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Research Article

Estimating Nielsen Numbers on Wedge Product Spaces

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Let $f : X \to X$ be a self-map of a finite polyhedron that is an aspherical wedge product space X. In this paper, we estimate the Nielsen number $N(f)$ of f. In particular, we study some algebraic properties of the free products and then estimate Nielsen numbers on torus wedge surface with boundary, Klein bottle wedge surface with boundary, and torus wedge torus.

1. Introduction

Let X be a space and let $f : X \to X$ be a self-map. Let $\text{Fix}(f) = \{x \in X : f(x) = x\}$ denote the fixed point set. The Nielsen number $N(f)$ provides a lower bound for

$$\min\{\#\text{Fix}(g) : g \simeq f\} \tag{1.1}$$

and is often sharp. But, in general, it is very difficult to compute $N(f)$ from its definition. For background on Nielsen fixed point theory, see [1–3].

For a given space X, algebraic properties of its fundamental group $\pi_1(X)$ are usually important to compute Nielsen numbers on it. However, if the fundamental group of X is free or a free product group, then computing Nielsen number on X is extremely difficult see [4].

In this paper, we estimate the Nielsen numbers on aspherical wedge product spaces, focusing on the following three cases:

(1) torus wedge surface with boundary;

(2) Klein bottle wedge surface with boundary;
(3) torus wedge torus.

It is well known that the fundamental group of the wedge product of spaces is the free product of the fundamental groups of the spaces. In Section 2, we present several properties of free product of groups which we use in the final section to classify all maps of the spaces above. Then we consider the Nielsen numbers on aspherical wedge product spaces in Section 3. As applications, we estimate the Nielsen numbers on the above three types of spaces in the final section.

2. Free products

Let $A*C$ be the free product of two groups A and C. The groups A and C are called the *free factors* of $A*C$. A *reduced sequence* (or *normal form*) is a sequence of elements $g_1, g_2, \dots, g_n$ from $A*C$ such that each $g_i \neq 1$, each g_i is in A or C, and successive g_i, g_{i+1} are not in the same free factor. It is well known that each element g of $A*C$ can be uniquely expressed as a product $g = g_1, g_2, \dots, g_n$, where $g_1, g_2, \dots, g_n$ is a reduced sequence, which is called the *reduced form* (or *normal form*) of g.

Let g be an element of $A*C$ with reduced form $g_1, g_2, \dots, g_n$. The *syllable length* $\lambda(g)$ of g is n and g is called *cyclically reduced* if g_1 and g_n are from different free factors or $n \leq 1$.

Theorem 2.1 [5, Theorem 4.2]. *Each element of $A*C$ is conjugate to a cyclically reduced element.*

Lemma 2.2. *Suppose that neither of u nor v is in the conjugate of a free factor. If $u^k = v^k$ in $A*C$, then $u = v$.*

Proof. Suppose that $u = u_1, u_2, \dots, u_m$ is cyclically reduced. Since u is not in the conjugate of a free factor, we have $m \geq 2$. Since u is cyclically reduced,

$$u^k = (u_1 u_2 \cdots u_m)(u_1 u_2 \cdots u_m) \cdots (u_1 u_2 \cdots u_m) \tag{2.1}$$

is also cyclically reduced and $\lambda(u^k) = km$. If v is not cyclically reduced, v^k is not either, and this contradicts the hypothesis $u^k = v^k$. Therefore, v is cyclically reduced and so v^k is cyclically reduced. Let $v = v_1, v_2, \dots, v_n$. Then, similarly to u, we have $n \geq 2$ and $\lambda(v^k) = kn$. Since $u^k = v^k$, this implies that $\lambda(u^k) = \lambda(v^k)$ and so $m = n$. Furthermore,

$$u^k v^{-k} = (u_1 \cdots u_m) \cdots (u_1 \cdots u_m)(v_n^{-1} \cdots v_1^{-1}) \cdots (v_n^{-1} \cdots v_1^{-1}) = 1. \tag{2.2}$$

Since u^k and v^{-k} are cyclically reduced and $m = n \geq 2$, we can say that $u^k v^{-k} = 1$ implies that each pair u_i and v_i^{-1} is in a same free factor and $u_i v_i^{-1} = 1$. Therefore, $u_i = v_i$ for all i and hence $u = v$.

Now, suppose that u is not cyclically reduced. Then, by Theorem 2.1, we can denote $u = wsw^{-1}$ for some element w and cyclically reduced element s in $A*C$. Since u is not in the conjugate of a free factor, the same is true of $s = w^{-1}uw$ and

$$s^k = (w^{-1}uw)^k = w^{-1}u^k w = w^{-1}v^k w = (w^{-1}vw)^k. \tag{2.3}$$

Since v is not in the conjugate of a free factor, this implies that $w^{-1}vw$ is not either, and hence the above paragraph applied to s and $w^{-1}vw$ in place of u and v shows that $s = w^{-1}vw$. Therefore, we have $u = wsw^{-1} = w(w^{-1}vw)w^{-1} = v$. □

Lemma 2.3. *Suppose that neither of u and v is in the conjugate of a free factor. If $u^m v^n = v^n u^m$ in $A*C$, then $uv = vu$.*

Proof. Suppose $u^m v^n = v^n u^m$, then $u^m = v^n u^m v^{-n} = (v^n u v^{-n})^m$. Since u and $v^n u v^{-n}$ are not in the conjugate of a free factor, by Lemma 2.2 we have $u = v^n u v^{-n}$. Then $v^n = u^{-1} v^n u = (u^{-1} v u)^n$. Applying Lemma 2.2 again, we obtain $v = u^{-1} v u$ and hence $uv = vu$. □

Lemma 2.4 [5, Corollary 4.1.5]. *If g is in $A*C$ and both $a \neq 1$ and gag^{-1} are in A, then g is in A.*

Theorem 2.5 [5, Corollary 4.1.6]. *If $uv = vu$ in $A*C$, then at least one of the following is true.*

(1) *u and v are in the same conjugate of a free factor.*

(2) *u and v are both powers of the same element in $A*C$.*

We say a group is 2-torsion free if it contains no elements of order 2.

Lemma 2.6. *Suppose that A and C are 2-torsion free groups. If g is in $A*C$ and both a and gag are in A, then g is in A.*

Proof. Let $g = g_1, g_2, \ldots, g_r$ be the reduced form of g in $A*C$. We use induction on r. If $r = 1$, then $g = g_1$ is either in A or in C. Suppose $g \in C$. If $a = 1$, then $gag = gg \notin A$, and if $a \neq 1$, then gag is a reduced form and has syllable length >1, and hence, $gag \notin A$, contrary to hypothesis. Therefore g is in A. Suppose that the lemma is true for $r = n-1$ and that $r = n$. We first show that $g_n \in A$. If g_1 and g_n are in C, then $gag = g_1, \ldots, g_n a g_1, \ldots, g_n$ has syllable length > 1. Thus $gag \notin A$, contrary to hypothesis. Suppose $g_1 \in A$ and $g_n \in C$. Since $gag \in A$, the terminal word g_n of $gag = g_1, \ldots, g_n a g_1, \ldots, g_n$ must cancel in gag. Let $n = 2k$ for some positive integer k. Then

$$gag = g_1, \ldots, g_{2k} a g_1, \ldots, g_{2k}. \tag{2.4}$$

Since the terminal g_{2k} cancels in gag, we have

$$g_1 = a^{-1}, \qquad g_2 = g_{2k}^{-1}, \ldots, g_{k+1} = g_{k+1}^{-1}, \ldots. \tag{2.5}$$

Then $g_{k+1}^2 = 1$ in A or C, which contradicts the hypothesis that A and C are 2-torsion free. Let $r = 2k+1$ for some positive integer k. Similarly to the case $n = 2k$, we have

$$g_1 = a^{-1}, \qquad g_2 = g_{2k+1}^{-1}, \ldots, g_{k+1} = g_{k+2}^{-1}, \ldots. \tag{2.6}$$

Thus g_{k+1} and g_{k+2} are in a same free factor. This is impossible because $g_1, g_2, \ldots, g_{2k+1}$ is a reduced sequence. Therefore, we can say that $g_n \in A$ and so $g_n a \in A$. Then, since

$$(g_1, g_2, \ldots, g_{n-1})(g_n a)(g_1, g_2, \ldots, g_{n-1}) = (gag) g_n^{-1} \tag{2.7}$$

is in A, by the induction hypothesis, $g_1, g_2, \dots, g_{n-1}$ is in A and hence $g = (g_1, g_2, \dots, g_{n-1})g_n$ is in A. $\square$

The following example illustrates the fact that the requirement that A and C are 2-torsion free is crucial in Lemma 2.6.

Example 2.7. Let $A = \langle a, b \mid aba^{-1}b \rangle$ and $C = \langle c \mid c^2 \rangle$. Then A is 2-torsion free and C is not. Let $g = a^{-1}bca$. Then, for $b \in A$,

$$gbg = (a^{-1}bca)b(a^{-1}bca) = a^{-1}bcca = a^{-1}ba = b^{-1} \tag{2.8}$$

is in A, but g is not in A.

Theorem 2.8. *Suppose that A and C are 2-torsion free groups. If $uv = v^{-1}u$ in $A * C$, then at least one of the following is true.*

(1) *u and v are in the same conjugate of a free factor.*

(2) $v = 1$.

Proof. We show that if $v \neq 1$, then u and v are in the same conjugate of a free factor. Since $v \neq 1$, then $u \neq 1$ because $uv = v^{-1}u$. If v is in the conjugate of some free factor, say, gAg^{-1}, then $g^{-1}vg \in A$ and $g^{-1}vg \neq 1$. Since $uvu^{-1} = v^{-1}$, we have

$$(g^{-1}ug)(g^{-1}vg)(g^{-1}ug)^{-1} = g^{-1}uvu^{-1}g = g^{-1}v^{-1}g = (g^{-1}vg)^{-1} \in A. \tag{2.9}$$

By Lemma 2.4, this implies that $g^{-1}ug$ is in A and hence u is in gAg^{-1}. Since A and C are 2-torsion free and $vuv = u$, using Lemma 2.6 instead of Lemma 2.4, we can similarly see that if u is in the conjugate of some free factor, then v is in the same conjugate of the free factor.

Now, suppose that neither u nor v is in the conjugate of a free factor. Since $uv = v^{-1}u$, this implies that

$$u^2v = uv^{-1}u = (v^{-1})^{-1}uu = vu^2 \tag{2.10}$$

and by Lemma 2.3, we have $uv = vu$. Thus $v = v^{-1}$ and hence $v = 1$, contrary to the assumption. Therefore, $v \neq 1$ implies that u and v are in the same conjugate of a free factor. $\square$

Let F be a finitely generated free group.

Corollary 2.9. *If $uv = v^{-1}u$ in F, then $v = 1$.*

Proof. Let $x_1, x_2, \dots, x_k$ be k-free generators of F. We use induction on k. Suppose $uv = v^{-1}u$ and $k = 1$. Then, since $F = \langle x_1 \rangle$ is abelian, we have $uv = vu$ and thus $v = 1$. Suppose that the corollary is true for $k = n - 1$ and that $k = n$. Since F is isomorphic to the free product of two free groups $A = \langle x_1, x_2, \dots, x_{n-1} \rangle$ and $C = \langle x_n \rangle$, by Theorem 2.8 there are only three cases possible.

(1) u and v are in the same conjugate of A. Let $u = waw^{-1}$ and $v = wbw^{-1}$ for some $w \in F$ and $a, b \in A$. Since $uv = v^{-1}u$, this implies that $ab = b^{-1}a$ in A. By the induction hypothesis, we have $b = 1$ and hence $v = ww^{-1} = 1$.

(2) u and v are in the same conjugate of C. Let $u = wx_n^p w^{-1}$ and $v = wx_n^q w^{-1}$ for some $w \in F$ and some integers p and q. Since $uv = v^{-1}u$, we have $wx_n^{p+q}w^{-1} = wx_n^{p-q}w^{-1}$. Therefore, $q = 0$ and hence $v = 1$.
(3) Neither of the above holds, but $v = 1$. $\square$

Let $G = A * C$, where $A = \langle a, b \mid aba^{-1}b \rangle$ and C is a 2-torsion free group and let $h : G \to G$ be an endomorphism. Since $h(a)h(b) = h(b)^{-1}h(a)$, we can classify h by applying Theorem 2.8.

Corollary 2.10. *At least, one of the following is true.*
(1) *$h(a)$ and $h(b)$ are in the same conjugate of a free factor.*
(2) *$h(b) = 1$.*

The following example illustrates the fact that if a group C has an order-2 element, then there is an endomorphism which does not satisfy both cases in Corollary 2.10.

Example 2.11. Let $C = \langle c \mid c^2 \rangle$ and $h : G \to G$ be a map defined by $h(a) = c$, $h(b) = a^{-1}cac$, and $h(c) = c$. Then

$$h(a)h(b) = c(a^{-1}cac) = (ca^{-1}ca)c = (a^{-1}cac)^{-1}c = h(b)^{-1}h(a). \tag{2.11}$$

Thus h is an endomorphism of G but $h(a)$ and $h(b)$ are not in the same conjugate of a free factor and clearly $h(b) \neq 1$.

3. Estimating Nielsen numbers on wedge product spaces

Let Y and Z be aspherical finite polyhedra and let $X = Y \vee Z$ be the wedge product of (Y, y_0) and (Z, z_0). We denote the intersection by x_0. Let $f : X \to X$ be a self-map such that $f(Y) \subseteq Y$. Using the homotopy extension property, we may assume that $f(x_0) = x_0$ and so f induces a homomorphism $f_\pi : \pi_1(X, x_0) \to \pi_1(X, x_0)$. Note that, up to homotopy, the condition $f(Y) \subseteq Y$ is equivalent to the condition $f_\pi(\pi_1(Y, x_0)) \subseteq \pi_1(Y, x_0)$. We will assume that Y and Z have those properties throughout this section unless stated otherwise.

Now consider a retraction $q : X \to Z$ sending Y to x_0, that is,

$$q = \mathrm{id}_Z \quad \text{on } Z, \qquad q = \overline{x}_0 \quad \text{on } Y, \tag{3.1}$$

where $\mathrm{id}_Z : Z \to Z$ is the identity map and $\overline{x}_0$ is the constant map at x_0. Let f_Z denote the restriction of f to Z.

Lemma 3.1. *The following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"] & X \arrow[d, "q"] \\ Z \arrow[r, "q \circ f_Z"] & Z \end{tikzcd} \tag{3.2}$$

that is, $q \circ f_Z \circ q = q \circ f : X \to Z$.

Proof. Suppose $x \in Y$. Since $f(Y) \subseteq Y$ and $q(Y) = x_0$, we have $q \circ f_Z \circ q(x) = q(f_Z(x_0)) = q(f(x_0)) = x_0$ and $q \circ f(x) = q(f(x)) = x_0$. On the other hand, if $x \in Z$, then $q(x) = x$ implies that $q \circ f_Z \circ q(x) = q(f_Z(x)) = q(f(x)) = q \circ f(x)$. □

In 1992, Woo and Kim [6] introduced the q-Nielsen number which is a lower bound for the Nielsen number. The map $q: X \to X_q$ in [6] is not our special map q defined above but an arbitrary map from X to a space X_q. Two fixed points x_0 and x_1 of f are in the same q-(fixed point) class if there exists a path c in X from x_0 to x_1 such that $q \circ c \simeq q \circ f \circ c$ in X_q. Using the ordinary fixed point index, they defined the q-Nielsen number $N_q(f)$. This is just the mod K-Nielsen number (see [2, Chapter III]) when $K = \ker q$ but it turns out to be more convenient when we consider the mod K-Nielsen number geometrically. Now we return to our map $q: X \to Z$.

By the definition of the map q, we have $\mathrm{Fix}(f) \cap Z = \mathrm{Fix}(q \circ f_Z)$.

Lemma 3.2. *Two fixed points z_1 and z_2 in $\mathrm{Fix}(f) \cap Z$ are in the same q-class of f if and only if they are in the same fixed point class of $(q \circ f_Z): Z \to Z$.*

Proof. Suppose that two fixed points z_1 and z_2 are in the same fixed point class of $q \circ f_Z$. Then there is a path γ in Z from z_1 to z_2 such that $\gamma \simeq q \circ f_Z \circ \gamma$ in Z. Since $q = \mathrm{id}_Z$ and $f_Z = f$ on Z, we may write $q \circ \gamma \simeq q \circ f \circ \gamma$ in Z. Therefore, two fixed points z_1 and z_2 are in the same q-class of f. Conversely, suppose that z_1 and z_2 are in the same q-class of f. Then there exists a path δ in X from z_1 to z_2 such that $q \circ \delta \simeq q \circ f \circ \delta$ in Z. Take $\delta_Z = q \circ \delta$. Then δ_Z is a path in Z from z_1 to z_2 and $\delta_Z \simeq q \circ f \circ \delta$ in Z. By Lemma 3.1, $q \circ f \circ \delta = q \circ f_Z \circ q \circ \delta = q \circ f_Z \circ \delta_Z$. This implies that $\delta_Z \simeq q \circ f_Z \circ \delta_Z$ in Z. □

For a fixed point $x \in \mathrm{Fix}(f)$, let $[x]$ (resp., $[x]_q$) denote the fixed point class (resp., q-class) of f containing x, and for $z \in \mathrm{Fix}(q \circ f_Z)$, let $[z]_Z$ denote the fixed point class of the map $q \circ f_Z$ containing z.

Lemma 3.3. *If $z \in \mathrm{Fix}(f) \cap Z$ and x_0 are not in the same fixed point class of $q \circ f_Z$, then the q-class of f containing z does not contain any fixed points of f in Y.*

Proof. The contrapositive statement of the lemma is as follows: if $[z]_q = [y]_q$ for some $y \in \mathrm{Fix}(f) \cap Y$, then $[z]_Z = [x_0]_Z$. Suppose $[z]_q = [y]_q$ for some $y \in \mathrm{Fix}(f) \cap Y$. Then there is a path γ in X from z to y such that $q \circ \gamma \simeq q \circ f \circ \gamma$ in Z. Let $\gamma_Z = q \circ \gamma$, then γ_Z is a path in Z from z to x_0 such that $\gamma_Z \simeq q \circ f \circ \gamma$ in Z. From Lemma 3.1, $q \circ f \circ \gamma = q \circ f_Z \circ q \circ \gamma = q \circ f_Z \circ \gamma_Z$ and hence

$$\gamma_Z \simeq q \circ f_Z \circ \gamma_Z \tag{3.3}$$

in Z. This means that $[z]_Z = [x_0]_Z$. □

Theorem 3.4. *The q-Nielsen number has a lower bound*

$$N_q(f) \geq N(q \circ f_Z) - 1. \tag{3.4}$$

Proof. Suppose $z \in \mathrm{Fix}(f) \cap Z = \mathrm{Fix}(q \circ f_Z)$. From Lemmas 3.2 and 3.3, we can say that if $[z]_q \neq [x_0]_q$, then $[z]_q = [z]_Z$ as a set. Furthermore since q is the identity map on Z, the corresponding indices of both classes $[z]_q$ and $[z]_Z$ are the same. Therefore $[z]_q \neq [x_0]_q$ is essential if and only if $[z]_Z$ is essential. □

Now, we consider another retraction $p : X \to Y$ sending Z to x_0 which is the same as the retraction $q : X \to Z$ with the roles of Y and Z exchanged. For a fixed point $x \in \mathrm{Fix}(f)$, let $[x]_p$ denote the p-class of f containing x, and for $y \in \mathrm{Fix}(p \circ f_Y)$, let $[y]_Y$ denote the fixed point class of the map $p \circ f_Y$ containing y. If $f(Y) \subseteq Y$ (resp., $f(Z) \subseteq Z$), then $p \circ f_Y = f_Y$ (resp., $q \circ f_Z = f_Z$).

In [7], Ferrario developed a formula for the Reidemeister trace of a pushout map. This is useful for union of spaces, quotient spaces, connected sums, and wedge product spaces. In particular, for a map $f : X \to X$ on a wedge product space $X = Y \vee Z$, he proved the following theorem. We obtain it again in a different way using our previous results.

Theorem 3.5. *If $f(Y) \subseteq Y$ and $f(Z) \subseteq Z$, then*

$$N(f_Y) + N(f_Z) - 2 \leq N(f) \leq N(f_Y) + N(f_Z) + 1. \tag{3.5}$$

Proof. In the proof of Theorem 3.4, we show that for $z \in \mathrm{Fix}(f) \cap Z$, if $[z]_q \neq [x_0]_q$, then $[z]_q = [z]_Z$ as a set. Since $f(Z) \subseteq Z$, we have the same result for $y \in \mathrm{Fix}(f) \cap Y$, that is, if $[y]_p \neq [x_0]_p$ then $[y]_p = [y]_Y$ as a set. Therefore $\mathrm{Fix}(f)$ has a decomposition

$$\mathrm{Fix}(f) = \Big(\bigcup [y]_Y\Big) \bigcup \Big(\bigcup [z]_Z\Big) \bigcup \Big([x_0]_p \cap [x_0]_q\Big). \tag{3.6}$$

Since $p \circ f_Y = f_Y$ and $q \circ f_Z = f_Z$ are restrictions of f, this implies that $[y]_Y \subseteq [y]$, $[z]_Z \subseteq [z]$, and $[x_0]_p \cap [x_0]_q \subseteq [x_0]$. Thus, we have $[y]_Y = [y]$, $[z]_Z = [z]$, and $[x_0]_p \cap [x_0]_q = [x_0]$. Furthermore, $[y]_Y$ (resp., $[z]_Z$) and $[y]$ (resp., $[z]$) have the same index. Consequently, we have

$$N(f) = N(f_Y) + N(f_Z) - \epsilon + \epsilon', \tag{3.7}$$

where

$$\begin{aligned} \epsilon &= \text{number of essential classes in } \left\{[x_0]_p, [x_0]_q\right\}, \\ \epsilon' &= \begin{cases} 1 & \text{if } [x_0] \text{ is essential,} \\ 0 & \text{if } [x_0] \text{ is inessential.} \end{cases} \end{aligned} \tag{3.8}$$

Therefore,

$$N(f_Y) + N(f_Z) - 2 \leq N(f) \leq N(f_Y) + N(f_Z) + 1. \tag{3.9}$$

□

Since the assumption of Theorem 3.5 is quite strong, it is necessary to generalize Theorem 3.5 in order to estimate the Nielsen number on wedge product spaces. First, we consider the condition $p_\pi \circ f_\pi(\pi_1(Z,x_0)) = 1$ instead of the condition $f(Z) \subseteq Z$ in the assumption of Theorem 3.5. Since the condition $f(Z) \subseteq Z$ implies that $p_\pi \circ f_\pi(\pi_1(Z,x_0)) = 1$ in $\pi_1(X,x_0)$ and the converse is not true, $p_\pi \circ f_\pi(\pi_1(Z,x_0)) = 1$ is a more generalized condition. But, unfortunately, under the assumptions $f(Y)\subseteq Y$ and $p_\pi\circ f_\pi(\pi_1(Z,x_0))=1$, the result of Theorem 3.5 in general form

$$N(f)\geq Nc(f_Y)+N(q\circ f_Z)-2 \tag{3.10}$$

is no longer true, as follows.

Example 3.6. Let $X=T\vee S^1$ be the wedge product of a torus and a circle at x_0. In this example, we have $Y=T$ and $Z=S^1$. Then $\pi_1(X,x_0)=\langle a,b,c\mid aba^{-1}b^{-1}\rangle$. Consider a map $f:X\to X$ with $f_\pi(a)=a^{-1}$, $f_\pi(b)=b^{-1}$, and $f_\pi(c)=a^{-1}cb^{-1}acb$. Then $p_\pi\circ f_\pi(c)=a^{-1}b^{-1}ab=1$ in $\pi_1(T,x_0)$. In order to compute the Nielsen number of f, we use the Fadell-Husseini formula for the Reidemeister trace in [8]. (See [8] or [9].) For the map f, the Reidemeister trace is

$$\mathrm{RT}(f,\widetilde{f})=[1]-\left(-[a^{-1}]-[b^{-1}]+[a^{-1}]+[a^{-1}cb^{-1}a]\right)+[a^{-1}b^{-1}], \tag{3.11}$$

where $[\cdot]$ denotes the Reidemeister class. Since $a^{-1}cb^{-1}a=f_\pi(c)b^{-1}c$, this implies that b^{-1} and $a^{-1}cb^{-1}a$ are Reidemeister equivalent and hence $[b^{-1}]=[a^{-1}cb^{-1}a]$. Consequently, we have $\mathrm{RT}(f,\widetilde{f})=[1]+[a^{-1}b^{-1}]$. Using the technique of abelianization, we know that these terms are distinct and thus that $N(f)=2$. But, we have $N(f_Y)=N(f_T)=4$ and $N(q\circ f_Z)=N(q\circ f_{S^1})=1$. Hence

$$N(f)=2\ngeq 4+1-2=N(f_Y)+N(q\circ f_Z)-2. \tag{3.12}$$

Lemma 3.7. *Suppose that $f(Y)\subseteq Y$, $p_\pi\circ f_\pi(\pi_1(Z,x_0))=1$, and two fixed points y_1 and y_2 are in $\mathrm{Fix}(f)\cap Y$. Then y_1 and y_2 are in the same p-class of f if and only if they are in the same fixed point class of f_Y.*

Proof. Suppose that y_1 and y_2 are in the same p-class of f, that is, there is a path $\gamma:I\to X$ from y_1 to y_2 such that $p\circ\gamma\simeq p\circ f\circ\gamma$ in Y. Let $\gamma_Y=p\circ\gamma$. Then γ_Y is a path in Y from y_1 to y_2 such that $\gamma_Y\simeq p\circ f\circ\gamma$ in Y. We show that $p\circ f\circ\gamma\simeq f_Y\circ\gamma_Y$ in Y and so y_1 and y_2 are in the same fixed point class of f_Y.

Modifying γ slightly as necessary, we can assume that γ is the product of finite numbers of the pathes $\gamma_1,\gamma_2,\ldots,\gamma_{2k+1}$ such that all γ_{2i+1} with $i=0,\ldots,k$ and γ_{2j} with $j=1,\ldots,k$ are pathes in Y and Z, respectively. Furthermore, all γ_{2j} are loops at x_0 in Z and so $\{\gamma_{2j}\}\in\pi_1(Z,x_0)$. Since $p_\pi\circ f_\pi(\pi_1(Z,x_0))=1$, we have $p_\pi\circ f_\pi(\{\gamma_{2j}\})=1$ for all j. Since Y is aspherical, it follows that $p\circ f\circ\gamma_{2j}\simeq\overline{x}_0$. Therefore,

$$\begin{aligned}
p\circ f\circ\gamma&=p\circ f\circ(\gamma_1\gamma_2\cdots\gamma_{2k+1})\\
&\simeq(p\circ f\circ\gamma_1)\overline{x}_0(p\circ f\circ\gamma_3)\overline{x}_0\cdots\overline{x}_0(p\circ f\circ\gamma_{2k+1})\\
&=p\circ f\circ p\circ\gamma=p\circ f_Y\circ p\circ\gamma=f_Y\circ\gamma_y.
\end{aligned} \tag{3.13}$$

Since $p=\mathrm{id}$ on Y, the converse is obvious. □

We now consider a map $f:X\to X$ with

$$f(Y)\subseteq Y,\qquad f_\pi(\pi_1(Z,x_0))\subseteq w\pi_1(Z,x_0)w^{-1}, \tag{3.14}$$

where $w = w_1, w_2, \ldots, w_k \in \pi_1(X, x_0)$ is cyclically reduced. We may assume that $w_1 \in \pi_1(Z, x_0)$ and $w_k \in \pi_1(Y, x_0)$ because if $w_1 \in \pi_1(Y, x_0)$, then since X is an aspherical space, we can homotope f to a map g such that $g_\pi(\cdot) = w_1^{-1} f_\pi(\cdot) w_1$ and if $w_k \in \pi_1(Z, x_0)$, then $w_k \pi_1(Z, x_0) w_k^{-1} \subseteq \pi_1(Z, x_0)$. For the map f, we also consider a map $\widehat{f} : X \to X$ which is homotopic to f such that

$$\widehat{f}_\pi(\cdot) = w^{-1} f_\pi(\cdot) w, \quad \text{so } \widehat{f}(Z) \subseteq Z. \tag{3.15}$$

The following theorem is a generalization of Theorem 3.5.

Theorem 3.8. *If a map f satisfies the conditions of (3.14), then*

$$N(f) \geq N(p \circ \widehat{f}_Y) + N(q \circ f_Z) - (k+2) \quad \text{where } k = \lambda(w). \tag{3.16}$$

Proof. For $I = [0,1]$, let $i_0 = 0$ and $i_k = 1$. Let $Y' = Y \vee I$ be the wedge product of Y and I at $i_0 \sim y_0$ and let $X' = Y' \vee Z$ be the wedge product of Y' and Z at $i_k \sim z_0$. We will construct a map $f' : X' \to X'$ which is the homotopy type of f. Since Y and Z are aspherical spaces, there are maps $f'_Y : (Y, i_0) \to (Y, i_0)$ and $f'_Z : (Z, i_k) \to (Z, i_k)$ such that

$$(f'_Y)_\pi = (f_Y)_\pi, \qquad (f'_Z)_\pi = (\widehat{f}_Z)_\pi. \tag{3.17}$$

To extend f' to all of X', we divide I into $2k+1$ equal closed intervals $I_0, J_1, I_1, \ldots, J_k, I_k$. Then for each integer r,

$$\begin{aligned} I_{2r} &\text{ is mapped homeomorphically onto } I \text{ from } i_0 \text{ to } i_k, \\ J_{2r+1} &\text{ is mapped onto a loop in } Z \text{ at } i_k \text{ representing } w_{2r+1}, \\ I_{2r+1} &\text{ is mapped homeomorphically onto } I \text{ from } i_k \text{ to } i_0, \\ J_{2r} &\text{ is mapped onto a loop in } Y \text{ at } i_0 \text{ representing } w_{2r}. \end{aligned} \tag{3.18}$$

By construction, there is exactly one fixed point in each interval I_i and no fixed point in any interval J_i. Therefore, f' has $k+1$ fixed points in I including i_0 and i_k.

We now show that f' is the same homotopy type as f. Choose a small neighborhood D of x_0 in X and let $\partial_Z D$ denote the intersection of the boundary of D with Z. Then there is a homotopy equivalence $\varphi : X \to X'$ such that $\varphi(x_0) = i_0$ and $\varphi(\partial_Z D) = i_k$. Then we see that

$$f'_\pi \circ \varphi_\pi = \varphi_\pi \circ f_\pi : \pi_1(X, x_0) \longrightarrow \pi_1(X', i_0), \tag{3.19}$$

because of the construction of f'. Thus f' is the homotopy type of f.

Let $Z' = I \vee Z \subseteq X'$ and let $p' : X' \to Y'$ and $q' : X' \to Z'$ retractions sending Z to i_k and Y to i_0, respectively. Let $\alpha = N(p' \circ f'_{Y'})$ and $\beta = N(q' \circ f'_{Z'})$. Since $f'(Y) \subseteq Y$, by Lemmas 3.2 and 3.3, there exist at least $\beta - 1$ essential q'-classes in Z' such that each of them is not $[i_0]_{q'}$.

On the other hand, since $f'(Z) \subseteq Z$, by Lemmas 3.2 and 3.3, there exist at least $\alpha - 1$ essential p'-classes such that each of them does not contain i_k and so, by Lemma 3.3, each of them does not contain any fixed points of f' in Z. Since $\mathrm{Fix}(f') \cap I$ has $k+1$ fixed points of f', we see that $|\mathrm{Fix}(f') \cap I \setminus \{i_k\}| = k$ and thus at least $(\alpha - 1) - k$ essential p'-classes do not intersect Z'.

Consequently, there exist at least $(\alpha - 1) + (\beta - 1) - k$ essential disjoint p' or q'-classes of f'. Therefore,

$$N(f') \geq N(p' \circ f'_{Y'}) + N(q' \circ f'_{Z'}) - (k+2). \tag{3.20}$$

Note that f and f' are of the same homotopy type and so are $p \circ \hat{f}_Y$ (resp., $q \circ f_Z$) and $p' \circ f'_{Y'}$ (resp., $q' \circ f'_{Z'}$). Since the Nielsen number is a homotopy type invariant (see [2]), we have

$$N(f) \geq N(p \circ \hat{f}_Y) + N(q \circ f_Z) - (k+2). \tag{3.21}$$

□

4. Applications

Let M be a surface with boundary, which is homotopy equivalent to a bouquet of k circles, and let $f : M \to M$ be a map. The fundamental group of M is the free group on k generators. In 1999, Wagner [10] provided a method for computing the Nielsen number $N(f)$ for a large class of maps of M by means of an algorithm that depends only on the induced endomorphism f_π of $\pi_1(M)$. In 2006, Kim [11] extended her results for another large class of maps of M. In the case of $k = 2$, Yi [12] extended Wagner's work using the idea of a mutant, which was introduced by Jiang in [13], and Kim [14] completed Yi's work so that the Nielsen number of all maps can be calculated.

The following spaces are examples of aspherical wedge product spaces for which we can classify the endomorphisms of the fundamental groups, and thus the self-maps. The general results in Section 3 along with existing technique allow us to estimate or calculate the Nielsen number on the following spaces:

(1) torus wedge surface with boundary;

(2) Klein bottle wedge surface with boundary;

(3) torus wedge torus;

except for some cases in (3) which satisfy the $L1$ condition in Theorem 4.5.

4.1. Torus wedge surface with boundary.

Let X be the wedge product of a torus T and a surface with boundary M at a point x_0. Let a and b be generators of the fundamental group of T and let $c_1, \ldots, c_k$ be generators of the fundamental group of M. Then the fundamental group of X at x_0 can be written as $G = A * C$, where $A = \langle a, b \mid ab = ba \rangle$ and $C = \langle c_1, \ldots, c_k \rangle$.

Theorem 4.1. *Every self-map f of X satisfies at least one of the following:*

(H1) *$f_\pi(a) = wa^{m_1}b^{n_1}w^{-1}$ and $f_\pi(b) = wa^{m_2}b^{n_2}w^{-1}$ for some $w \in G$;*

(H2) *$f_\pi(a) = g^s$ and $f_\pi(b) = g^t$ for some element $g \in G$ and integers s and t.*

Proof. From Theorem 2.5, it is enough to show that if $f_\pi(a)$ and $f_\pi(b)$ are in the same conjugate of C, then both are powers of the same element in G. Suppose $f_\pi(a) = wg_1w^{-1}$ and $f_\pi(b) = wg_2w^{-1}$ for some $w \in G$ and $g_1, g_2 \in C$. Since f_π is an endomorphism of G, then $f_\pi(a)f_\pi(b) = f_\pi(b)f_\pi(a)$ and therefore g_1 and g_2 are commuting elements of a free group, hence elements of a cyclic subgroup [5, page 42] (but a simpler proof is to note that the subgroup of free group C generated by g_1 and g_2 is abelian so, since it must be free, it is cyclic). □

4.1.1. (H1) condition. If a map f satisfies the condition (H1), then there is a map f' which is (freely) homotopic to f such that $f'_\pi(\cdot) = w^{-1} f_\pi(\cdot) w$, which satisfies $f'_\pi(A) \subseteq A$. Thus we can estimate the Nielsen number of f' using Theorems 3.4 and 3.5 if we can compute the Nielsen number $N(q \circ f'_M)$ on the surface with boundary M.

Example 4.2. Let $X = T \vee M$ be the wedge product of a torus and a surface with boundary, with

$$\pi_1(X, x_0) = \langle a, b, c_1, c_2 \mid ab = ba \rangle. \tag{4.1}$$

Let f be a map that induces the endomorphism described by the following four words:

$$\begin{aligned} f_\pi(a) &= c_1^{-1} a^2 b c_2 a^2 b^3 c_2^{-1} b^{-1} a^{-2} c_1, \\ f_\pi(b) &= c_1^{-1} a^2 b c_2 a^{-1} b c_2^{-1} b^{-1} a^{-2} c_1, \\ f_\pi(c_1) &= a c_1^7 a^{-1} c_1 b, \\ f_\pi(c_2) &= c_1 c_2^3. \end{aligned} \tag{4.2}$$

Then, the map f satisfies the (H1) condition with $w = c_1^{-1} a^2 b c_2$ and

$$\begin{aligned} q_\pi \circ f'_\pi(c_1) &= c_2^{-1} c_1^8 c_2, \\ q_\pi \circ f'_\pi(c_2) &= c_2^{-1} c_1^2 c_2^3 c_1^{-1} c_2. \end{aligned} \tag{4.3}$$

Using Wagner's algorithm [10] on $q \circ f'_M$, we have

$$N(q \circ f'_M) = 8. \tag{4.4}$$

Thus, by Theorem 3.4,

$$N(f) = N(f') \geq 8 - 1 = 7. \tag{4.5}$$

4.1.2. (H2) condition. If a map f satisfies the condition (H2), then

$$\begin{aligned} f_\pi(a) &= g^s, \\ f_\pi(b) &= g^t, \\ f_\pi(c_i) &= z_i, \quad 1 \leq i \leq k, \end{aligned} \tag{4.6}$$

where $g, z_i \in \pi_1(X, x_0)$.

Let S be the wedge of $(k+1)$ circles. Then the fundamental group $\pi_1(S)$ is the free group on $(k+1)$ generators $r_0, r_1, \dots, r_k$. Let φ be a map from X to S such that

$$\begin{aligned} \varphi_\pi(a) &= r_0^s, \\ \varphi_\pi(b) &= r_0^t, \\ \varphi_\pi(c_i) &= r_i, \quad 1 \le i \le k. \end{aligned} \tag{4.7}$$

Let ψ be a map from S to X such that

$$\begin{aligned} \psi_\pi(r_0) &= g, \\ \psi_\pi(r_i) &= z_i, \quad 1 \le i \le k. \end{aligned} \tag{4.8}$$

Then $f = \psi \circ \varphi$ and from the commutativity of the Nielsen number, we have

$$N(f) = N(\psi \circ \varphi) = N(\varphi \circ \psi), \tag{4.9}$$

where $\varphi \circ \psi$ is a map from S to itself. Thus calculating the Nielsen number $N(f)$ for any map which satisfies condition (H2) is now reduced to the calculation of the Nielsen number on surfaces with boundary. However, as we mentioned at the beginning of this section, the calculation of the Nielsen number on surfaces with boundary is still an open problem except in the case of figure-eight space.

Example 4.3. Let $X = T \vee M$ be the wedge product of a torus and a circle, with

$$\pi_1(X, x_0) = \langle a, b, c \mid ab = ba \rangle. \tag{4.10}$$

Let f be a map that induces the endomorphism described by the following three words:

$$\begin{aligned} f_\pi(a) &= bc^2a, \\ f_\pi(b) &= (bc^2a)^2, \\ f_\pi(c) &= a^2c^2. \end{aligned} \tag{4.11}$$

Then, the map f satisfies the (H2) condition and factors through figure-eight space S, that is, $f = \psi \circ \varphi$, where $\varphi : X \to S$ is

$$\begin{aligned} \varphi_\pi(a) &= r_0, \\ \varphi_\pi(b) &= r_0^2, \\ \varphi_\pi(c) &= r_1, \end{aligned} \tag{4.12}$$

and $\psi : S \to X$ is

$$\begin{aligned} \psi_\pi(r_0) &= bc^2a, \\ \psi_\pi(r_1) &= a^2c^2. \end{aligned} \tag{4.13}$$

Hence by the commutativity of the Nielsen number and the Wagner, Yi, and Kim algorithms for computing the Nielsen numbers on S (see [10, 12, 14]), we have

$$N(f) = N(\psi \circ \varphi) = N(\varphi \circ \psi) = 2. \tag{4.14}$$

4.2. Klein bottle wedge surface with boundary. Let X be the wedge product of the Klein bottle K and a surface with boundary M at a point x_0. Let a and b be generators of the fundamental group of K and let $c_1,\dots,c_k$ be generators of the fundamental group of M. Then the fundamental group of X at x_0 is $G = A*C$, where $A = \langle a,b \mid ab = b^{-1}a\rangle$ and $C = \langle c_1,\dots,c_k\rangle$.

Theorem 4.4. *Every self-map of X satisfies at least one of the following:*
(K1) $f_\pi(a) = wa_1w^{-1}$ *and* $f_\pi(b) = wa_2w^{-1}$ *for some* $a_1,a_2 \in A$ *and* $w \in G$;
(K2) $f_\pi(b) = 1$.

Proof. Since C is a free group, by Corollary 2.10, it is enough to show that if $f_\pi(a)$ and $f_\pi(b)$ are in a conjugate of C, then $f_\pi(b) = 1$. Suppose $f_\pi(a) = xux^{-1}$ and $f_\pi(b) = xvx^{-1}$ for some $x \in G$ and $u,v \in C$. Since $f_\pi(a)f_\pi(b) = f_\pi(b)^{-1}f_\pi(a)$, this implies that $uv = v^{-1}u$ in C. Since C is a finitely generated free group, by Corollary 2.9, we have $v = 1$ and therefore $f_\pi(b) = 1$. □

4.2.1. (K1) condition. We can use Theorems 3.4 and 3.5 to estimate the Nielsen number of the map with this condition. In fact, it is the same technique as that of the map with condition (H1) of the map on a wedge product of a torus and a surface with boundary.

4.2.2. (K2) condition. We can use the commutativity property of the Nielsen number to calculate the Nielsen number of a map satisfying this condition. If a map f satisfies condition (K2), then for $f_\pi(a) = g$, we have $f_\pi(b) = 1 = g^0$. Thus, it is the same technique as the one we used above for the map with condition (H2) on a wedge product of a torus and a surface with boundary.

4.3. Torus wedge torus. Let $X = T_1 \vee T_2$ be the wedge product of two tori T_1 and T_2 at a point x_0. Let a, b and c, d be generators of the fundamental group of T_1 and T_2, respectively. Then the fundamental group of X at x_0 is $G = A*C$, where $A = \langle a,b \mid ab = ba\rangle$ and $C = \langle c,d \mid cd = dc\rangle$. Then, by Theorem 2.5, we know the following.

Theorem 4.5. *Every self-map f of X satisfies at least one of the following:*
(L1) $f_\pi(a)$ *and* $f_\pi(b)$ *are in the same conjugate of a free factor and so are* $f_\pi(c)$ *and* $f_\pi(d)$;
(L2) $f_\pi(a)$ *and* $f_\pi(b)$ *are in the same conjugate of a free factor, and* $f_\pi(c)$ *and* $f_\pi(d)$ *are both powers of the same element in* G;
(L2′) $f_\pi(a)$ *and* $f_\pi(b)$ *are both powers of the same element in* G, *and* $f_\pi(c)$ *and* $f_\pi(d)$ *are in the same conjugate of a free factor;*
(L3) $f_\pi(a)$ *and* $f_\pi(b)$ *are both powers of the same element in* G *and so are* $f_\pi(c)$ *and* $f_\pi(d)$.

4.3.1. (L1) condition. If a map f satisfies the condition (L1), then

$$f_\pi(a)=\sigma g_1\sigma^{-1},\qquad f_\pi(b)=\sigma g_2\sigma^{-1},\qquad f_\pi(c)=\tau h_1\tau^{-1},\qquad f_\pi(d)=\tau h_2\tau^{-1}, \tag{4.15}$$

where σ and τ are in G, both g_1 and g_2 are either in A or C, and so are h_1 and h_2. We will consider only the case that $g_1,g_2\in A$ and $h_1,h_2\in C$. Then there is a map f' which is homotopic to f such that $f'_\pi(\cdot)=\sigma^{-1}f_\pi(\cdot)\sigma$, that is,

$$f'_\pi(a)=g_1,\qquad f'_\pi(b)=g_2,\qquad f'_\pi(c)=\lambda h_1\lambda^{-1},\qquad f'_\pi(d)=\lambda h_2\lambda^{-1}, \tag{4.16}$$

where $\lambda=\sigma^{-1}\tau$. Let $\lambda=\lambda_1,\lambda_2,\dots,\lambda_\ell$ denote the reduced form of λ in G. If $\lambda\neq 1$, then let $w=\mu^{-1}\lambda\nu^{-1}$, where

$$\mu=\begin{cases}\lambda_1 & \text{if } \lambda_1\in A,\\ 1 & \text{if } \lambda_1\in C,\end{cases}\qquad \nu=\begin{cases}1 & \text{if } \lambda_\ell\in A,\\ \lambda_\ell & \text{if } \lambda_\ell\in C,\end{cases} \tag{4.17}$$

and if $\lambda=1$, then let $w=1$. Then w is a cyclically reduced word in G with the first free factor in A and there is a map f'' which is homotopic to f such that

$$f''_\pi(a)=a_1,\qquad f''_\pi(b)=a_2,\qquad f'_\pi(c)=wc_1w^{-1},\qquad f'_\pi(d)=wc_2w^{-1}, \tag{4.18}$$

where $a_i=\mu g_i\mu^{-1}=g_i\in A$ and $c_i=\nu h_i\nu^{-1}=h_i\in C$ for $i=1,2$. Therefore, by Theorem 3.8,

$$N(f)=N(f'')\geq N\big(p\circ\widehat{f''_{T_1}}\big)+N\big(q\circ f''_{T_2}\big)-(k+2), \tag{4.19}$$

where $k=\lambda(w)$ is the syllable length of w in G. The following example illustrates the estimation for the Nielsen number in this case.

Example 4.6. Let $f:X\to X$ defined by

$$\begin{aligned}f_\pi(a)&=c^5d^{-3}a^{-2}bd^3c^{-5},\\ f_\pi(b)&=c^5d^{-3}b^{-3}d^3c^{-5},\\ f_\pi(c)&=c^7da^{-3}b^{10}c^2d^{-2}b^{-10}a^3d^{-1}c^{-7},\\ f_\pi(d)&=c^7da^{-3}b^{10}c^3d^4b^{-10}a^3d^{-1}c^{-7}.\end{aligned} \tag{4.20}$$

Then

$$\begin{aligned}f''_\pi(a)&=a^{-2}b,\\ f''_\pi(b)&=b^{-3},\\ f''_\pi(c)&=wc^2d^{-2}w^{-1},\\ f''_\pi(d)&=wc^3d^4w^{-1},\end{aligned} \tag{4.21}$$

where $w = c^2d^4a^{-3}b^{10}$. Thus $k = 2$,

$$N(p \circ \hat{f}''_{T_1}) = |\det(I - M_1)| = 12, \qquad N(q \circ f''_{T_2}) = |\det(I - M_2)| = 9, \tag{4.22}$$

where I is the identity matrix, $M_1 = \left[\begin{smallmatrix} -2 & 1 \\ 0 & -3 \end{smallmatrix}\right]$, and $M_2 = \left[\begin{smallmatrix} 2 & -2 \\ 3 & 4 \end{smallmatrix}\right]$; see [15]. Therefore,

$$N(f) \geq 12 + 9 - (2 + 2) = 17. \tag{4.23}$$

4.3.2. (L2) and (L2′) conditions. For a map f satisfying the (L2) or (L2′) conditions, using a similar technique as for the (H2) condition, we can make f factor through the torus wedge circle. By the commutativity of the Nielsen number, the problem is now reduced to computing the Nielsen number of the corresponding map of the torus wedge circle.

4.3.3. (L3) condition. The commutativity of the Nielsen number is also very useful in this case because every map satisfying the (L3) condition factors through figure eight. For the corresponding map of figure eight, we can always compute the Nielsen number using the Wagner, Yi, and Kim algorithms in [10, 12, 14].

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