \le 2\varepsilon. \tag{2.13}$$

Hence, $\lim_{n\to\infty} f(\widetilde{x}^{(n)}) = 0$. Since Ω is compact, the sequence $\widetilde{x}^{(n)}$ has an accumulation point in Ω, say x^*. Without loss of generality, we assume that $\lim_{n\to\infty}\widetilde{x}^{(n)} = x^*$ holds. On the one hand, it follows that $\lim_{n\to\infty} f(\widetilde{x}^{(n)}) = f(x^*)$, since f is continuous. On the other hand, it follows that $f(x^*) = 0$, since the limit is unique. □

Next, we prove the fixed point version of Theorem 1.1 in l^2.

Theorem 2.2. *Let $L = \{l_i\}_{i=1}^{\infty} \in l^2$, $l_i \ge 0$, for all $i \in \mathbb{N}$. Let $\Omega = \{x \in l^2 : |x_i| \le l_i, \forall i \in \mathbb{N}\}$ and suppose that the mapping $g : \Omega \to l^2$ is continuous satisfying*

$$\begin{aligned} &g_i(x_1, x_2, \dots, x_{i-1}, -l_i, x_{i+1}, \dots) \ge 0, \\ &g_i(x_1, x_2, \dots, x_{i-1}, +l_i, x_{i+1}, \dots) \le 0, \end{aligned} \quad \forall i \in \mathbb{N}. \tag{2.14}$$

Then, $g(x) = x$ has a solution in Ω.

Proof. We consider the continuous function

$$f(x) := g(x) - x, \quad x \in \Omega. \tag{2.15}$$

Since for all $i \in \mathbb{N}$

$$\begin{aligned} f_i(x_1, \dots, x_{i-1}, -l_i, x_{i+1}, \dots) &= g_i(x_1, \dots, x_{i-1}, -l_i, x_{i+1}, \dots) + l_i \ge 0, \\ f_i(x_1, \dots, x_{i-1}, +l_i, x_{i+1}, \dots) &= g_i(x_1, \dots, x_{i-1}, +l_i, x_{i+1}, \dots) - l_i \le 0, \end{aligned} \tag{2.16}$$

due to Theorem 2.1 there exists $x \in \Omega$ satisfying $f(x) = 0$; that is, $g(x) = x$. □

Example 2.3. Let $b \in l^2$ and $A = (a_{ik})$ satisfying $\sum_{i,k=1}^{\infty} |a_{ik}|^2 < \infty$. Then, the mapping

$$g(x) := \left(b_1 - \sum_{k=1}^{\infty} a_{1k}x_k, b_2 - \sum_{k=1}^{\infty} a_{2k}x_k, \dots\right) \tag{2.17}$$

is (even) a compact mapping from l^2 to l^2. Now, if A is some kind of diagonally dominant in the sense that there exists some $L = \{l_i\}_{i=1}^{\infty} \in l^2$such that for all $i \in \mathbb{N}$

$$a_{ii} \cdot l_i \ge |b_i| + \sum_{k=1, k\neq i}^{\infty} |a_{ik}| \cdot l_k, \tag{2.18}$$

then by Theorem 2.1 there exists some $\xi \in \Omega = \{x \in l^2 : |x_i| \le l_i, \forall i \in \mathbb{N}\}$ with $A\xi = b$. By Theorem 2.2 it follows that there exists $\eta \in \Omega$ satisfying $\eta = b - A\eta$.

Remark 2.4. Note that in Theorem 2.2 it is not necessary that g is a self-mapping as it is assumed in many other fixed point theorems.

Remark 2.5. Theorem 2.2 is also valid in $\mathbb{R}^n$ of course. Note, however, that the conditions (2.14) cannot be changed analogously as the conditions (1.1) have been changed to (1.3). We demonstrate this in Figure 2.1 for $n = 1$.

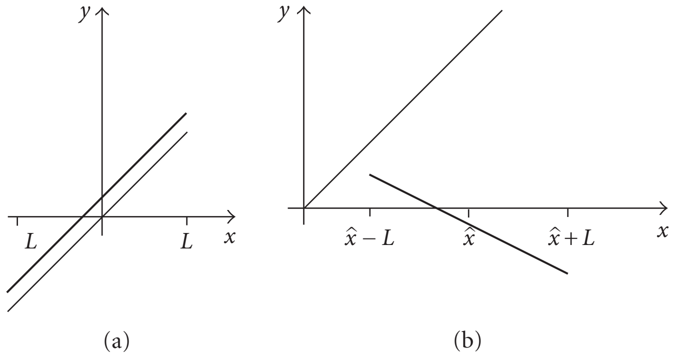

Figure 2.1. In both pictures the thick line is the graph of a function $y = g(x)$, $x \in \Omega$. In the left picture, $\Omega = [-L, L]$ and $g(-L) < 0$, $g(L) > 0$. According to Corollary 1.2 $g(x)$ has a zero in Ω. However, $g(x)$ has no fixed point in Ω, which is no contradiction to Theorem (2.2), since $g(-L) \geq 0$, $g(L) \leq 0$ is not valid, here. In the right picture, $\Omega = [\hat{x} - L, \hat{x} + L]$ and $g(\hat{x} - L) > 0$, $g(\hat{x} + L) < 0$. According to Corollary 1.2, $g(x)$ has a zero in Ω. However, $g(x)$ has no fixed point in Ω.

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