Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2008, Article ID 418971, 4 pages
doi:10.1155/2008/418971

Research Article

T-Stability of Picard Iteration in Metric Spaces

Yuan Qing[1] and B. E. Rhoades[2]

[1] *Department of Mathematics, Beijing University of Aeronautics and Astronautics, Beijing 100083, China*
[2] *Department of Mathematics, Indiana University, Bloomington, IN 47405-7106, USA*

Correspondence should be addressed to Yuan Qing, yuanqingbuaa@hotmail.com

Received 10 July 2007; Accepted 11 January 2008

Recommended by Hélène Frankowska

We establish a general result for the stability of Picard's iteration. Several theorems in the literature are obtained as special cases.

Copyright © 2008 Y. Qing and B. E. Rhoades. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let (X,d) be a complete metric space and T a self-map of X. Let $x_{n+1}=f(T,x_n)$ be some iteration procedure. Suppose that $F(T)$, the fixed point set of T, is nonempty and that x_n converges to a point $q\in F(T)$. Let $\{y_n\}\subset X$ and define $\epsilon_n=d(y_{n+1},f(T,y_n))$. If $\lim\epsilon_n=0$ implies that $\lim y_n=q$, then the iteration procedure $x_{n+1}=f(T,x_n)$ is said to be T-stable. Without loss of generality, we may assume that $\{y_n\}$ is bounded, for if $\{y_n\}$ is not bounded, then it cannot possibly converge. If these conditions hold for $x_{n+1}=Tx_n$, that is, Picard's iteration, then we will say that Picard's iteration is T-stable.

We will obtain sufficient conditions that Picard's iteration is T-stable for an arbitrary self-map, and then demonstrate that a number of contractive conditions are Picard T-stable.

We will need the following lemma from [1].

Lemma 1. *Let $\{x_n\}$, $\{\epsilon_n\}$ be nonnegative sequences satisfying $x_{n+1}\le hx_n+\epsilon_n$, for all $n\in\mathbb{N}$, $0\le h<1$, $\lim\epsilon_n=0$. Then, $\lim x_n=0$.*

Theorem 1. *Let (X,d) be a nonempty complete metric space and T a self-map of X with $F(T)\neq\varnothing$. If there exist numbers $L\ge 0$, $0\le h<1$, such that*

$$d(Tx,q)\le Ld(x,Tx)+hd(x,q) \tag{1}$$

for each $x \in X$, $q \in F(T)$, and, in addition,

$$\lim d(y_n, Ty_n) = 0, \tag{2}$$

then Picard's iteration is T-stable.

Proof. First, we show that the fixed point q of T is unique. Suppose p is another fixed point of T, then

$$d(p,q) = d(Tp,q) \leq Ld(p,Tp) + hd(p,q) = hd(p,q). \tag{3}$$

Since $0 \leq h < 1$, so $d(p,q) = 0$, that is, $p = q$.

Let $\{y_n\} \subset X$, $\epsilon_n = d(y_{n+1}, Ty_n)$, and $\lim \epsilon_n = 0$. We need to show that $\lim y_n = q$.

Using (1), (2), and Lemma 1,

$$d(y_{n+1}, q) \leq d(y_{n+1}, Ty_n) + d(Ty_n, q) \leq \epsilon_n + Ld(y_n, Ty_n) + hd(y_n, q), \tag{4}$$

and $\lim y_n = q$. □

Corollary 1. *Let (X,d) be a nonempty complete metric space and T a self-map of X satisfying the following: there exists $0 \leq h < 1$, such that, for each $x, y \in X$,*

$$d(Tx,Ty) \leq h \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \tag{5}$$

Then, Picard's iteration is T-stable.

Proof. From [2, Theorem 11], T has a unique fixed point q. Also, T satisfies (1). It remains to show that (2) is satisfied.

Define p_n to be the diameter of the orbit of y_n; that is, $p_n = \delta(O(y_n, Ty_n, \ldots))$. First, we show that p_n is bounded:

$$\begin{aligned} d(Ty_n, q) &\leq h \max\{d(y_n,q), d(y_n,Ty_n), d(y_n,Tq), d(q,Ty_n), d(q,Tq)\} \\ &\leq h \max\{d(y_n,q), d(y_n,Ty_n), d(y_n,q), d(q,Ty_n), 0\} \\ &= h \max\{d(y_n,q), d(y_n,Ty_n), d(y_n,q), d(q,Ty_n)\}. \end{aligned} \tag{6}$$

Hence, $d(Ty_n, q) \leq hd(y_n, q)$ or $d(Ty_n, q) \leq hd(y_n, Ty_n)$ or $d(Ty_n, q) \leq hd(q, Ty_n)$.

If $d(Ty_n, q) \leq hd(y_n, q)$, it is clear that

$$d(Ty_n, q) \leq hd(y_n, q) \leq \frac{h}{1-h} d(y_n, q). \tag{7}$$

If $d(Ty_n, q) \leq hd(q, Ty_n)$, then

$$d(Ty_n, q) = 0 \leq \frac{h}{1-h} d(y_n, q). \tag{8}$$

If $d(Ty_n, q) \leq hd(y_n, Ty_n)$, then

$$d(y_n, Ty_n) \leq d(Ty_n, q) + d(y_n, q) \leq hd(y_n, Ty_n) + d(y_n, q). \tag{9}$$

Hence, $d(Ty_n, q) \leq (h/(1-h))d(y_n, q)$. Now it is easy to see that $\{Ty_n\}$ is bounded and so is $\{p_n\}$, since $\{y_n\}$ is bounded.

For any $i, j \geq n$, using (5),

$$d(Ty_i, Ty_j) \leq h \max\{d(y_i, y_j), d(y_i, Ty_i), d(y_j, Ty_j), d(y_i, Ty_j), d(y_j, Ty_i)\} \leq hp_n. \tag{10}$$

Thus,

$$d(y_i, Ty_j) \leq d(y_i, Ty_{i-1}) + d(Ty_{i-1}, Ty_j) \leq \epsilon_{i-1} + hp_{n-1}. \tag{11}$$

But

$$d(y_i, y_j) \leq d(y_i, Ty_{i-1}) + d(Ty_{i-1}, Ty_{j-1}) + d(Ty_{j-1}, y_j) \leq \epsilon_{i-1} + hp_{n-1} + \epsilon_{i-1}, \tag{12}$$

which implies that

$$p_n \leq 2\epsilon_{i-1} + hp_{n-1}, \tag{13}$$

and $\lim p_n = 0$ by Lemma 1. Since $d(y_n, Ty_n) \leq p_n$, $\lim d(y_n, Ty_n) = 0$.

The conclusion now follows from Theorem 1. □

Corollary 2 (see [3, Theorem 1]). *Let (X, d) be a nonempty complete metric space and T a self-map of X satisfying*

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y) \tag{14}$$

for all $x, y \in X$, where $L \geq 0$, $0 \leq a < 1$. Suppose that T has a fixed point p. Then, T is Picard T-stable.

Proof. Since T satisfies (14) for all $x, y \in X$, then T satisfies inequality (1) of our paper. Let $\{y_n\} \subset X$ and define $\epsilon_n = d(y_{n+1}, y_n)$. From the proof of Theorem 1 of [3], $\lim d(y_n, Ty_n) = 0$. Therefore, by our theorem (Theorem 1), T is Picard T-stable. □

Definition (5) of this paper is actually Definition (24) of [2]. Therefore, many contractive conditions are special cases of (5), and, for each of these, Picard's iteration is T-stable. For example, Theorems 1 and 2 of [4] and Theorem 1 of [5] are special cases of Corollary 1.

We will not examine the analogues of Theorem 1 for Mann, Ishikawa, Kirk, or any other iteration scheme since, if one obtains convergence to a fixed point for a map using Picard's iteration, there is no point in considering any other more complicated iteration procedure.

Acknowledgment

This article is partly supported by the National Natural Science Foundation of China (no. 10271012).

References

[1] Q. Liu, "A convergence theorem of the sequence of Ishikawa iterates for quasi-contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 146, no. 2, pp. 301–305, 1990.

[2] B. E. Rhoades, "A comparison of various definitions of contractive mappings," *Transactions of the American Mathematical Society*, vol. 226, pp. 257–290, 1977.

[3] M. O. Osilike, "Stability results for fixed point iteration procedures," *Journal of the Nigerian Mathematical Society*, vol. 14-15, pp. 17–29, 1995.

[4] A. M. Harder and T. L. Hicks, "Stability results for fixed point iteration procedures," *Mathematica Japonica*, vol. 33, no. 5, pp. 693–706, 1988.

[5] B. E. Rhoades, "Fixed point theorems and stability results for fixed point iteration procedures," *Indian Journal of Pure and Applied Mathematics*, vol. 21, no. 1, pp. 1–9, 1990.