

## Research Article

# Strong Convergence of Two Iterative Algorithms for Nonexpansive Mappings in Hilbert Spaces

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Received 6 April 2009; Accepted 12 September 2009

Recommended by Simeon Reich

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## 1. Introduction

Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Recall that a mapping  $T : C \rightarrow C$  is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad (1.1)$$

for all  $x, y \in C$ . We use  $\text{Fix}(T)$  to denote the set of fixed points of  $T$ .

Construction of fixed points of nonlinear mappings is an important and active research area. In particular, iterative algorithms for finding fixed points of nonexpansive mappings have received vast investigation (cf. [1, 2]) since these algorithms find applications in a variety of applied areas of inverse problem, partial differential equations, image recovery, and signal processing see; [3–8]. Iterative methods for nonexpansive mappings have been extensively investigated in the literature; see [1–7, 9–21].

It is our purpose in this paper to introduce two iterative algorithms for nonexpansive mappings in Hilbert spaces. We prove that the proposed algorithms strongly converge to a fixed point of nonexpansive mapping  $T$ .

## 2. Preliminaries

Let  $C$  be a nonempty closed convex subset of  $H$ . For every point  $x \in H$ , there exists a unique nearest point in  $C$ , denoted by  $P_C x$  such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \quad (2.1)$$

The mapping  $P_C$  is called the metric projection of  $H$  onto  $C$ . It is well known that  $P_C$  is a nonexpansive mapping.

In order to prove our main results, we need the following well-known lemmas.

**Lemma 2.1** (see [22], Demiclosed principle). *Let  $C$  be a nonempty closed convex of a real Hilbert space  $H$ . Let  $T : C \rightarrow C$  be a nonexpansive mapping. Then  $I - T$  is demiclosed at 0, that is, if  $x_n \rightarrow x \in C$  and  $x_n - T x_n \rightarrow 0$ , then  $x = T x$ .*

**Lemma 2.2** (see [20]). *Let  $\{x_n\}, \{z_n\}$  be bounded sequences in a Banach space  $E$ , and let  $\{\beta_n\}$  be a sequence in  $[0, 1]$  which satisfies the following condition:  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ . Suppose that  $x_{n+1} = (1 - \beta_n)x_n + \beta_n z_n$  for all  $n \geq 0$  and  $\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$ , then  $\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0$ .*

**Lemma 2.3** (see [22]). *Assume, that  $\{a_n\}$  is a sequence of nonnegative real numbers such that  $a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, n \geq 0$ , where  $\{\gamma_n\}$  is a sequence in  $(0, 1)$  and  $\{\delta_n\}$  is a sequence in  $\mathbb{R}$  such that*

- (i)  $\sum_{n=0}^{\infty} \gamma_n = \infty$ ,
- (ii)  $\limsup_{n \rightarrow \infty} \delta_n \leq 0$  or  $\sum_{n=0}^{\infty} |\delta_n \gamma_n| < \infty$ ,

then  $\lim_{n \rightarrow \infty} a_n = 0$ .

## 3. Main Results

Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $T : C \rightarrow C$  be a nonexpansive mapping. For each  $t \in (0, 1)$ , we consider the following mapping  $T_t$  given by

$$T_t x = T P_C[(1 - t)x], \quad \forall x \in C. \quad (3.1)$$

It is easy to check that  $\|T_t x - T_t y\| \leq (1 - t)\|x - y\|$  which implies that  $T_t$  is a contraction. Using the Banach contraction principle, there exists a unique fixed point  $x_t$  of  $T_t$  in  $C$ , that is,

$$x_t = T P_C[(1 - t)x_t]. \quad (3.2)$$

**Theorem 3.1.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $T : C \rightarrow C$  be a nonexpansive mapping with  $\text{Fix}(T) \neq \emptyset$ . For each  $t \in (0, 1)$ , let the net  $\{x_t\}$  be generated by (3.2). Then, as  $t \rightarrow 0$ , the net  $\{x_t\}$  converges strongly to a fixed point of  $T$ .*

*Proof.* First, we prove that  $\{x_t\}$  is bounded. Take  $u \in \text{Fix}(T)$ . From (3.2), we have

$$\|x_t - u\| = \|T P_C[(1 - t)x_t] - T P_C u\| \leq (1 - t)\|x_t - u\| + t\|u\|, \quad (3.3)$$

that is,

$$\|x_t - u\| \leq \|u\|. \quad (3.4)$$

Hence,  $\{x_t\}$  is bounded.

Again from (3.2), we obtain

$$\|x_t - Tx_t\| = \|TP_C[(1-t)x_t] - TP_Cx_t\| \leq t\|x_t\| \longrightarrow 0, \quad \text{as } t \longrightarrow 0. \quad (3.5)$$

Next we show that  $\{x_t\}$  is relatively norm compact as  $t \rightarrow 0$ . Let  $\{t_n\} \subset (0, 1)$  be a sequence such that  $t_n \rightarrow 0$  as  $n \rightarrow \infty$ . Put  $x_n := x_{t_n}$ . From (3.5), we have

$$\|x_n - Tx_n\| \longrightarrow 0. \quad (3.6)$$

From (3.2), we get, for  $u \in \text{Fix}(T)$ ,

$$\begin{aligned} \|x_t - u\|^2 &= \|TP_C[(1-t)x_t] - Tu\|^2 \\ &\leq \|x_t - u - tx_t\|^2 \\ &= \|x_t - u\|^2 - 2t\langle x_t, x_t - u \rangle + t^2\|x_t\|^2 \\ &= \|x_t - u\|^2 - 2t\langle x_t - u, x_t - u \rangle - 2t\langle u, x_t - u \rangle + t^2\|x_t\|^2. \end{aligned} \quad (3.7)$$

Hence,

$$\|x_t - u\|^2 \leq \langle u, u - x_t \rangle + \frac{t}{2}\|x_t\|^2 \leq \langle u, u - x_t \rangle + \frac{t}{2}M, \quad (3.8)$$

where  $M > 0$  is a constant such that  $\sup_t \{\|x_t\|\} \leq M$ . In particular,

$$\|x_n - u\|^2 \leq \langle u, u - x_n \rangle + \frac{t_n}{2}M, \quad u \in \text{Fix}(T). \quad (3.9)$$

Since  $\{x_n\}$  is bounded, without loss of generality, we may assume that  $\{x_n\}$  converges weakly to a point  $x^* \in C$ . Noticing (3.6) we can use Lemma 2.1 to get  $x^* \in \text{Fix}(T)$ . Therefore we can substitute  $x^*$  for  $u$  in (3.9) to get

$$\|x_n - x^*\|^2 \leq \langle x^*, x^* - x_n \rangle + \frac{t_n}{2}M. \quad (3.10)$$

Hence, the weak convergence of  $\{x_n\}$  to  $x^*$  actually implies that  $x_n \rightarrow x^*$  strongly. This has proved the relative norm compactness of the net  $\{x_t\}$  as  $t \rightarrow 0$ .

To show that the entire net  $\{x_t\}$  converges to  $x^*$ , assume  $x_{t_m} \rightarrow \tilde{x} \in \text{Fix}(T)$ , where  $t_m \rightarrow 0$ . Put  $x_m = x_{t_m}$ . Similarly we have

$$\|x_m - x^*\|^2 \leq \langle x^*, x^* - x_m \rangle + \frac{t_m}{2}M. \quad (3.11)$$

Therefore,

$$\|\tilde{x} - x^*\|^2 \leq \langle x^*, x^* - \tilde{x} \rangle. \quad (3.12)$$

Interchange  $x^*$  and  $\tilde{x}$  to obtain

$$\|x^* - \tilde{x}\|^2 \leq \langle \tilde{x}, \tilde{x} - x^* \rangle. \quad (3.13)$$

Adding up (3.12) and (3.13) yields

$$2\|x^* - \tilde{x}\|^2 \leq \|x^* - \tilde{x}\|^2, \quad (3.14)$$

which implies that  $\tilde{x} = x^*$ . This completes the proof.  $\square$

**Theorem 3.2.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $T : C \rightarrow C$  be a nonexpansive mapping such that  $\text{Fix}(T) \neq \emptyset$ . Let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be two real sequences in  $(0, 1)$ . For given  $x_0 \in C$  arbitrarily, let the sequence  $\{x_n\}$ ,  $n \geq 0$ , be generated iteratively by*

$$y_n = P_C[(1 - \alpha_n)x_n], \quad x_{n+1} = (1 - \beta_n)x_n + \beta_n T y_n. \quad (3.15)$$

Suppose that the following conditions are satisfied:

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ,
- (ii)  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ ,

then the sequence  $\{x_n\}$  generated by (3.15) strongly converges to a fixed point of  $T$ .

*Proof.* First, we prove that the sequence  $\{x_n\}$  is bounded. Take  $u \in \text{Fix}(T)$ . From (3.15), we have

$$\begin{aligned} \|x_{n+1} - u\| &= \|(1 - \beta_n)(x_n - u) + \beta_n(T y_n - u)\| \\ &\leq (1 - \beta_n)\|x_n - u\| + \beta_n\|y_n - u\| \\ &\leq (1 - \beta_n)\|x_n - u\| + \beta_n[(1 - \alpha_n)\|x_n - u\| + \alpha_n\|u\|] \\ &= (1 - \alpha_n\beta_n)\|x_n - u\| + \alpha_n\beta_n\|u\| \\ &\leq \max\{\|x_n - u\|, \|u\|\}. \end{aligned} \quad (3.16)$$

Hence,  $\{x_n\}$  is bounded and so is  $\{T x_n\}$ .

Set  $z_n = T y_n$ ,  $n \geq 0$ . It follows that

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|T y_{n+1} - T y_n\| \\ &\leq \|y_{n+1} - y_n\| \\ &\leq \|(1 - \alpha_{n+1})x_{n+1} - (1 - \alpha_n)x_n\| \\ &\leq \|x_{n+1} - x_n\| + \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\|. \end{aligned} \quad (3.17)$$

Hence,

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \quad (3.18)$$

This together with Lemma 2.2 implies that

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0. \quad (3.19)$$

Therefore,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \beta_n \|x_n - z_n\| = 0. \quad (3.20)$$

We observe that

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\ &\leq \|x_n - x_{n+1}\| + (1 - \beta_n) \|x_n - Tx_n\| + \beta_n \|Ty_n - Tx_n\| \\ &\leq \|x_n - x_{n+1}\| + (1 - \beta_n) \|x_n - Tx_n\| + \beta_n \|y_n - x_n\| \\ &\leq \|x_n - x_{n+1}\| + (1 - \beta_n) \|x_n - Tx_n\| + \alpha_n \|x_n\|, \end{aligned} \quad (3.21)$$

that is,

$$\|x_n - Tx_n\| \leq \frac{1}{\beta_n} \{\|x_{n+1} - x_n\| + \alpha_n \|x_n\|\} \rightarrow 0. \quad (3.22)$$

Let the net  $\{x_t\}$  be defined by (3.2). By Theorem 3.1, we have  $x_t \rightarrow x^*$  as  $t \rightarrow 0$ . Next we prove  $\limsup_{n \rightarrow \infty} \langle x^*, x^* - x_n \rangle \leq 0$ . Indeed,

$$\begin{aligned} \|x_t - x_n\|^2 &= \|x_t - Tx_n + Tx_n - x_n\|^2 \\ &= \|x_t - Tx_n\|^2 + 2\langle x_t - Tx_n, Tx_n - x_n \rangle + \|Tx_n - x_n\|^2 \\ &\leq \|x_t - Tx_n\|^2 + M\|x_n - Tx_n\| \\ &\leq \|(1-t)x_t - x_n\|^2 + M\|x_n - Tx_n\| \\ &= \|x_t - x_n\|^2 - 2t\langle x_t, x_t - x_n \rangle + t^2\|x_t\|^2 + M\|x_n - Tx_n\| \\ &\leq \|x_t - x_n\|^2 - 2t\langle x_t, x_t - x_n \rangle + t^2M + M\|x_n - Tx_n\|, \end{aligned} \quad (3.23)$$

where  $M > 0$  such that  $\sup\{\|x_t\|^2, 2\|x_t - Tx_n\|, \|x_t - x_n\|, t \in (0, 1), n \geq 0\} \leq M$ . It follows that

$$\langle x_t, x_t - x_n \rangle \leq \frac{t}{2}M + \frac{M}{2t}\|Tx_n - x_n\|. \quad (3.24)$$

Therefore,

$$\limsup_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \langle x_t, x_t - x_n \rangle \leq 0. \quad (3.25)$$

We note that

$$\begin{aligned} \langle x^*, x^* - x_n \rangle &= \langle x^*, x^* - x_t \rangle + \langle x^* - x_t, x_t - x_n \rangle + \langle x_t, x_t - x_n \rangle \\ &\leq \langle x^*, x^* - x_t \rangle + \|x^* - x_t\| \|x_t - x_n\| + \langle x_t, x_t - x_n \rangle \\ &\leq \langle x^*, x^* - x_t \rangle + \|x^* - x_t\| M + \langle x_t, x_t - x_n \rangle. \end{aligned} \quad (3.26)$$

This together with  $x_t \rightarrow x^*$  and (3.25) implies that

$$\limsup_{n \rightarrow \infty} \langle x^*, x^* - x_n \rangle \leq 0. \quad (3.27)$$

Finally we show that  $x_n \rightarrow x^*$ . From (3.15), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \beta_n) \|x_n - x^*\|^2 + \beta_n \|y_n - x^*\|^2 \\ &\leq (1 - \beta_n) \|x_n - x^*\|^2 + \beta_n \|(1 - \alpha_n)(x_n - x^*) - \alpha_n x^*\|^2 \\ &\leq (1 - \beta_n) \|x_n - x^*\|^2 + \beta_n \left[ (1 - \alpha_n) \|x_n - x^*\|^2 - 2\alpha_n (1 - \alpha_n) \langle x^*, x_n - x^* \rangle + \alpha_n^2 \|x^*\|^2 \right] \\ &\leq (1 - \alpha_n \beta_n) \|x_n - x^*\|^2 + \alpha_n \beta_n \left[ 2(1 - \alpha_n) \langle x^*, x^* - x_n \rangle + \frac{\alpha_n}{\beta_n} \|x^*\|^2 \right]. \end{aligned} \quad (3.28)$$

We can check that all assumptions of Lemma 2.3 are satisfied. Therefore,  $x_n \rightarrow x^*$ . This completes the proof.  $\square$

## Acknowledgment

The second author was partially supported by the Grant NSC 98-2622-E-230-006-CC3 and NSC 98-2923-E-110-003-MY3.

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