Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2009, Article ID 369215, 23 pages
doi:10.1155/2009/369215

Research Article

A General Iterative Method for Solving the Variational Inequality Problem and Fixed Point Problem of an Infinite Family of Nonexpansive Mappings in Hilbert Spaces

Rabian Wangkeeree and Uthai Kamraksa

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

Correspondence should be addressed to Rabian Wangkeeree, rabianw@nu.ac.th

Received 3 November 2008; Accepted 16 January 2009

Recommended by Anthony Lau

We introduce an iterative scheme for finding a common element of the set of common fixed points of a family of infinitely nonexpansive mappings, and the set of solutions of the variational inequality for an inverse-strongly monotone mapping in a Hilbert space. Under suitable conditions, some strong convergence theorems for approximating a common element of the above two sets are obtained. As applications, at the end of the paper we utilize our results to study the problem of finding a common element of the set of fixed points of a family of infinitely nonexpansive mappings and the set of fixed points of a finite family of k-strictly pseudocontractive mappings. The results presented in the paper improve some recent results of Qin and Cho (2008).

1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, C is a nonempty closed convex subset of H, and P_C is the metric projection of H onto C. In the following, we denote by $\to$ strong convergence and by $\rightharpoonup$ weak convergence. Recall that a mapping $T : C \to C$ is called nonexpansive if

$$\|Tu - Tv\| \leq \|u - v\|, \quad \forall u, v \in C. \tag{1.1}$$

We denote by $F(T)$ the set of fixed points of T. Recall that a mapping $B : C \to H$ is said to be

(i) monotone if $\langle Bu - Bv, u - v \rangle \geq 0$, for all $u, v \in C$;

(ii) L-Lipschitz if there exists a constant $L > 0$ such that $\|Bu - Bv\| \leq L\|u - v\|$, for all $u, v \in C$;

(iii) α-inverse-strongly monotone [1, 2] if there exists a positive real number α such that

$$\langle Bu - Bv, u - v\rangle \geq \alpha \|Bu - Bv\|^2, \quad \forall u, v \in C. \tag{1.2}$$

Remark 1.1. It is obvious that any α-inverse-strongly monotone mapping B is monotone and $(1/\alpha)$-Lipschitz continuous.

Let $B : C \to H$ be a mapping. The classical variational inequality problem is to find a $u \in C$ such that

$$\langle Bu, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.3}$$

The set of solutions of variational inequality (1.3) is denoted by $\mathrm{VI}(B, C)$. The variational inequality has been extensively studied in the literature; see, for example, [3, 4] and the references therein.

A self-mapping $f : C \to C$ is a contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(u) - f(v)\| \leq \alpha \|u - v\|, \quad \forall u, v \in C. \tag{1.4}$$

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [5–8] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space:

$$\theta(x) = \min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.5}$$

where A is a linear bounded operator, C is the fixed point set of a nonexpansive mapping T, and b is a given point in H. Let H be a real Hilbert space. Recall that a linear bounded operator B is strongly positive if there is a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.6}$$

Recently, Marino and Xu [9] introduced the following general iterative scheme based on the viscosity approximation method introduced by Moudafi [10]:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.7}$$

where A is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.7) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in C, \tag{1.8}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.9}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

On the other hand, two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced by Mann [11] and is defined as follows:

$$\begin{aligned} x_1 &= x \in C \text{ chosen arbitrary,} \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1, \end{aligned} \tag{1.10}$$

where the sequence $\{\alpha_n\}$ is in the interval $(0, 1)$.

The second iteration process is referred to as Ishikawa's iteration process [12] which is defined recursively by

$$\begin{aligned} x_1 &= x \in C \text{ chosen arbitrary,} \\ y_n &= \beta_n x_n + (1-\beta_n)T x_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \geq 1, \end{aligned} \tag{1.11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $(0, 1)$. However, both (1.16) and (1.11) have only weak convergence in general (see [13], e.g.). Very recently, Qin and Cho [14] introduced a composite iterative algorithm $\{x_n\}$ defined as follows:

$$\begin{aligned} x_1 &= x \in C \text{ chosen arbitrary,} \\ z_n &= \gamma_n x_n + (1-\gamma_n)T x_n, \\ y_n &= \beta_n x_n + (1-\beta_n)T z_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)y_n, \quad n \geq 1, \end{aligned} \tag{1.12}$$

where f is a contraction, T is a nonexpansive mapping, and A is a strongly positive linear bounded self-adjoint operator, proved that, under certain appropriate assumptions on the parameters, $\{x_n\}$ defined by (1.12) converges strongly to a fixed point of T, which solves some variational inequality and is also the optimality condition for the minimization problem (1.9).

On the other hand, for finding an element of $F(T) \cap \mathrm{VI}(B, C)$, under the assumption that a set $C \subseteq H$ is nonempty, closed, and convex, a mapping $T : C \to C$ is nonexpansive and a mapping $B : C \to H$ is α-inverse-strongly monotone, Takahashi and Toyoda [15] introduced the following iterative scheme:

$$\begin{aligned} x_1 &= x \in C \text{ chosen arbitrary,} \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n)T P_C(x_n - \eta_n B x_n), \quad n \geq 1, \end{aligned} \tag{1.13}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\eta_n\}$ is a sequence in $(0,2\alpha)$. They proved that if $F(T)\cap \mathrm{VI}(B,C)\neq\emptyset$, then the sequence $\{x_n\}$ generated by (1.13) converges weakly to some $z\in F(T)\cap \mathrm{VI}(B,C)$. Recently, Iiduka and Takahashi [16] proposed another iterative scheme as follows

$$\begin{gathered} x_1 = x \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x + (1-\alpha_n) T P_C(x_n - \eta_n B x_n), \quad n \geq 1, \end{gathered} \tag{1.14}$$

where B is an α-inverse strongly monotone mapping, $\{\alpha_n\}\subseteq(0,1)$ and $\{\lambda_n\}\subseteq(0,2\alpha)$ satisfy some parameters controlling conditions. They showed that if $F(T)\cap\mathrm{VI}(B,C)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.14) converges strongly to some $z\in F(T)\cap\mathrm{VI}(B,C)$.

The existence of common fixed points for a finite family of nonexpansive mappings has been considered by many authors (see [17–20] and the references therein). The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings (see [21, 22]). The problem of finding an optimal point that minimizes a given cost function over the common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance (see [18, 23]). A simple algorithmic solution to the problem of minimizing a quadratic function over the common set of fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation (see [23, 24]).

In this paper, we study the mapping W_n defined by

$$\begin{gathered} U_{n,n+1} = I, \\ U_{n,n} = \mu_n T_n U_{n,n+1} + (1-\mu_n) I, \\ U_{n,n-1} = \mu_{n-1} T_{n-1} U_{n,n} + (1-\mu_{n-1}) I, \\ \vdots \\ U_{n,k} = \mu_k T_k U_{n,k+1} + (1-\mu_k) I, \\ U_{n,k-1} = \mu_{k-1} T_{k-1} U_{n,k} + (1-\mu_{k-1}) I, \\ \vdots \\ U_{n,2} = \mu_2 T_2 U_{n,3} + (1-\mu_2) I, \\ W_n = U_{n,1} = \mu_1 T_1 U_{n,2} + (1-\mu_1) I, \end{gathered} \tag{1.15}$$

where $\{\mu_i\}$ is a nonnegative real sequence with $0\leq\mu_i<1$, for all $i\geq 1$, $T_1,T_2,\ldots$, form a family of infinitely nonexpansive mappings of C into itself. Nonexpansivity of each T_i ensures the nonexpansivity of W_n. Such a W_n is nonexpansive from C to C and it is called a W-mapping generated by $T_1,T_2,\ldots,T_n$ and $\mu_1,\mu_2,\ldots,\mu_n$.

In this paper, motivated and inspired by Su et al. [25], Marino and Xu [9], Takahashi and Toyoda [15], and Iiduka and Takahashi [16], we will introduce a new iterative scheme:

$$\begin{gathered} x_1 = x \in C \text{ chosen arbitrary}, \\ z_n = \gamma_n x_n + (1-\gamma_n) W_n x_n, \\ y_n = \beta_n x_n + (1-\beta_n) W_n z_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n) I - \alpha_n A) P_C(y_n - \lambda_n B y_n), \end{gathered} \tag{1.16}$$

where W_n is a mapping defined by (1.15), f is a contraction, A is strongly positive linear bounded self-adjoint operator, B is a α-inverse strongly monotone, and we prove that under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$, the sequences $\{x_n\}$ defined by (1.16) converge strongly to a common element of the set of common fixed points of a family of $\{T_n\}$ and the set of solutions of the variational inequality for an inverse-strongly monotone mapping, which solves some variational inequality and is also the optimality condition for the minimization problem (1.9).

2. Preliminaries

Let H be a real Hilbert space. It is well known that for any $\lambda \in [0,1]$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

Let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x-y\|, \quad \forall y \in C. \tag{2.2}$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.3}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{gathered}\langle x - P_C x, y - P_C x\rangle \leq 0,\\ \|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2,\end{gathered} \tag{2.4}$$

for all $x \in H$, $y \in C$. It is easy to see that the following is true:

$$u \in \mathrm{VI}(B,C) \Longleftrightarrow u = P_C(u - \lambda B u), \quad \lambda > 0. \tag{2.5}$$

A Banach space X is said to satisfy the Opial's condition if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|, \quad y \neq x. \tag{2.6}$$

It is well known that each Hilbert space satisfies the Opial's condition.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle u - v, w\rangle \geq 0$, for all $u \in C\}$ and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C; \\ \varnothing, & v \notin C. \end{cases} \tag{2.7}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(B, C)$; see [26].

Now we collect some useful lemmas for proving the convergence result of this paper.

Lemma 2.1. *In a Hilbert space H. Then the following inequality holds*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, (x + y)\rangle, \quad \forall x, y \in H. \tag{2.8}$$

Lemma 2.2 (see [27]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|z_n - x_n\| = 0$.*

Lemma 2.3 (see [28]). *Assume $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \quad n \geq 1, \tag{2.9}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$

(2) $\limsup_{n\to\infty}(\delta_n/\gamma_n) \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.4 (see [9]). *Assume that A is a strongly positive linear bounded self-adjoint operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Throughout this paper, we will assume that $0 < \mu_n \leq \mu < 1$, for all $n \geq 1$. Concerning W_n defined by (1.15), we have the following lemmas which are important to prove our main result.

Lemma 2.5 (see [29]). *Let C be a nonempty closed convex subset of a Hilbert space H, let $T_i : C \to C$ be a family of infinitely nonexpansive mapping with $\bigcap_{i=1}^{\infty} F(T_i) \neq \varnothing$, and let $\{\mu_i\}$ be a real sequence such that $0 < \mu_i \leq \mu < 1$, for all $i \geq 1$. Then*

(1) *W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(T_i)$ for each $n \geq 1$;*

(2) *for each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} U_{n,k}x$ exists;*

(3) *the mapping $W : C \to C$ define by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C, \tag{2.10}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$ and it is called the W-mapping generated by $T_1, T_2, \ldots,$ and $\mu_1, \mu_2, \ldots$.

Lemma 2.6 (see [30]). *Let C be a nonempty closed convex subset of a Hilbert space H, let $\{T_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, and let $\{\mu_i\}$ be a real sequence such that $0 < \mu_i \leq \mu < 1$, for all $i \geq 1$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \| Wx - W_n x \| = 0. \tag{2.11}$$

3. Main Results

Now we are in a position to state and prove the main result in this paper.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H, let f be a contraction of C into itself, let B be an α-inverse strongly monotone mapping of C into H, and let $\{T_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $F := \bigcap_{n=1}^{\infty} F(T_i) \cap VI(B, C) \neq \emptyset$. Let A be a strongly positive linear bounded self-adjoint operator with the coefficient $\overline{\gamma} > 0$ such that $\|A\| \leq 1$. Assume that $0 < \gamma \leq \overline{\gamma}/\alpha$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $[0, 1]$ satisfying the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$,

(C3) $(1 + \beta_n)\gamma_n - 2\beta_n > d$ *for some* $d \in (0, 1)$,

(C4) $\lim_{n\to\infty}|\beta_{n+1} - \beta_n| = \lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0$,

(C5) $\sum_{n=1}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ *and* $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 2\alpha)$.

Then the sequence $\{x_n\}$ defined by (1.16) *converges strongly to $q \in F$, where $q = P_F(\gamma f + (I - A))q$ which solves the following variational inequality:*

$$\langle \gamma f(q) - Ap, p - q \rangle \leq 0, \quad \forall p \in F. \tag{3.1}$$

Proof. Since $\alpha_n \to 0$ as $n \to \infty$ by the condition (C1), we may assume, without loss of generality that $\alpha_n < (1-\delta_n)\|A\|^{-1}$ for all $n \geq 0$. First, we will show that $I-\lambda_n B$ is nonexpansive. Indeed, for all $x, y \in C$ and $\lambda_n \in [0, 2\alpha]$,

$$\begin{aligned} \left\| (I - \lambda_n B)x - (I - \lambda_n B)y \right\|^2 &= \left\| (x - y) - \lambda_n (Ax - Ay) \right\|^2 \\ &= \|x - y\|^2 - 2\lambda_n \langle x - y, Ax - Ay \rangle + \lambda_n^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \lambda_n (\lambda_n - 2\alpha) \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2, \end{aligned} \tag{3.2}$$

which implies that $I - \lambda_n B$ is nonexpansive. Noticing that A is a linear bounded self-adjoint operator, one has

$$\|A\| = \sup\left\{ \left| \langle Ax, x \rangle \right| : x \in H, \ \|x\| = 1 \right\}. \tag{3.3}$$

Observing that

$$\begin{aligned}\langle((1-\delta_n)I-\alpha_n A)x,x\rangle &= 1-\delta_n-\alpha_n\langle Ax,x\rangle\\ &\le 1-\delta_n-\alpha_n\|A\|\\ &\le 0,\end{aligned}$$

we obtain $(1-\delta_n)I-\alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\delta_n)I-\alpha_n A\| &= \sup\{\langle((1-\delta_n)I-\alpha_n A)x,x\rangle : x\in H,\ \|x\|=1\}\\ &= \sup\{1-\delta_n-\alpha_n\langle Ax,x\rangle : x\in H,\ \|x\|=1\}\\ &\le 1-\delta_n-\alpha_n\overline{\gamma}.\end{aligned}$$

Next, we observe that $\{x_n\}$ is bounded. Indeed, pick $p\in\bigcap_{i=1}^{\infty}F(T_i)\cap \mathrm{VI}(B,C)$ and notice that

$$\begin{aligned}\|z_n-p\| &\le \gamma_n\|x_n-p\|+(1-\gamma_n)\|W_n x_n-p\|\le\|x_n-p\|,\\ \|y_n-p\| &\le \beta_n\|x_n-p\|+(1-\beta_n)\|W_n z_n-p\|\\ &\le \beta_n\|x_n-p\|+(1-\beta_n)\|z_n-p\|\\ &\le \|x_n-p\|.\end{aligned} \tag{3.4}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n\gamma f(x_n)+\delta_n x_n+((1-\delta_n)I-\alpha_n A)P_C(y_n-\lambda B y_n)-p\|\\ &= \|\alpha_n(\gamma f(x_n)-Ap)+\delta_n(x_n-p)+((1-\delta_n)I-\alpha_n A)P_C(y_n-\lambda B y_n)-p\|\\ &\le \alpha_n\|\gamma f(x_n)-Ap\|+\delta_n\|x_n-p\|+(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-p\|\\ &\le \alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-Ap\|+(1-\alpha_n\overline{\gamma})\|x_n-p\|\\ &= [1-\alpha_n(\overline{\gamma}-\gamma\alpha)]\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|\\ &= [1-\alpha_n(\overline{\gamma}-\gamma\alpha)]\|x_n-p\|+\alpha_n(\overline{\gamma}-\gamma\alpha)\frac{\|\gamma f(p)-Ap\|}{\overline{\gamma}-\gamma\alpha}.\end{aligned} \tag{3.5}$$

By simple induction, we have

$$\|x_n-p\|\le\max\left\{\|x_0-p\|,\frac{\|Ap-\gamma f(p)\|}{\overline{\gamma}-\gamma\alpha}\right\}, \tag{3.6}$$

which gives that the sequence $\{x_n\}$ is bounded, and so are $\{y_n\}$ and $\{z_n\}$.

Next, we claim that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.7}$$

Since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}\|W_nx_n - W_{n-1}x_n\| &= \|U_{n,1}x_n - U_{n-1,1}x_n\| \\ &= \|\mu_1T_1U_{n,2}x_n - (1-\mu_1)x_n - \mu_1T_1U_{n-1,2}x_n - (1-\mu_1)x_n\| \\ &\leq \mu_1\|U_{n,2}x_n - U_{n-1,2}x_n\| \\ &= \mu_1\|\mu_2T_2U_{n,3}x_n - (1-\mu_2)x_n - \mu_2T_2U_{n-1,3}x_n - (1-\mu_2)x_n\| \\ &\leq \mu_1\mu_2\|U_{n,3}x_n - U_{n-1,3}x_n\| \\ &\quad\vdots \\ &\leq \left(\prod_{i=1}^{n}\mu_i\right)\|U_{n,n}x_n - U_{n-1,n}x_n\| \\ &\leq M_1\left(\prod_{i=1}^{n}\mu_i\right),\end{aligned} \tag{3.8}$$

where $M_1 \geq 0$ is a constant such that $\|U_{n,n}x_n - U_{n-1,n}x_n\| \leq M_1$. Similarly, there exists $M_2 \geq 0$ such that $\|U_{n,n}y_n - U_{n-1,n}y_n\| \leq M_2$.

Observing that

$$\begin{aligned}z_n &= \gamma_nx_n + (1-\gamma_n)W_nx_n, \\ z_{n-1} &= \gamma_{n-1}x_{n-1} + (1-\gamma_{n-1})W_{n-1}x_{n-1},\end{aligned} \tag{3.9}$$

we obtain that

$$z_n - z_{n-1} = (1-\gamma_n)(W_nx_n - W_{n-1}x_{n-1}) + \gamma_n(x_n - x_{n-1}) + (\gamma_{n-1} - \gamma_n)(W_{n-1}x_{n-1} - x_{n-1}). \tag{3.10}$$

It follows that

$$\begin{aligned}\|z_n - z_{n-1}\| &\leq (1-\gamma_n)\|W_nx_n - W_{n-1}x_{n-1}\| + \gamma_n\|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|W_nx_{n-1} - x_{n-1}\| \\ &\leq (1-\gamma_n)\|W_nx_n - W_{n-1}x_n\| + (1-\gamma_n)\|W_{n-1}x_n - W_{n-1}x_{n-1}\| \\ &\quad + \gamma_n\|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|W_nx_{n-1} - x_{n-1}\| \\ &\leq (1-\gamma_n)\|W_nx_n - W_{n-1}x_n\| + (1-\gamma_n)\|x_n - x_{n-1}\| \\ &\quad + \gamma_n\|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|W_nx_{n-1} - x_{n-1}\| \\ &= (1-\gamma_n)\|W_nx_n - W_{n-1}x_n\| + \|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|W_nx_{n-1} - x_{n-1}\| \\ &\leq (1-\gamma_n)M_1\prod_{i=1}^{n}\mu_i + \|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|W_nx_{n-1} - x_{n-1}\|.\end{aligned} \tag{3.11}$$

Noticing that

$$y_n = \beta_n x_n + (1-\beta_n) W_n z_n,$$
$$y_{n-1} = \beta_{n-1} x_{n-1} + (1-\beta_{n-1}) W_{n-1} z_{n-1}, \tag{3.12}$$

we obtain

$$y_n - y_{n-1} = (1-\beta_n)(W_n z_n - W_{n-1} z_{n-1}) + \beta_n (x_n - x_{n-1}) + (W_{n-1} z_{n-1} - x_{n-1})(\beta_{n-1} - \beta_n). \tag{3.13}$$

It follows that

$$\begin{aligned}
\|y_n - y_{n-1}\| &\le (1-\beta_n)\|W_n z_n - W_n z_{n-1}\| + \beta_n \|x_n - x_{n-1}\| + \|W_n z_{n-1} - x_{n-1}\| |\beta_{n-1} - \beta_n| \\
&\le (1-\beta_n)\|W_n z_n - W_{n-1} z_n\| + (1-\beta_n)\|W_{n-1} z_n - W_{n-1} z_{n-1}\| \\
&\quad + \beta_n \|x_n - x_{n-1}\| + \|W_{n-1} z_{n-1} - x_{n-1}\| |\beta_{n-1} - \beta_n| \\
&\le (1-\beta_n)\|W_n z_n - W_{n-1} z_n\| + (1-\beta_n)\|z_n - z_{n-1}\| \\
&\quad + \beta_n \|x_n - x_{n-1}\| + \|W_{n-1} z_{n-1} - x_{n-1}\| |\beta_{n-1} - \beta_n| \\
&\le (1-\beta_n) M_2 \prod_{i=1}^{n} \mu_i + (1-\beta_n)\|z_n - z_{n-1}\| + \beta_n \|x_n - x_{n-1}\| \\
&\quad + \|W_{n-1} z_{n-1} - x_{n-1}\| |\beta_{n-1} - \beta_n|.
\end{aligned} \tag{3.14}$$

Substituting (3.11) into (3.14), we get

$$\begin{aligned}
\|y_n - y_{n-1}\| &\le (1-\beta_n) M_2 \prod_{i=1}^{n} \mu_i + (1-\beta_n)(1-\gamma_n) M_1 \prod_{i=1}^{n} \mu_i \\
&\quad + (1-\beta_n)\|x_n - x_{n-1}\| + (1-\beta_n)|\gamma_{n-1} - \gamma_n| \|W_{n-1} x_{n-1} - x_{n-1}\| \\
&\quad + \beta_n \|x_n - x_{n-1}\| + |\beta_{n-1} - \beta_n| \|W_{n-1} z_{n-1} - x_{n-1}\| \\
&= (1-\beta_n) M_2 \prod_{i=1}^{n} \mu_i + (1-\beta_n)(1-\gamma_n) M_1 \prod_{i=1}^{n} \mu_i + \|x_n - x_{n-1}\| \\
&\quad + M_3 \left((1-\beta_n)|\gamma_{n-1} - \gamma_n| + |\beta_{n-1} - \beta_n| \right),
\end{aligned} \tag{3.15}$$

where M_3 is an appropriate constant such that

$$M_3 \ge \max\left\{ \sup_{n\ge1} \|W_{n-1} x_{n-1} - x_{n-1}\|, \sup_{n\ge1} \|W_{n-1} z_{n-1} - x_{n-1}\| \right\}. \tag{3.16}$$

Putting $l_n = (x_{n+1} - \delta_n x_n)/(1-\delta_n)$, we get, $x_{n+1} = (1-\delta_n) l_n + \delta_n x_n$.

Now, we compute $l_{n+1} - l_n$. Observing that

$$\begin{aligned} l_{n+1} - l_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\delta_{n+1})I - \alpha_{n+1}A)P_C(y_{n+1} - \lambda_{n+1}By_{n+1})}{1-\delta_{n+1}} \\ &\quad - \frac{\alpha_n\gamma f(x_n) + ((1-\delta_n)I - \alpha_n A)P_C(y_n - \lambda_n By_n)}{1-\delta_n} \\ &= \frac{\alpha_{n+1}}{1-\delta_{n+1}}(\gamma f(x_{n+1}) - AP_C(y_{n+1} - \lambda_{n+1}By_{n+1})) \\ &\quad + \frac{\alpha_n}{1-\delta_n}(AP_C(y_n - \lambda_n By_n) - \gamma f(x_n)) \\ &\quad + P_C(y_{n+1} - \lambda_{n+1}By_{n+1}) - P_C(y_n - \lambda_n By_n). \end{aligned} \tag{3.17}$$

It follows from (3.15) that

$$\begin{aligned} \|l_{n+1} - l_n\| &\le \frac{\alpha_{n+1}}{1-\delta_{n+1}}\|\gamma f(x_{n+1}) - AP_C(y_{n+1} - \lambda_{n+1}By_{n+1})\| \\ &\quad + \frac{\alpha_n}{1-\delta_n}\|AP_C(y_n - \lambda_n By_n) - \gamma f(x_n)\| + \|y_{n+1} - y_n\| \\ &\le \frac{\alpha_{n+1}}{1-\delta_{n+1}}\|\gamma f(x_{n+1}) - AP_C(y_{n+1} - \lambda_{n+1}By_{n+1})\| \\ &\quad + \frac{\alpha_n}{1-\delta_n}\|AP_C(y_n - \lambda_n By_n) - \gamma f(x_n)\| \\ &\quad + (1-\beta_n)M_2\prod_{i=1}^{n}\mu_i + (1-\beta_n)(1-\gamma_n)M_1\prod_{i=1}^{n}\mu_i \\ &\quad + \|x_n - x_{n-1}\| + M_3((1-\beta_n)|\gamma_{n-1} - \gamma_n| + |\beta_{n-1} - \beta_n|). \end{aligned} \tag{3.18}$$

It follows that

$$\begin{aligned} \|l_{n+1} - l_n\| - \|x_n - x_{n-1}\| &\le \frac{\alpha_{n+1}}{1-\delta_{n+1}}\|\gamma f(x_{n+1}) - AP_C(y_{n+1} - \lambda_{n+1}By_{n+1})\| \\ &\quad + \frac{\alpha_n}{1-\delta_n}\|AP_C(y_n - \lambda_n By_n) - \gamma f(x_n)\| \\ &\quad + (1-\beta_n)M_2\prod_{i=1}^{n}\mu_i + (1-\beta_n)(1-\gamma_n)M_1\prod_{i=1}^{n}\mu_i \\ &\quad + M_3((1-\beta_n)|\gamma_{n-1} - \gamma_n| + |\beta_{n-1} - \beta_n|). \end{aligned} \tag{3.19}$$

Observing the conditions (C1) and (C4) and taking the superior limit as $n \to \infty$, we get

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_n - x_{n-1}\|) \le 0. \tag{3.20}$$

We can obtain $\lim_{n\to\infty}\|l_n - x_n\| = 0$ easily by Lemma 2.2 since

$$x_{n+1} - x_n = (1-\delta_n)(l_n - x_n), \tag{3.21}$$

one obtains that (3.7) holds. Setting $t_n = P_C(y_n - \lambda_n y_n)$, we have

$$x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)t_n. \tag{3.22}$$

Observing that

$$\begin{aligned} x_n - t_n &= x_n - x_{n+1} + x_{n+1} - t_n \\ &= x_n - x_{n+1} + \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)t_n - t_n \\ &= x_n - x_{n+1} + \alpha_n(\gamma f(x_n) - At_n) + \delta_n(x_n - t_n), \end{aligned} \tag{3.23}$$

we arrive at

$$(1-\delta_n)(x_n - t_n) = x_n - x_{n+1} + \alpha_n(\gamma f(x_n) - At_n). \tag{3.24}$$

This implies

$$(1-\delta_n)\|x_n - t_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - At_n\|. \tag{3.25}$$

From (3.7) and (C1) we obtain that

$$\lim_{n\to\infty}\|x_n - t_n\| = 0. \tag{3.26}$$

Next, we will show that $\|By_n - Bp\| \to 0$ as $n \to \infty$ for any $p \in F$. Observe that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|((1-\delta_n)I - \alpha_n A)(t_n - p) + \delta_n(x_n - p) + \alpha_n(\gamma f(x_n) - Ap)\|^2 \\ &= \|((1-\delta_n)I - \alpha_n A)(t_n - p) + \delta_n(x_n - p)\|^2 + \alpha_n^2\|\gamma f(x_n) - Ap\|^2 \\ &\quad + 2\delta_n\alpha_n\langle x_n - p, \gamma f(x_n) - Ap\rangle + 2\alpha_n\langle((1-\delta_n)I - \alpha_n A)(t_n - p), \gamma f(x_n) - Ap\rangle \\ &\le ((1-\delta_n - \alpha_n\overline{\gamma})\|t_n - p\| + \delta_n\|x_n - p\|)^2 + \alpha_n^2\|\gamma f(x_n) - Ap\|^2 \\ &\quad + 2\delta_n\alpha_n\langle x_n - p, \gamma f(x_n) - Ap\rangle + 2\alpha_n\langle((1-\delta_n)I - \alpha_n A)(t_n - p), \gamma f(x_n) - Ap\rangle \\ &= (1-\delta_n - \alpha_n\overline{\gamma})^2\|t_n - p\|^2 + \delta_n^2\|x_n - p\|^2 \\ &\quad + 2(1-\delta_n - \alpha_n\overline{\gamma})\delta_n\|t_n - p\|\|x_n - p\| + c_n \\ &\le (1-\delta_n - \alpha_n\overline{\gamma})^2\|t_n - p\|^2 + \delta_n^2\|x_n - p\|^2 \\ &\quad + (1-\delta_n - \alpha_n\overline{\gamma})\delta_n\left(\|t_n - p\|^2 + \|x_n - p\|^2\right) + c_n \\ &= \left[(1-\alpha_n\overline{\gamma})^2 - 2(1-\alpha_n\overline{\gamma})\delta_n + \delta_n^2\right]\|t_n - p\|^2 + \delta_n^2\|x_n - p\|^2 \\ &\quad + ((1-\alpha_n\overline{\gamma})\delta_n - \delta_n^2)\left(\|t_n - p\|^2 + \|x_n - p\|^2\right) + c_n \\ &= (1-\alpha_n\overline{\gamma})^2\|t_n - p\|^2 - (1-\alpha_n\overline{\gamma})\delta_n\|t_n - p\|^2 + (1-\alpha_n\overline{\gamma})\delta_n\|x_n - p\|^2 + c_n \end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|t_n-p\|^2 + (1-\alpha_n\overline{\gamma})\delta_n\|x_n-p\|^2 + c_n \\
&\leq (1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\left[\|(y_n-\lambda_n By_n)-(p-\lambda_n Bp)\|^2\right] \\
&\quad + (1-\alpha_n\overline{\gamma})\delta_n\|x_n-p\|^2 + c_n \\
&\leq (1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\left[\|y_n-p\|^2 + \lambda_n(\lambda_n-2\alpha)\|By_n-Bp\|^2\right] \\
&\quad + (1-\alpha_n\overline{\gamma})\delta_n\|x_n-p\|^2 + c_n \\
&\leq \|x_n-p\|^2 + b(b-2\alpha)\|By_n-Bp\|^2 + c_n,
\end{aligned}
\tag{3.27}
$$

where

$$
\begin{aligned}
c_n = {} & \alpha_n^2\|\gamma f(x_n)-Ap\|^2 + 2\delta_n\alpha_n\langle x_n-p,\gamma f(x_n)-Ap\rangle \\
& + 2\alpha_n\langle((1-\delta_n)I-\alpha_n A)(t_n-p),\gamma f(x_n)-Ap\rangle.
\end{aligned}
\tag{3.28}
$$

This impies that

$$
\begin{aligned}
-b(b-2\alpha)\|By_n-Bp\|^2 &\leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + c_n \\
&\leq \|x_n-x_{n+1}\|(\|x_n-p\| + \|x_{n+1}-p\|) + c_n.
\end{aligned}
\tag{3.29}
$$

Since $\lim_{n\to\infty}c_n = 0$ and from (3.7), we obtain

$$
\lim_{n\to\infty}\|By_n-Bp\| = 0. \tag{3.30}
$$

From (2.3), we have

$$
\begin{aligned}
\|t_n-p\|^2 &= \|P_C(y_n-\lambda_n By_n) - P_C(p-\lambda_n Bp)\|^2 \\
&\leq \langle (y_n-\lambda_n By_n)-(p-\lambda_n Bp), t_n-p\rangle \\
&= \frac{1}{2}\Big\{\|(y_n-\lambda_n By_n)-(p-\lambda_n Bp)\|^2 + \|t_n-p\|^2 \\
&\qquad -\|(y_n-\lambda_n By_n)-(p-\lambda_n Bp)-(t_n-p)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|y_n-p\|^2 + \|t_n-p\|^2 - \|(y_n-t_n)-\lambda_n(By_n-Bp)\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|y_n-p\|^2 + \|t_n-p\|^2 - \|y_n-t_n\|^2 + 2\lambda_n\langle y_n-t_n, By_n-Bp\rangle - \lambda_n^2\|By_n-Bp\|^2\Big\},
\end{aligned}
\tag{3.31}
$$

so, we obtain

$$
\|t_n-p\|^2 \leq \|y_n-p\|^2 - \|y_n-t_n\|^2 + 2\lambda_n\langle y_n-t_n, By_n-Bp\rangle - \lambda_n^2\|By_n-Bp\|^2. \tag{3.32}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|t_n-p\|^2+(1-\alpha_n\overline{\gamma})\delta_n\|x_n-p\|^2+c_n\\
&\le (1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\\
&\quad\times\left[\|y_n-p\|^2-\|y_n-t_n\|^2+2\lambda_n\langle y_n-t_n,By_n-Bp\rangle-\lambda_n^2\|By_n-Bp\|^2\right]\\
&\quad+(1-\alpha_n\overline{\gamma})\delta_n\|x_n-p\|^2+c_n\\
&\le (1-\alpha_n\overline{\gamma})\|x_n-p\|^2-(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-t_n\|^2\\
&\quad+2\lambda_n(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-t_n\|\|By_n-Bp\|\\
&\quad-\lambda_n^2(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|By_n-Bp\|^2+c_n,
\end{aligned}
\tag{3.33}
$$

which implies that

$$
\begin{aligned}
&(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-t_n\|^2\\
&\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\lambda_n(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-t_n\|\|By_n-Bp\|\\
&\qquad-\lambda_n^2(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|By_n-Bp\|^2+c_n\\
&\quad\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)\\
&\qquad+2\lambda_n(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-t_n\|\|By_n-Bp\|\\
&\qquad-\lambda_n^2(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|By_n-Bp\|^2+c_n.
\end{aligned}
\tag{3.34}
$$

Applying (3.7), (3.30), and $\lim_{n\to\infty}c_n=0$ to the last inequality, we obtain that

$$
\lim_{n\to\infty}\|y_n-t_n\|=0.
\tag{3.35}
$$

It follows from (3.26) and (3.35) that

$$
\|x_n-y_n\|\le\|x_n-t_n\|+\|t_n-y_n\|\longrightarrow 0\quad\text{as } n\longrightarrow\infty.
\tag{3.36}
$$

On the other hand, one has

$$
\begin{aligned}
\|W_nx_n-x_n\| &\le \|x_n-y_n\|+\|y_n-W_nx_n\|\\
&\le \|x_n-y_n\|+\|y_n-W_nz_n\|+\|W_nz_n-W_nx_n\|\\
&\le \|x_n-y_n\|+\beta_n\|x_n-W_nz_n\|+\|z_n-x_n\|\\
&\le \|x_n-y_n\|+\beta_n\|x_n-W_nx_n\|+\beta_n\|W_nx_n-W_nz_n\|+\|z_n-x_n\|\\
&\le \|x_n-y_n\|+\beta_n\|x_n-W_nx_n\|+(1+\beta_n)\|z_n-x_n\|\\
&\le \|x_n-y_n\|+\beta_n\|x_n-W_nx_n\|+(1+\beta_n)(1-\gamma_n)\|W_nx_n-x_n\|\\
&= \|x_n-y_n\|-\left[(1+\beta_n)\gamma_n-2\beta_n-1\right]\|W_nx_n-x_n\|,
\end{aligned}
\tag{3.37}
$$

which implies

$$\left[(1+\beta_n)\gamma_n - 2\beta_n\right]\left\|W_n x_n - x_n\right\| \le \left\|x_n - y_n\right\|. \tag{3.38}$$

From the conditions (C3), it follows that

$$\lim_{n\to\infty}\left\|W_n x_n - x_n\right\| = 0. \tag{3.39}$$

Applying Lemma 2.6 and (3.39), we obtain that

$$\begin{aligned}\left\|W x_n - x_n\right\| &\le \left\|W x_n - W_n x_n\right\| + \left\|W_n x_n - x_n\right\| \\ &\le \sup_{x\in\{x_n\}}\left\|W x - W_n x\right\| + \left\|W_n x_n - x_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.40}$$

It follows from (3.26) and (3.40) that

$$\begin{aligned}\left\|W t_n - t_n\right\| &\le \left\|W t_n - W x_n\right\| + \left\|W x_n - x_n\right\| + \left\|x_n - t_n\right\| \\ &\le 2\left\|t_n - x_n\right\| + \left\|W x_n - x_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.41}$$

We observe that $P_F(\gamma f + (I - A))$ is a contraction. Indeed, for all $x, y \in H$, we have

$$\begin{aligned}&\left\|P_F(\gamma f + (I - A))(x) - P_F(\gamma f + (I - A))(y)\right\| \\ &\quad\le \left\|(\gamma f + (I - A))(x) - (\gamma f + (I - A))(y)\right\| \\ &\quad\le \gamma\left\|f(x) - f(y)\right\| + \|I - A\|\|x - y\| \\ &\quad\le \gamma\alpha\|x - y\| + (1 - \overline{\gamma})\|x - y\| \\ &\quad< \gamma\|x - y\|.\end{aligned} \tag{3.42}$$

Banach's Contraction Mapping Principle guarantees that $P_F(\gamma f + (I - A))$ has a unique fixed point, say $q \in H$. That is, $q = P_F(\gamma f + (I - A))(q)$.

Next, we claim that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Aq, t_n - q\rangle \le 0. \tag{3.43}$$

Indeed, we choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Aq, W t_n - q\rangle = \lim_{i\to\infty}\langle \gamma f(q) - Aq, W t_{n_i} - q\rangle. \tag{3.44}$$

Since $\{t_{n_i}\}$ is bounded, there exists a subsequence $\{t_{n_{i_j}}\}$ of $\{t_{n_i}\}$ which converges weakly to $z \in C$. Without loss of generality, we can assume that $t_{n_i} \rightharpoonup z$. From $\|Wt_{n_i} - t_{n_i}\| \to 0$, we obtain $Wt_{n_i} \rightharpoonup z$. Therefore, we have

$$\begin{aligned} \limsup_{n\to\infty}\langle \gamma f(q) - Aq, Wt_n - q\rangle &= \lim_{i\to\infty}\langle \gamma f(q) - Aq, Wt_{n_i} - q\rangle \\ &= \langle \gamma f(q) - Aq, z - q\rangle. \end{aligned} \tag{3.45}$$

Next we prove that $z \in F := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{VI}(B, C)$.

First, we prove that $z \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$.

Suppose the contrary, $z \notin F(W)$, that is, $Wz \neq z$. Since $t_{n_i} \rightharpoonup z$, by the Opial's condition and (3.41), we have

$$\begin{aligned} \liminf_{i\to\infty}\|t_{n_i} - z\| &< \liminf_{i\to\infty}\|t_{n_i} - Wz\| \\ &\leq \liminf_{i\to\infty}\{\|t_{n_i} - Wt_{n_i}\| + \|Wt_{n_i} - Wz\|\} \\ &\leq \liminf_{i\to\infty}\{\|t_{n_i} - Wt_{n_i}\| + \|t_{n_i} - z\|\} \\ &= \liminf_{i\to\infty}\|t_{n_i} - z\|. \end{aligned} \tag{3.46}$$

This is a contradiction, which shows that $z \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$.

Next, we prove $z \in \mathrm{VI}(B, C)$. For this purpose, let T be the maximal monotone mapping defined by (2.7):

$$Tv = \begin{cases} Bv + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases} \tag{3.47}$$

For any given $(v, w) \in G(T)$, hence $w - Bv \in N_C(v)$. Since $t_n \in C$, we have

$$\langle v - t_n, w - Bv\rangle \geq 0. \tag{3.48}$$

On the other hand, from $t_n = P_C(y_n - \lambda_n B y_n)$, we have

$$\langle v - t_n, t_n - (y_n - \lambda_n B y_n)\rangle \geq 0, \tag{3.49}$$

that is,

$$\left\langle v - t_n, \frac{t_n - y_n}{\lambda_n} + By_n \right\rangle \geq 0. \tag{3.50}$$

Therefore, we obtian

$$\begin{aligned}\langle v-t_{n_i},w\rangle &\geq \langle v-t_{n_i},Bv\rangle\\ &\geq \langle v-t_{n_i},Bv\rangle-\left\langle v-t_{n_i},\frac{t_{n_i}-y_{n_i}}{\lambda_{n_i}}+By_{n_i}\right\rangle\\ &=\left\langle v-t_{n_i},Bv-By_{n_i}-\frac{t_{n_i}-y_{n_i}}{\lambda_{n_i}}\right\rangle\\ &=\langle v-t_{n_i},Bv-Bt_{n_i}\rangle+\langle v-t_{n_i},Bt_{n_i}-By_{n_i}\rangle-\left\langle v-t_{n_i},\frac{t_{n_i}-y_{n_i}}{\lambda_{n_i}}\right\rangle\\ &\geq \langle v-t_{n_i},Bt_{n_i}\rangle-\left\langle v-t_{n_i},\frac{t_{n_i}-y_{n_i}}{\lambda_{n_i}}+By_{n_i}\right\rangle\\ &=\langle v-t_{n_i},Bt_{n_i}-By_{n_i}\rangle-\left\langle v-t_{n_i},\frac{t_{n_i}-y_{n_i}}{\lambda_{n_i}}\right\rangle.\end{aligned}\tag{3.51}$$

Noting that $\|t_{n_i}-y_{n_i}\|\to 0$ as $n\to\infty$ and B is Lipschitz continuous, hence from (3.18), we obtain

$$\langle v-z,w\rangle\geq 0.\tag{3.52}$$

Since T is maximal monotone, we have $z\in T^{-1}0$, and hence $z\in \mathrm{VI}(B,C)$.

The conclusion $z\in\bigcap_{i=1}^{\infty}F(T_i)\cap \mathrm{VI}(B,C)$ is proved.

Hence by (3.45), we obtain

$$\limsup_{n\to\infty}\langle\gamma f(q)-Aq,Wt_n-q\rangle=\langle\gamma f(q)-Aq,z-q\rangle\leq 0.\tag{3.53}$$

Since $q=P_F f(q)$, it follows from (3.39), (3.41), and (3.53) that

$$\begin{aligned}\limsup_{n\to\infty}\langle\gamma f(q)-Aq,t_n-q\rangle&=\limsup_{n\to\infty}\langle\gamma f(q)-Aq,(t_n-Wt_n)+(Wt_n-q)\rangle\\ &\leq\limsup_{n\to\infty}\langle\gamma f(q)-Aq,Wt_n-q\rangle\\ &\leq 0.\end{aligned}\tag{3.54}$$

Hence (3.43) holds. Using (3.26) and (3.54), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle\gamma f(q)-Aq,x_n-q\rangle&=\limsup_{n\to\infty}\langle\gamma f(q)-Aq,(x_n-t_n)+(t_n-q)\rangle\\ &\leq\limsup_{n\to\infty}\langle\gamma f(q)-Aq,t_n-q\rangle\\ &\leq 0.\end{aligned}\tag{3.55}$$

Now, from Lemma 2.1, it follows that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\alpha_n\gamma f(x_n)+\delta_n x_n+((1-\delta_n)I-\alpha_n A)t_n-q\|^2 \\
&= \|((1-\delta_n)I-\alpha_n A)(t_n-q)+\delta_n(x_n-q)+\alpha_n(\gamma f(x_n)-Aq)\|^2 \\
&= \|((1-\delta_n)I-\alpha_n A)(t_n-q)+\delta_n(x_n-q)\|^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2 \\
&\quad +2\delta_n\alpha_n\langle x_n-q,\gamma f(x_n)-Aq\rangle \\
&\quad +2\alpha_n\langle((1-\delta_n)I-\alpha_n A)(t_n-q),\gamma f(x_n)-Aq\rangle \\
&\le ((1-\delta_n-\alpha_n\overline{\gamma})\|t_n-q\|+\delta_n\|x_n-q\|)^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2 \\
&\quad +2\delta_n\alpha_n\gamma\langle x_n-q,f(x_n)-f(q)\rangle+2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-A(q)\rangle \\
&\quad +2(1-\delta_n)\gamma\alpha_n\langle t_n-q,f(x_n)-f(q)\rangle+2(1-\delta_n)\alpha_n\langle t_n-q,\gamma f(q)-Aq\rangle \\
&\quad -2\alpha_n^2\langle A(t_n-q),\gamma f(q)-Aq\rangle \\
&\le ((1-\delta_n-\alpha_n\overline{\gamma})\|x_n-q\|+\delta_n\|x_n-q\|)^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2 \\
&\quad +2\delta_n\alpha_n\gamma\alpha\|x_n-q\|^2+2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle \\
&\quad +2(1-\delta_n)\gamma\alpha_n\alpha\|x_n-q\|^2+2(1-\delta_n)\alpha_n\langle t_n-q,\gamma f(q)-Aq\rangle \\
&\quad -2\alpha_n^2\langle A(t_n-q),\gamma f(q)-Aq\rangle \\
&= \left[(1-\alpha_n\overline{\gamma})^2+2\delta_n\alpha_n\gamma\alpha+2(1-\delta_n)\gamma\alpha_n\alpha\right]\|x_n-q\|^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2 \\
&\quad +2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle+2(1-\delta_n)\alpha_n\langle t_n-q,\gamma f(q)-Aq\rangle \\
&\quad -2\alpha_n^2\langle A(t_n-q),\gamma f(q)-Aq\rangle \\
&\le \left[1-2(\overline{\gamma}-\alpha_n\gamma)\alpha_n\right]\|x_n-q\|^2+\overline{\gamma}^2\alpha_n^2\|x_n-q\|^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2 \\
&\quad +2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle+2(1-\delta_n)\alpha_n\langle t_n-q,\gamma f(q)-Aq\rangle \\
&\quad +2\alpha_n^2\|A(t_n-q)\|\|\gamma f(q)-Aq\| \\
&= \left[1-2(\overline{\gamma}-\alpha_n\gamma)\alpha_n\right]\|x_n-q\|^2 \\
&\quad +\alpha_n\Big\{\alpha_n(\overline{\gamma}^2\|x_n-q\|^2+\|\gamma f(x_n)-Aq\|^2 \\
&\qquad +2\|A(t_n-q)\|\|\gamma f(q)-Aq\|)+2\delta_n\langle x_n-q,\gamma f(q)-Aq\rangle \\
&\qquad +2(1-\delta_n)\langle t_n-q,\gamma f(q)-Aq\rangle\Big\}.
\end{aligned} \tag{3.56}$$

Since $\{x_n\}$, $\{f(x_n)\}$, and t_n are bounded, we can take a constant $M_5>0$ such that

$$\overline{\gamma}^2\|x_n-q\|^2+\|\gamma f(x_n)-Aq\|^2+2\|A(t_n-q)\|\|\gamma f(q)-Aq\|\le M_5, \tag{3.57}$$

for all $n \geq 0$. It then follows that

$$\|x_{n+1} - q\|^2 \leq \left[1 - 2(\overline{\gamma} - \alpha_n\gamma)\alpha_n\right] \|x_n - q\|^2 + \alpha_n\sigma_n, \tag{3.58}$$

where

$$\sigma_n = 2\delta_n\langle x_n - q, \gamma f(q) - Aq\rangle + 2(1 - \delta_n)\langle t_n - q, \gamma f(q) - Aq\rangle + \alpha_n M_4. \tag{3.59}$$

Using (C1), (3.54), and (3.55), we get $\limsup_{n\to\infty}\delta_n \leq 0$. Now applying Lemma 2.3 to (3.58), we conclude that $x_n \to q$. This completes the proof. □

Remark 3.2. Theorem 3.1 mainly improve the results of Qin and Cho [14] from a single nonexpansive mapping to an infinite family of nonexpansive mappings.

4. Applications

In this section, we obtain two results by using a special case of the proposed method.

Theorem 4.1. *Let H be a real Hilbert space, let B be an α-inverse strongly monotone mapping on H, let $\{T_i : H \to H\}$ be a family of infinitely nonexpansive mappings with $F := \bigcap_{i=1}^{\infty} F(T_i) \cap B^{-1}(0) \neq \emptyset$. Let $f : H \to H$ a contraction with coefficient $\alpha \in (0,1)$, and let A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Suppose the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated by*

$$\begin{aligned} x_1 &= x \in H \text{ chosen arbitrary}, \\ z_n &= \gamma_n x_n + (1 - \gamma_n) W_n x_n, \\ y_n &= \beta_n x_n + (1 - \beta_n) W_n z_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n) I - \alpha_n A)(y_n - \lambda_n B y_n), \end{aligned} \tag{4.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

- (C1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (C2) $0 < \liminf_{n\to\infty}\delta_n \leq \limsup_{n\to\infty}\delta_n < 1$,
- (C3) $(1 + \beta_n)\gamma_n - 2\beta_n > a$ *for some* $a \in (0,1)$,
- (C4) $\lim_{n\to\infty}|\beta_{n+1} - \beta_n| = \lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0$,
- (C5) $\sum_{n=1}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ *and* $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\alpha)$.

Then $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $q = P_F(\gamma f + (I - A))(q)$ which solves the variational inequality:

$$\langle (A - \gamma f)q, q - z\rangle \leq 0, \quad z \in F. \tag{4.2}$$

Proof. We have $B^{-1}(0) = \mathrm{VI}(B, H)$ and $P_H = I$. Applying Theorem 3.1, we obtain the desired result. □

Next, we will apply the main results to the problem for finding a common element of the set of fixed points of a family of infinitely nonexpansive mappings and the set of fixed points of a finite family of k-strictly pseudocontractive mappings.

Definition 4.2. A mappings $S : C \to H$ is said to be a k-strictly pseudocontractive mapping if there exists $k \in [0,1)$ such that

$$\|Sx - Sy\|^2 \le \|x-y\|^2 + k\left\|(I-S)x - (I-S)y\right\|^2, \quad \forall x, y \in C. \tag{4.3}$$

The following lemmas can be obtained from [31, Proposition 2.6] by Acedo and Xu, easily.

Lemma 4.3. *Let H be a Hilbert space, let C be a closed convex subset of H. For any integer $N \ge 1$, assume that, for each $1 \le i \le N$, $S_i : C \to H$ is a k_i-strictly pseudocontractive mapping for some $0 \le k_i < 1$. Assume that $\{\varphi_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \varphi_i = 1$. Then $\sum_{i=1}^N \varphi_i S_i$ is a k-strictly pseudocontractive mapping with $k = \max\{k_i : 1 \le i \le N\}$.*

Lemma 4.4. *Let $\{S_i\}_{i=1}^N$ and $\{\varphi_i\}_{i=1}^N$ be as in Lemma 4.3. Suppose that $\{S_i\}_{i=1}^N$ has a common fixed point in C. Then $F(\sum_{i=1}^N \varphi_i S_i) = \bigcap_{i=1}^N F(S_i)$.*

Let $S_i : C \to H$ be a k_i-strictly pseudocontractive mapping for some $0 \le k_i < 1$. We define a mapping $A = I - \sum_{i=1}^N \varphi_i S_i : C \to H$, where $\{\varphi_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \varphi_i = 1$, then A is a $((1-k)/2)$-inverse-strongly monotone mapping with $k = \max\{k_i : 1 \le i \le N\}$. In fact, from Lemma 4.3, we have

$$\left\|\sum_{i=1}^N \varphi_i S_i x - \sum_{i=1}^N \varphi_i S_i y\right\|^2 \le \|x-y\|^2 + k\left\|\left(I - \sum_{i=1}^N \varphi_i S_i\right)x - \left(I - \sum_{i=1}^N \varphi_i S_i\right)y\right\|^2, \quad \forall x, y \in C. \tag{4.4}$$

That is,

$$\left\|(I-A)x - (I-A)y\right\|^2 \le \|x-y\|^2 + k\|Ax - Ay\|^2. \tag{4.5}$$

On the other hand

$$\left\|(I-A)x - (I-A)y\right\|^2 = \|x-y\|^2 - 2\langle x-y, Ax-Ay\rangle + \|Ax-Ay\|^2. \tag{4.6}$$

Hence, we have

$$\langle x-y, Ax-Ay\rangle \ge \frac{1-k}{2}\|Ax-Ay\|^2. \tag{4.7}$$

This shows that A is $((1-k)/2)$-inverse-strongly monotone.

Theorem 4.5. *Let C be a closed convex subset of a real Hilbert space H. For any integer $N > 1$, assume that, for each $1 \le i \le N$, $S_i : C \to H$ is a k_i-strictly pseudocontractive mapping for* *some* $0 \leq k_i < 1$. *Let* $\{T_i : C \to C\}$ *be a family of infinitely nonexpansive mappings with* $F := \bigcap_{i=1}^{\infty} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i) \neq \emptyset$. *Let* $f : C \to C$ *a contraction with coefficient* $\alpha \in (0,1)$ *and let* A *be a strongly positive bounded linear operator with coefficient* $\overline{\gamma} > 0$ *and* $0 < \gamma < \overline{\gamma}/\alpha$. *Let the sequences* $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *be generated by*

$$\begin{aligned} x_1 &= x \in H \text{ chosen arbitrary}, \\ z_n &= \gamma_n x_n + (1-\gamma_n) W_n x_n, \\ y_n &= \beta_n x_n + (1-\beta_n) W_n z_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A) P_C\left((1-\lambda_n) y_n - \lambda_n \sum_{i=1}^{N} \varphi_i S_i y_n\right), \end{aligned} \tag{4.8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\lambda_n\}$ *are the sequences in* $[0,1]$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty}\delta_n \leq \limsup_{n\to\infty}\delta_n < 1$,

(C3) $(1+\beta_n)\gamma_n - 2\beta_n > a$ *for some* $a \in (0,1)$,

(C4) $\lim_{n\to\infty}|\beta_{n+1} - \beta_n| = \lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0$,

(C5) $\sum_{n=1}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ *and* $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\alpha)$.

Then $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *converge strongly to* $q = P_F(\gamma f + (I - A))(q)$ *which solves the variational inequality:*

$$\langle (A - \gamma f) q, q - z \rangle \leq 0, \quad z \in F. \tag{4.9}$$

Proof. Taking $B = I - \sum_{i=1}^{N} \varphi_i S_i : C \to H$, we know that $B : C \to H$ is α-inverse strongly monotone with $\alpha = (1-k)/2$. Hence, B is a monotone L-Lipschitz continuous mapping with $L = 2/(1-k)$. From Lemma 4.4, we know that $\sum_{i=1}^{N} \varphi_i S_i$ is a k-strictly pseudocontractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$ and then $F(\sum_{i=1}^{N} \varphi_i S_i) = \mathrm{VI}(B, C)$ by Chang [30, Proposition 1.3.5]. Observe that

$$P_C(y_n - \lambda_n B y_n) = P_C\left((1-\lambda_n) y_n - \lambda_n \sum_{i=1}^{N} \varphi_i S_i y_n\right). \tag{4.10}$$

The conclusion of Theorem 4.5 can be obtained from Theorem 3.1. □

Remark 4.6. Theorem 4.5 is a generalization and improvement of the theorems by Qin and Cho [14], Iiduka and Takahashi [16, Thorem 3.1], and Takahashi and Toyoda [15].

Acknowledgments

The authors would like to thank the referee for the comments which improve the manuscript. R. Wangkeeree was supported for CHE-PhD-THA-SUP/2551 from the Commission on Higher Education and the Thailand Research Fund under Grant TRG5280011.

References

[1] F. E. Browder and W. V. Petryshyn, “Construction of fixed points of nonlinear mappings in Hilbert space,” *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[2] F. Liu and M. Z. Nashed, “Regularization of nonlinear ill-posed variational inequalities and convergence rates,” *Set-Valued Analysis*, vol. 6, no. 4, pp. 313–344, 1998.

[3] J.-C. Yao and O. Chadli, “Pseudomonotone complementarity problems and variational inequalities,” in *Handbook of Generalized Convexity and Generalized Monotonicity*, vol. 76 of *Nonconvex Optimization and Its Applications*, pp. 501–558, Springer, New York, NY, USA, 2005.

[4] L. C. Zeng, S. Schaible, and J. C. Yao, “Iterative algorithm for generalized set-valued strongly nonlinear mixed variational-like inequalities,” *Journal of Optimization Theory and Applications*, vol. 124, no. 3, pp. 725–738, 2005.

[5] F. Deutsch and I. Yamada, “Minimizing certain convex functions over the intersection of the fixed point sets of nonexpansive mappings,” *Numerical Functional Analysis and Optimization*, vol. 19, no. 1-2, pp. 33–56, 1998.

[6] H.-K. Xu, “Iterative algorithms for nonlinear operators,” *Journal of the London Mathematical Society. Second Series*, vol. 66, no. 1, pp. 240–256, 2002.

[7] H. K. Xu, “An iterative approach to quadratic optimization,” *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.

[8] I. Yamada, “The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings,” in *Inherently Parallel Algorithm for Feasibility and Optimization and Their Applications*, D. Butnariu, Y. Censor, and S. Reich, Eds., vol. 8 of *Studies in Computational Mathematics*, pp. 473–504, North-Holland, Amsterdam, The Netherlands, 2001.

[9] G. Marino and H.-K. Xu, “A general iterative method for nonexpansive mappings in Hilbert spaces,” *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 43–52, 2006.

[10] A. Moudafi, “Viscosity approximation methods for fixed-points problems,” *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.

[11] W. R. Mann, “Mean value methods in iteration,” *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.

[12] S. Ishikawa, “Fixed points by a new iteration method,” *Proceedings of the American Mathematical Society*, vol. 44, pp. 147–150, 1974.

[13] A. Genel and J. Lindenstrauss, “An example concerning fixed points,” *Israel Journal of Mathematics*, vol. 22, no. 1, pp. 81–86, 1975.

[14] X. Qin and Y. Cho, “Convergence of a general iterative method for nonexpansive mappings in Hilbert spaces,” *Journal of Computational and Applied Mathematics*. In press.

[15] W. Takahashi and M. Toyoda, “Weak convergence theorems for nonexpansive mappings and monotone mappings,” *Journal of Optimization Theory and Applications*, vol. 118, no. 2, pp. 417–428, 2003.

[16] H. Iiduka and W. Takahashi, “Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 61, no. 3, pp. 341–350, 2005.

[17] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, “Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 8, pp. 2350–2360, 2007.

[18] H. H. Bauschke, “The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space,” *Journal of Mathematical Analysis and Applications*, vol. 202, no. 1, pp. 150–159, 1996.

[19] M. Shang, Y. Su, and X. Qin, “Strong convergence theorems for a finite family of nonexpansive mappings,” *Fixed Point Theory and Applications*, vol. 2007, Article ID 76971, 9 pages, 2007.

[20] K. Shimoji and W. Takahashi, “Strong convergence to common fixed points of infinite nonexpansive mappings and applications,” *Taiwanese Journal of Mathematics*, vol. 5, no. 2, pp. 387–404, 2001.

[21] H. H. Bauschke and J. M. Borwein, “On projection algorithms for solving convex feasibility problems,” *SIAM Review*, vol. 38, no. 3, pp. 367–426, 1996.

[22] P. L. Combettes, “The foundations of set theoretic estimation,” *Proceedings of the IEEE*, vol. 81, no. 2, pp. 182–208, 1993.

[23] D.C. Youla, “Mathematical theory of image restoration by the method of convex projections,” in *Image Recovery: Theory and Application*, H. Star, Ed., pp. 29–77, Academic Press, Orlando, Fla, USA, 1987.

[24] A. N. Iusem and A. R. De Pierro, “On the convergence of Han’s method for convex programming with quadratic objective,” *Mathematical Programming Series B*, vol. 52, no. 1–3, pp. 265–284, 1991.

[25] Y. Su, M. Shang, and X. Qin, “A general iterative scheme for nonexpansive mappings and inverse-strongly monotone mappings,” *Journal of Applied Mathematics and Computing*, vol. 28, no. 1-2, pp. 283–294, 2008.

[26] R. T. Rockafellar, “On the maximality of sums of nonlinear monotone operators,” *Transactions of the American Mathematical Society*, vol. 149, no. 1, pp. 75–88, 1970.

[27] T. Suzuki, “Strong convergence of Krasnoselskii and Mann’s type sequences for one-parameter nonexpansive semigroups without Bochner integrals,” *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[28] H.-K. Xu, “Viscosity approximation methods for nonexpansive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[29] S. Takahashi and W. Takahashi, “Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces,” *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[30] S. S. Chang, *Variational Inequalities and Related Problems*, Chongqing Publishing House, Chongqing, China, 2007.

[31] G. L. Acedo and H.-K. Xu, “Iterative methods for strict pseudo-contractions in Hilbert spaces,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 7, pp. 2258–2271, 2007.