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Research Article

A Generalization of Kannan's Fixed Point Theorem

Yusuke Enjouji, Masato Nakanishi, and Tomonari Suzuki

Department of Mathematics, Kyushu Institute of Technology, Tobata, Kitakyushu 804-8550, Japan

Correspondence should be addressed to Tomonari Suzuki, suzuki-t@mns.kyutech.ac.jp

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In order to observe the condition of Kannan mappings, we prove a generalization of Kannan's fixed point theorem. Our theorem involves constants and we obtain the best constants to ensure a fixed point.

1. Introduction

A mapping T on a metric space (X, d) is called *Kannan* if there exists $\alpha \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty) \tag{1.1}$$

for all $x, y \in X$. Kannan [1] proved that if X is complete, then every Kannan mapping has a fixed point. It is interesting that Kannan's theorem is independent of the Banach contraction principle [2]. Also, Kannan's fixed point theorem is very important because Subrahmanyam [3] proved that Kannan's theorem characterizes the metric completeness. That is, a metric space X is complete if and only if every Kannan mapping on X has a fixed point. Recently, Kikkawa and Suzuki proved a generalization of Kannan's fixed point theorem. See also [4–8].

Theorem 1.1 (see [9]). *Define a nonincreasing function φ from $[0, 1/2)$ into $(1/2, 1]$ by*

$$\varphi(\alpha) = \begin{cases} 1 & \text{if } 0 \leq \alpha < \sqrt{2} - 1, \\ 1 - \alpha & \text{if } \sqrt{2} - 1 \leq \alpha < \dfrac{1}{2}. \end{cases} \tag{1.2}$$

Let T be a mapping on a complete metric space (X, d). Assume that there exists $\alpha \in [0, 1/2)$ such that

$$\varphi(\alpha)d(x,Tx) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq \alpha d(x,Tx) + \alpha d(y,Ty) \tag{1.3}$$

for all $x, y \in X$. Then T has a unique fixed point z. Moreover $\lim_n T^n x = z$ holds for every $x \in X$.
Remark 1.2. $\varphi(\alpha)$ is the best constant for every $\alpha \in [0, 1/2)$.

From this theorem, we can tell that a Kannan mapping with $\alpha < \sqrt{2}-1$ is much stronger than a Kannan mapping with $\alpha \geq \sqrt{2}-1$.

While x and y play the same role in (1.1), x and y do not play the same role in (1.3). So we can consider "$\alpha d(x,Tx) + \beta d(y,Ty)$" instead of "$\alpha d(x,Tx) + \alpha d(y,Ty)$." And it is a quite natural question of what is the best constant for each pair (α, β). In this paper, we give the complete answer to this question.

2. Preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

We use two lemmas. The first lemma is essentially proved in [5].

Lemma 2.1 (see [5, 9]). *Let (X, d) be a metric space and let T be a mapping on X. Let $x \in X$ satisfy $d(Tx, T^2x) \leq r d(x, Tx)$ for some $r \in [0, 1)$. Then for $y \in X$, either*

$$(1+r)^{-1}d(x,Tx) \leq d(x,y) \quad \text{or} \quad (1+r)^{-1}d\left(Tx,T^2x\right) \leq d(Tx,y) \tag{2.1}$$

holds.

The second lemma is obvious. We use this lemma several times in the proof of Theorem 4.1.

Lemma 2.2. *Let a, A, b, and B be four real numbers such that $a \leq A$ and $b \leq B$. Then $aB + Ab \leq ab + AB$ holds.*

3. Fixed Point Theorem

In this section, we prove a fixed point theorem.

We first put Δ and Δ_j $(j = 1, \ldots, 4)$ by

$$\Delta = \{(\alpha,\beta) : \alpha \geq 0,\ \beta \geq 0,\ \alpha+\beta < 1\},$$
$$\Delta_1 = \left\{(\alpha,\beta) \in \Delta : \alpha \leq \beta,\ \alpha+\beta+\alpha^2 < 1\right\},$$

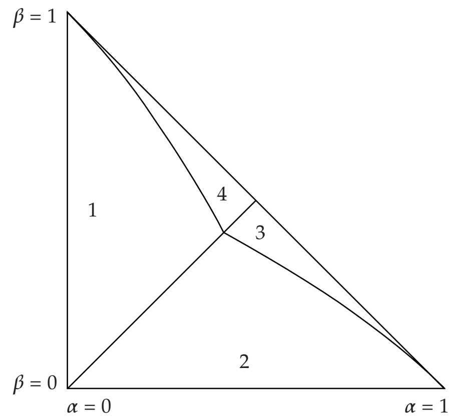

Figure 1: Δ_j $(j = 1, \ldots, 4)$

$$\Delta_2 = \left\{ (\alpha, \beta) \in \Delta \colon \alpha \geq \beta,\ \alpha + \beta + \beta^2 < 1 \right\},$$
$$\Delta_3 = \left\{ (\alpha, \beta) \in \Delta \colon \alpha \geq \beta,\ \alpha + \beta + \beta^2 \geq 1 \right\},$$
$$\Delta_4 = \left\{ (\alpha, \beta) \in \Delta \colon \alpha \leq \beta,\ \alpha + \beta + \alpha^2 \geq 1 \right\}. \tag{3.1}$$

See Figure 1.

Theorem 3.1. *Define a nonincreasing function φ from Δ into $(1/2, 1]$ by*

$$\varphi(\alpha, \beta) = \begin{cases} 1 & \text{if } (\alpha, \beta) \in \Delta_1, \\ 1 & \text{if } (\alpha, \beta) \in \Delta_2, \\ 1 - \beta & \text{if } (\alpha, \beta) \in \Delta_3, \\ \dfrac{1 - \beta}{1 - \beta + \alpha} & \text{if } (\alpha, \beta) \in \Delta_4. \end{cases} \tag{3.2}$$

Let T be a mapping on a complete metric space (X, d). Assume that there exists $(\alpha, \beta) \in \Delta$ such that

$$\varphi(\alpha, \beta) d(x, Tx) \leq d(x, y) \text{implies } d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty) \tag{3.3}$$

for all $x, y \in X$. Then T has a unique fixed point z. Moreover $\lim_n T^n x = z$ holds for every $x \in X$.

Proof. We put

$$q := \frac{\beta}{1 - \alpha} \in [0, 1), \qquad r := \frac{\alpha}{1 - \beta} \in [0, 1). \tag{3.4}$$

Since $\psi(\alpha,\beta) \le 1$, $\psi(\alpha,\beta)d(x,Tx) \le d(x,Tx)$ holds. From the assumption, we have

$$d\left(Tx,T^2x\right) \le \alpha d(x,Tx) + \beta d\left(Tx,T^2x\right) \tag{3.5}$$

and hence

$$d\left(Tx,T^2x\right) \le rd(x,Tx) \tag{3.6}$$

for all $x \in X$. Since

$$\psi(\alpha,\beta)d\left(Tx,T^2x\right) \le d\left(Tx,T^2x\right) \le rd(x,Tx) \le d(Tx,x), \tag{3.7}$$

we have

$$d\left(T^2x,Tx\right) \le \alpha d\left(Tx,T^2x\right) + \beta d(x,Tx) \tag{3.8}$$

and hence

$$d\left(Tx,T^2x\right) \le qd(x,Tx) \tag{3.9}$$

for all $x \in X$.

Fix $u \in X$ and put $u_n = T^n u$ for $n \in \mathbb{N}$. From (3.6), we have

$$\sum_{n=1}^{\infty} d(u_n,u_{n+1}) \le \sum_{n=1}^{\infty} r^n d(u,Tu) < \infty. \tag{3.10}$$

So $\{u_n\}$ is a Cauchy sequence in X. Since X is complete, $\{u_n\}$ converges to some point $z \in X$.

We next show

$$d(z,Tx) \le \beta d(x,Tx) \quad \forall x \in X \setminus \{z\}. \tag{3.11}$$

Since $\{u_n\}$ converges, for sufficiently large $n \in \mathbb{N}$, we have

$$\psi(\alpha,\beta)d(u_n,Tu_n) \le d(u_n,u_{n+1}) \le d(u_n,x) \tag{3.12}$$

and hence

$$d(Tu_n,Tx) \le \alpha d(u_n,Tu_n) + \beta d(x,Tx). \tag{3.13}$$

Therefore we obtain

$$\begin{aligned} d(z,Tx) &= \lim_{n\to\infty} d(u_{n+1},Tx) = \lim_{n\to\infty} d(Tu_n,Tx) \\ &\le \lim_{n\to\infty} \left(\alpha d(u_n,Tu_n) + \beta d(x,Tx)\right) = \beta d(x,Tx) \end{aligned} \tag{3.14}$$

for all $x \in X \setminus \{z\}$. By (3.11), we have

$$d(x,Tx) \le d(x,z) + d(z,Tx) \le d(x,z) + \beta d(x,Tx) \tag{3.15}$$

and hence

$$(1-\beta)d(x,Tx) \le d(x,z) \quad \forall x \in X \setminus \{z\}. \tag{3.16}$$

Let us prove that z is a fixed point of T. In the case where $(\alpha,\beta) \in \Delta_1$, arguing by contradiction, we assume $Tz \neq z$. Then we have

$$d\left(Tz,T^2z\right) \le rd(z,Tz) < d(z,Tz) = \lim_{n\to\infty} d(Tz,u_n). \tag{3.17}$$

So for sufficiently large $n \in \mathbb{N}$,

$$\varphi(\alpha,\beta)d\left(Tz,T^2z\right) = d\left(Tz,T^2z\right) \le d(Tz,u_n) \tag{3.18}$$

holds and hence

$$\begin{aligned} d\left(T^2z,z\right) &= \lim_{n\to\infty} d\left(T^2z,Tu_n\right) \\ &\le \lim_{n\to\infty} \left(\alpha d\left(Tz,T^2z\right) + \beta d(u_n,Tu_n)\right) = \alpha d\left(Tz,T^2z\right). \end{aligned} \tag{3.19}$$

Thus we obtain

$$\begin{aligned} d(z,Tz) &\le d\left(z,T^2z\right) + d\left(Tz,T^2z\right) \le (1+\alpha)d\left(Tz,T^2z\right) \\ &\le (1+\alpha)rd(z,Tz) = \frac{\alpha+\alpha^2}{1-\beta}d(z,Tz) \\ &< d(z,Tz), \end{aligned} \tag{3.20}$$

which is a contradiction. Therefore we obtain $Tz = z$.

In the case where $(\alpha, \beta) \in \Delta_2$, if we assume $Tz \neq z$, then we have

$$
\begin{aligned}
d(z, Tz) &\le d\left(z, T^2 z\right) + d\left(Tz, T^2 z\right) \le (1+\beta) d\left(Tz, T^2 z\right) \\
&\le (1+\beta) q d(z, Tz) = \frac{\beta + \beta^2}{1-\alpha} d(z, Tz) \\
&< d(z, Tz),
\end{aligned}
\tag{3.21}
$$

which is a contradiction. Therefore $Tz = z$ holds.

In the case where $(\alpha, \beta) \in \Delta_3$, we consider the following two cases.

(i) There exist at least two natural numbers n satisfying $u_n = z$.

(ii) $u_n \neq z$ for sufficiently large $n \in \mathbb{N}$.

In the first case, if we assume $Tz \neq z$, then $\{u_n\}$ cannot be Cauchy. Therefore $Tz = z$. In the second case, we have by (3.16), $\psi(\alpha, \beta) d(u_n, Tu_n) \le d(u_n, z)$ for sufficiently large $n \in \mathbb{N}$. From the assumption,

$$
d(z, Tz) = \lim_{n \to \infty} d(Tu_n, Tz) \le \lim_{n \to \infty} \left(\alpha d(u_n, Tu_n) + \beta d(z, Tz)\right) = \beta d(z, Tz). \tag{3.22}
$$

Since $\beta < 1$, we obtain $Tz = z$.

In the case where $(\alpha, \beta) \in \Delta_4$, we note that $\psi(\alpha, \beta) = (1+r)^{-1}$. By Lemma 2.1, either

$$
\psi(\alpha, \beta) d(u_n, Tu_n) \le d(u_n, z) \quad \text{or} \quad \psi(\alpha, \beta) d\left(Tu_n, T^2 u_n\right) \le d(Tu_n, z) \tag{3.23}
$$

holds for every $n \in \mathbb{N}$. Thus there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$
\psi(\alpha, \beta) d\left(u_{n_j}, Tu_{n_j}\right) \le d\left(u_{n_j}, z\right) \tag{3.24}
$$

for $j \in \mathbb{N}$. From the assumption, we have

$$
d(z, Tz) = \lim_{j \to \infty} d\left(Tu_{n_j}, Tz\right) \le \lim_{j \to \infty} \left(\alpha d\left(u_{n_j}, Tu_{n_j}\right) + \beta d(z, Tz)\right) = \beta d(z, Tz). \tag{3.25}
$$

Since $\beta < 1$, we obtain $Tz = z$. Therefore we have shown $Tz = z$ in all cases.

From (3.11), the fixed point z is unique. □

Remark 3.2. We have shown $Tz = z$, dividing four cases. It is interesting that the four methods are all different. We can rewrite ψ by

$$
\psi(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha + \beta + \min\{\alpha, \beta\}^2 < 1, \\ \dfrac{1-\beta}{1-\beta+\min\{\alpha, \beta\}} & \text{if } \alpha + \beta + \min\{\alpha, \beta\}^2 \ge 1. \end{cases} \tag{3.26}
$$

4. The Best Constants

In this section, we prove the following theorem, which informs that $\psi(\alpha,\beta)$ is the best constant for every $(\alpha,\beta) \in \Delta$.

Theorem 4.1. *Define a function ψ as in Theorem 3.1. For every $(\alpha,\beta) \in \Delta$, there exist a complete metric space (X,d) and a mapping T on X such that T has no fixed points and*

$$\psi(\alpha,\beta)d(x,Tx) < d(x,y) \text{implies } d(Tx,Ty) \leq \alpha d(x,Tx) + \beta d(y,Ty) \tag{4.1}$$

for all $x,y \in X$.

Proof. We put q and r by (3.4).

In the case where $(\alpha,\beta) \in \Delta_1 \cup \Delta_2$, define a complete subset X of the Euclidean space $\mathbb{R}$ by $X = \{-1,1\}$. We also define a mapping T on X by $Tx = -x$ for $x \in X$. Then T does not have any fixed points and

$$\psi(\alpha,\beta)d(x,Tx) = 2 \geq d(x,y) \tag{4.2}$$

for all $x,y \in X$.

In the case where $(\alpha,\beta) \in \Delta_3$, we put

$$p := \frac{\beta}{1-\beta} \in (0,1). \tag{4.3}$$

We note that $\psi(\alpha,\beta)(1+p) = 1$. Define a complete subset X of the Euclidean space $\mathbb{R}$ by

$$X = \{0,1\} \cup \{x_n : n \in \mathbb{N} \cup \{0\}\}, \tag{4.4}$$

where $x_n = (1-q)(-p)^n$ for $n \in \mathbb{N} \cup \{0\}$. Define a mapping T on X by $T0 = 1$, $T1 = x_0$, and $Tx_n = x_{n+1}$ for $n \in \mathbb{N} \cup \{0\}$. Then we have

$$\begin{gathered} d(T1,T0) = q = \alpha d(1,T1) + \beta d(0,T0) \leq \alpha d(0,T0) + \beta d(1,T1), \\ \psi(\alpha,\beta)d(0,T0) > \psi(\alpha,\beta)d(x_n,Tx_n) = (1-q)p^n = d(0,x_n) \end{gathered} \tag{4.5}$$

for $n \in \mathbb{N} \cup \{0\}$. Since

$$\begin{aligned} &d(Tx_n,T1) - (\alpha d(x_n,Tx_n) + \beta d(1,T1)) \\ &\quad = (1-q)\left(1-(-p)^{n+1} - \frac{\alpha}{\beta}p^{n+1} - \frac{\beta^2}{1-\alpha-\beta}\right) \\ &\quad \leq (1-q)\left(1-\frac{\beta^2}{1-\alpha-\beta}\right) + (1-q)p^{n+1}\left(1-\frac{\alpha}{\beta}\right) \leq 0, \end{aligned} \tag{4.6}$$

we have

$$d(Tx_n, T1) \le \alpha d(x_n, Tx_n) + \beta d(1, T1) \le \alpha d(1, T1) + \beta d(x_n, Tx_n) \tag{4.7}$$

for $n \in \mathbb{N} \cup \{0\}$. For $m, n \in \mathbb{N} \cup \{0\}$ with $m < n$, since

$$\begin{aligned} &d(Tx_n, Tx_m) - (\alpha d(x_n, Tx_n) + \beta d(x_m, Tx_m)) \\ &\quad = (1-q)\left(\left|(-p)^{n+1} - (-p)^{m+1}\right| - \frac{\alpha}{\beta}p^{n+1} - p^{m+1}\right) \\ &\quad \le (1-q)\left(p^{n+1} + p^{m+1} - \frac{\alpha}{\beta}p^{n+1} - p^{m+1}\right) \le 0, \end{aligned} \tag{4.8}$$

we have

$$d(Tx_n, Tx_m) \le \alpha d(x_n, Tx_n) + \beta d(x_m, Tx_m) \le \alpha d(x_m, Tx_m) + \beta d(x_n, Tx_n). \tag{4.9}$$

In the case where $(\alpha, \beta) \in \Delta_4$, we note that $\psi(\alpha, \beta)(1 + r) = 1$. We also note that $r \ge 2^{-1/2} > 1/2$. Let ℓ_∞ be the Banach space consisting of all functions f from $\mathbb{N}$ into $\mathbb{R}$ (i.e., f is a real sequence) such that $\|f\| := \sup_n |f(n)| < \infty$. Let $\{e_n\}$ be the canonical basis of ℓ_∞. Define a complete subset X of ℓ_∞ by

$$X = \{0, e_1\} \cup \{x_n : n \in \mathbb{N} \cup \{0\}\}, \tag{4.10}$$

where

$$x_n = (1-r)r^n e_{n+1} - (1-r)r^n e_{n+2} \tag{4.11}$$

for $n \in \mathbb{N} \cup \{0\}$. We note that

$$d(x_m, x_n) = \begin{cases} (1-r^2)r^m & \text{if } m+1 = n, \\ (1-r)r^m & \text{if } m+1 < n, \end{cases} \tag{4.12}$$

for $m, n \in \mathbb{N}$ with $m < n$. Define a mapping T on X by $T0 = e_1$, $Te_1 = x_0$, and $Tx_n = x_{n+1}$ for $n \in \mathbb{N} \cup \{0\}$. Then we have

$$\begin{gathered} d(T0, Te_1) = r = \alpha d(0, T0) + \beta d(e_1, Te_1) \le \alpha d(e_1, Te_1) + \beta d(0, T0), \\ \psi(\alpha, \beta) d(0, T0) > \psi(\alpha, \beta) d(x_n, Tx_n) = (1-r)r^n = d(0, x_n) \end{gathered} \tag{4.13}$$

for $n \in \mathbb{N} \cup \{0\}$. Since

$$d(Te_1, Tx_0) - (\alpha d(e_1, Te_1) + \beta d(x_0, Tx_0)) = (1-\beta)\left(1 - 2r^2\right) \le 0, \tag{4.14}$$

we have

$$d(Te_1, Tx_0) \leq \alpha d(e_1, Te_1) + \beta d(x_0, Tx_0) \leq \alpha d(x_0, Tx_0) + \beta d(e_1, Te_1). \tag{4.15}$$

Since $\alpha + \beta + \alpha^2 \geq 1$, we have

$$\begin{aligned} d(Te_1, Tx_n) &= 1 - r \leq \alpha r = \alpha d(e_1, Te_1) \\ &< \alpha d(e_1, Te_1) + \beta d(x_n, Tx_n) \leq \alpha d(x_n, Tx_n) + \beta d(e_1, Te_1) \end{aligned} \tag{4.16}$$

for $n \in \mathbb{N}$. We have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= \left(1 - r^2\right) r^{n+1} = \alpha d(x_n, Tx_n) + \beta d(x_{n+1}, Tx_{n+1}) \\ &\leq \alpha d(x_{n+1}, Tx_{n+1}) + \beta d(x_n, Tx_n) \end{aligned} \tag{4.17}$$

for $n \in \mathbb{N} \cup \{0\}$. For $m, n \in \mathbb{N} \cup \{0\}$ with $m + 1 < n$, we have

$$\begin{aligned} \psi(\alpha, \beta) d(x_m, Tx_m) &= (1 - r) r^m = d(x_m, x_n), \\ d(Tx_n, Tx_m) - \left(\alpha d(x_n, Tx_n) + \beta d(x_m, Tx_m)\right) &< d(Tx_n, Tx_m) - \beta d(x_m, Tx_m) \\ &= r^{m+1}(1 - r) - \beta r^m \left(1 - r^2\right) \\ &= r^m (1 - r)(\alpha - \beta) \leq 0. \end{aligned} \tag{4.18}$$

This completes the proof. □

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References

[1] R. Kannan, "Some results on fixed points. II," *American Mathematical Monthly*, vol. 76, pp. 405–408, 1969.

[2] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[3] P. V. Subrahmanyam, "Completeness and fixed-points," *Monatshefte für Mathematik*, vol. 80, no. 4, pp. 325–330, 1975.

[4] M. Kikkawa and T. Suzuki, "Three fixed point theorems for generalized contractions with constants in complete metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 9, pp. 2942–2949, 2008.

[5] T. Suzuki, "A generalized Banach contraction principle that characterizes metric completeness," *Proceedings of the American Mathematical Society*, vol. 136, no. 5, pp. 1861–1869, 2008.

[6] T. Suzuki, "Fixed point theorems and convergence theorems for some generalized nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 2, pp. 1088–1095, 2008.

[7] T. Suzuki and M. Kikkawa, "Some remarks on a recent generalization of the Banach contraction principle," in *Proceedings of the 8th International Conference on Fixed Point Theory and Its Applications (ICFPTA '08)*, S. Dhompongsa, K. Goebel, W. A. Kirk, S. Plubtieng, B. Sims, and S. Suantai, Eds., pp. 151–161, Yokohama, Chiang Mai, Thailand, July 2008.

[8] T. Suzuki and C. Vetro, "Three existence theorems for weak contractions of Matkowski type," *International Journal of Mathematics and Statistics*, vol. 6, supplement 10, pp. 110–120, 2010.

[9] M. Kikkawa and T. Suzuki, "Some similarity between contractions and Kannan mappings," *Fixed Point Theory and Applications*, vol. 2008, Article ID 649749, 8 pages, 2008.