Hindawi Publishing Corporation Fixed Point Theory and Applications Volume 2009, Article ID 393245, 9 pages doi:10.1155/2009/393245

Review Article

T-Stability Approach to Variational Iteration Method for Solving Integral Equations

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Received 16 February 2009; Accepted 26 August 2009

Recommended by Nan-jing Huang

We consider *T*-stability definition according to Y. Qing and B. E. Rhoades (2008) and we show that the variational iteration method for solving integral equations is *T*-stable. Finally, we present some text examples to illustrate our result.

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1. Introduction and Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space and T a self-map of X. Let $x_{n+1} = f(T, x_n)$ be some iteration procedure. Suppose that F(T), the fixed point set of T, is nonempty and that x_n converges to a point $q \in F(T)$. Let $\{y_n\} \subseteq X$ and define $e_n = \|y_{n+1} - f(T, y_n)\|$. If $\lim e_n = 0$ implies that $\lim y_n = q$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T-stable. Without loss of generality, we may assume that $\{y_n\}$ is bounded, for if $\{y_n\}$ is not bounded, then it cannot possibly converge. If these conditions hold for $x_{n+1} = Tx_n$, that is, Picard's iteration, then we will say that Picard's iteration is T-stable.

Theorem 1.1 (see [1]). Let $(X, \|\cdot\|)$ be a Banach space and T a self-map of X satisfying

$$||Tx - Ty|| \le L||x - Tx|| + \alpha ||x - y|| \tag{1.1}$$

for all $x, y \in X$, where $L \ge 0$, $0 \le \alpha < 1$. Suppose that T has a fixed point p. Then, T is Picard T-stable.

Various kinds of analytical methods and numerical methods [2–10] were used to solve integral equations. To illustrate the basic idea of the method, we consider the general

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nonlinear system:

$$L[u(t)] + N[u(t)] = g(t),$$
 (1.2)

where L is a linear operator, N is a nonlinear operator, and g(t) is a given continuous function. The basic character of the method is to construct a functional for the system, which reads

$$u_{n+1}(x) = u_n(x) + \int_0^t \lambda(s) \left[Lu_n(s) + N\widetilde{u}_n(s) - g(s) \right] ds, \tag{1.3}$$

where λ is a Lagrange multiplier which can be identified optimally via variational theory, u_n is the nth approximate solution, and \tilde{u}_n denotes a restricted variation; that is, $\delta \tilde{u}_n = 0$.

Now, we consider the Fredholm integral equation of second kind in the general case, which reads

$$u(x) = f(x) + \lambda \int_{a}^{b} K(x,t)u(t)dt,$$
(1.4)

where K(x,t) is the kernel of the integral equation. There is a simple iteration formula for (1.4) in the form

$$u_{n+1}(x) = f(x) + \lambda \int_{a}^{b} K(x,t)u_n(t)dt.$$
 (1.5)

Now, we show that the nonlinear mapping *T*, defined by

$$u_{n+1}(x) = T(u_n(x)) = f(x) + \lambda \int_a^b K(x,t)u_n(t)dt,$$
 (1.6)

is *T*-stable in $L^2[a,b]$.

First, we show that the nonlinear mapping *T* has a fixed point. For $m, n \in \mathbb{N}$ we have

$$||T(u_{m}(x)) - T(u_{n}(x))|| = ||u_{m+1}(x) - u_{n+1}(x)||$$

$$= \left\| \lambda \int_{a}^{b} K(x,t)(u_{m}(t) - u_{n}(t))dt \right\|$$

$$\leq |\lambda| \left[\iint_{a}^{b} K^{2}(x,t)dxdt \right]^{1/2} ||u_{m}(x) - u_{n}(x)||.$$
(1.7)

Therefore, if

$$|\lambda| < \left[\iint_a^b K^2(x, t) dx dt \right]^{-1/2}, \tag{1.8}$$

then, the nonlinear mapping *T* has a fixed point.

Second, we show that the nonlinear mapping T satisfies (1.1). Let (1.6) hold. Putting L=0 and $\alpha=|\lambda|[\int \int_a^b K^2(x,t)dxdt]^{1/2}$ shows that (1.1) holds for the nonlinear mapping T. All of the conditions of Theorem 1.1 hold for the nonlinear mapping T and hence it is

All of the conditions of Theorem 1.1 hold for the nonlinear mapping *T* and hence it is *T*-stable. As a result, we can state the following theorem.

Theorem 1.2. Use the iteration scheme

$$u_0(x) = f(x),$$

$$u_{n+1}(x) = T(u_n(x)) = f(x) + \lambda \int_a^b K(x,t)u_n(t)dt,$$
(1.9)

for n = 0, 1, 2, ..., to construct a sequence of successive iterations $\{u_n(x)\}$ to the solution of (1.4). In addition, if

$$|\lambda| < \left[\iint_a^b K^2(x, t) dx dt \right]^{-1/2}, \tag{1.10}$$

L = 0 and $\alpha = |\lambda| \left[\int_a^b \int_a^b K^2(x,t) dx dt \right]^{1/2}$. Then the nonlinear mapping T, in the norm of $L^2(a,b)$, is T-stable.

Theorem 1.3 (see [11]). Use the iteration scheme

$$u_0(x) = f(x),$$

$$u_{n+1}(x) = f(x) + \lambda \int_a^b K(x,t) u_n(t) dt,$$
(1.11)

for n = 0, 1, 2, ..., to construct a sequence of successive iteration $\{u_n(x)\}$ to the solution of (1.4). In addition, let

$$\iint_{a}^{b} K^{2}(x,t) dx dt = B^{2} < \infty, \tag{1.12}$$

and assume that $f(x) \in L^2(a,b)$. Then, if $|\lambda| < 1/B$, the above iteration converges, in the norm of $L^2(a,b)$ to the solution of (1.4).

Corollary 1.4. *Consider the iteration scheme*

$$u_0(x) = f(x),$$

$$u_{n+1}(x) = T(u_n(x)) = f(x) + \lambda \int_a^b K(x,t)u_n(t)dt,$$
(1.13)

for $n = 0, 1, 2, \dots$ If

$$|\lambda| < \left[\iint_a^b K^2(x, t) dx dt \right]^{-1/2}, \tag{1.14}$$

L=0 and $\alpha=|\lambda|[\int_a^b\int_a^bK^2(x,t)dxdt]^{1/2}$, then stability of the nonlinear mapping T in the norm of $L^2(a,b)$ is a coefficient condition for the above iteration to converge in the norm of $L^2(a,b)$, and to the solution of (1.4).

2. Test Examples

In this section we present some test examples to show that the stability of the iteration method is a coefficient condition for the convergence in the norm of $L^2(a, b)$ to the solution of (1.4). In fact the stability interval is a subset of converges interval.

Example 2.1 (see [12]). Consider the integral equation

$$u(x) = \sqrt{x} + \lambda \int_{0}^{1} x t u(t) dt.$$
 (2.1)

The iteration formula reads

$$u_{n+1}(x) = \sqrt{x} + \lambda \int_0^1 x t u_n(t) dt, \qquad (2.2)$$

$$u_0(x) = \sqrt{x}. (2.3)$$

Substituting (2.3) into (2.2), we have the following results:

$$u_1(x) = \sqrt{x} + \lambda \int_0^1 xt\sqrt{t}dt = \sqrt{x} + \frac{2\lambda x}{5},$$

$$u_2(x) = \sqrt{x} + \lambda \int_0^1 xt \left[\sqrt{t} + \frac{2\lambda t}{5}\right]dt = \sqrt{x} + \left[\frac{2\lambda}{5} + \frac{2\lambda^2}{15}\right]x,$$

$$u_3(x) = \sqrt{x} + \lambda \int_0^1 xt \left[\sqrt{t} + \left(\frac{2\lambda}{5} + \frac{2\lambda^2}{15}\right)t\right]dt = \sqrt{x} + \left[\frac{2\lambda}{5} + \frac{2\lambda^2}{15} + \frac{2\lambda^3}{45}\right]x.$$

$$(2.4)$$

Continuing this way ad infinitum, we obtain

$$u_n(x) = \sqrt{x} + \left[\frac{2}{5 \cdot 3^0} \lambda + \frac{2}{5 \cdot 3^1} \lambda^2 + \frac{2}{5 \cdot 3^2} \lambda^3 + \cdots\right] x,$$
 (2.5)

then

$$u_n(x) = \sqrt{x} + \left(\frac{2}{5} \sum_{i=1}^n \frac{\lambda^i}{3^{i-1}}\right) x.$$
 (2.6)

The above sequence is convergent if $|\lambda|$ < 3, and the exact solution is

$$\lim_{n \to \infty} u_n(x) = \sqrt{x} + \frac{6\lambda}{5(3-\lambda)} x = u(x). \tag{2.7}$$

On the other hand we have

$$\left[\iint_{a}^{b} K^{2}(x,t)dxdt\right]^{1/2} = \left[\iint_{0}^{1} (xt)^{2} dxdt\right]^{1/2} = \frac{1}{3}.$$
 (2.8)

Then if $|\lambda|$ < 3 for mapping

$$u_{n+1}(x) = T(u_n(x)) = \sqrt{x} + \lambda \int_0^1 x t u_n(t) dt,$$
 (2.9)

we have

$$||T(u_{m}(x)) - T(u_{n}(x))|| = ||u_{m+1}(x) - u_{n+1}(x)||$$

$$= \left\| \lambda \int_{0}^{1} xt(u_{m}(t) - u_{n}(t)) dt \right\|$$

$$\leq |\lambda| \left[\iint_{0}^{1} (xt)^{2} dx dt \right]^{1/2} ||u_{m}(x) - u_{n}(x)||$$

$$\leq \frac{|\lambda|}{3} ||u_{m}(x) - u_{n}(x)||,$$
(2.10)

which implies that T has a fixed point. Also, putting L = 0 and $\alpha = |\lambda|/3$ shows that (1.1) holds for the nonlinear mapping T. All of the conditions of Theorem 1.1 hold for the nonlinear mapping T and hence it is T-stable.

Example 2.2 (see [12]). Consider the integral equation

$$u(x) = x + \lambda \int_{0}^{1} (1 - 3xt)u(t)dt,$$
 (2.11)

its iteration formula reads

$$u_{n+1}(x) = x + \lambda \int_0^1 (1 - 3xt) u_n(t) dt,$$

$$u_0(x) = x.$$
(2.12)

Then we have

is

$$u_n(x) = x + \sum_{j=1}^n \lambda^j \int_0^1 \cdots \int_0^1 (1 - 3xt_1)(1 - 3t_1t_2) \cdots (1 - 3t_{j-1}t_j)t_j dt_j \cdots dt_1.$$
 (2.13)

By (2.13), we have the following results:

$$u_{1}(x) = x + \lambda \int_{0}^{1} (1 - 3xt)t dt = (1 - \lambda)x + \frac{1}{2}\lambda,$$

$$u_{2}(x) = x + \lambda \int_{0}^{1} (1 - 3xt) \left[(1 - \lambda)t + \frac{1}{2}\lambda \right] dt$$

$$= (1 - \lambda)x + \frac{1}{2}\lambda + \frac{\lambda^{2}}{4}x,$$

$$u_{3}(x) = x + \lambda \int_{0}^{1} (1 - 3xt) \left[(1 - \lambda)t + \frac{1}{2}\lambda + \frac{\lambda^{2}}{4}t \right] dt$$

$$= (1 - \lambda)x + \frac{\lambda^{2}}{4}(1 - \lambda)x + \frac{1}{2}\lambda + \frac{\lambda^{3}}{8}.$$
(2.14)

Continuing this way ad infinitum, we obtain

$$u_n(x) = \sum_{j=0}^n \frac{3(-1)^j - 1}{2} \left(\frac{\lambda}{2}\right)^j x + \left(\frac{1 + (-1)^{j+1}}{2}\right) \left(\frac{\lambda}{2}\right)^j.$$
 (2.15)

The above sequence is convergent if $|\lambda/2| < 1$, that is, $-2 < \lambda < 2$ and the exact solution

$$\lim_{n \to \infty} u_n(x) = \frac{2\lambda}{4 - \lambda^2} + \frac{4(1 - \lambda)}{4 - \lambda^2} x = u(x). \tag{2.16}$$

On the other hand we have

$$\left[\iint_{a}^{b} K^{2}(x,t)dxdt\right]^{1/2} = \left[\iint_{0}^{1} (1-3xt)^{2} dxdt\right]^{1/2} = \frac{1}{\sqrt{2}}.$$
 (2.17)

Then if $|\lambda| < \sqrt{2}$, for mapping

$$u_{n+1}(x) = T(u_n(x)) = x + \lambda \int_0^1 (1 - 3xt) u_n(t) dt,$$
 (2.18)

we have

$$||T(u_{m}(x)) - T(u_{n}(x))|| = ||u_{m+1}(x) - u_{n+1}(x)||$$

$$= \left\| \lambda \int_{0}^{1} xt(u_{m}(t) - u_{n}(t))dt \right\|$$

$$\leq |\lambda| \left[\iint_{0}^{1} (1 - 3xt)^{2} dx dt \right]^{1/2} ||u_{m}(x) - u_{n}(x)||$$

$$\leq \frac{|\lambda|}{\sqrt{2}} ||u_{m}(x) - u_{n}(x)||,$$
(2.19)

which implies that T has a fixed point. Also, putting L = 0 and $\alpha = |\lambda|/\sqrt{2}$ shows that (1.1) holds for the nonlinear mapping T. All of conditions of Theorem 1.1 hold for the nonlinear mapping T and hence it is T-stable.

Example 2.3. Consider the integral equation

$$u(x) = \sin ax + \lambda \frac{a}{2} \int_0^{\pi/2a} \cos(ax) u(t) dt,$$
 (2.20)

its iteration formula reads

$$u_{n+1}(x) = \sin ax + \lambda \frac{a}{2} \int_{0}^{\pi/2a} \cos(ax) u_n(t) dt,$$
 (2.21)

$$u_0(x) = \sin ax. \tag{2.22}$$

Substituting (2.22) into (2.21), we have the following results:

$$u_{1}(x) = \sin ax + \lambda \frac{a}{2} \int_{0}^{\pi/2a} \cos(ax) \sin(at) dt = \sin(ax) + \frac{\lambda}{2} \cos(ax),$$

$$u_{2}(x) = \sin(ax) + \lambda \frac{a}{2} \int_{0}^{\pi/2a} \cos(ax) \left[\sin(at) + \frac{\lambda}{2} \cos(at) \right] dt$$

$$= \sin(ax) + \cos(ax) \left[\frac{\lambda}{2} + \frac{\lambda^{2}}{4} \right],$$

$$u_{3}(x) = \sin(ax) + \lambda \frac{a}{2} \int_{0}^{\pi/2a} \cos(ax) \left[\sin(at) + \left[\frac{\lambda}{2} + \frac{\lambda^{2}}{4} \right] \cos(at) \right] dt$$

$$= \sin(ax) + \cos(ax) \left[\frac{\lambda}{2} + \frac{\lambda^{2}}{4} + \frac{\lambda^{3}}{8} \right].$$

$$(2.23)$$

Continuing this way ad infinitum, we obtain

$$u_n(x) = \sin(ax) + \cos(ax) \sum_{i=1}^{\infty} \left(\frac{\lambda}{2}\right)^i.$$
 (2.24)

The above sequence is convergent if $|\lambda/2| < 1$; that is, $-2 < \lambda < 2$, and the exact solution is

$$\lim_{n \to \infty} u_n(x) = \sin(ax) + \frac{\lambda}{2 - \lambda} \cos(ax) = u(x). \tag{2.25}$$

On the other hand we have

$$\left[\iint_{a}^{b} K^{2}(x,t) dx dt \right]^{1/2} = \left[\iint_{0}^{\pi/2a} \left(\frac{a}{2} \cos(ax) \right)^{2} dx dt \right]^{1/2} = \sqrt{\frac{\pi^{2}}{32}}.$$
 (2.26)

Then if $|\lambda| < 1/\sqrt{\pi^2/32} \cong 1.800$, for mapping

$$u_{n+1}(x) = T(u_n(x)) = x + \lambda \frac{a}{2} \int_0^{\pi/2a} \cos(ax) u_n(t) dt,$$
 (2.27)

we have

$$||T(u_{m}(x)) - T(u_{n}(x))|| = ||u_{m+1}(x) - u_{n+1}(x)||$$

$$= \left\| \lambda \int_{0}^{1} xt(u_{m}(t) - u_{n}(t))dt \right\|$$

$$\leq |\lambda| \left[\iint_{0}^{\pi/2a} \left(\frac{a}{2} \cos(ax) \right)^{2} dx dt \right]^{1/2} ||u_{m}(x) - u_{n}(x)||$$

$$\leq |\lambda| \sqrt{\frac{\pi^{2}}{32}} ||u_{m}(x) - u_{n}(x)||,$$
(2.28)

which implies that T has a fixed point. Also, putting L=0 and $\alpha=|\lambda|\sqrt{\pi^2/32}$ shows that (1.1) holds for the nonlinear mapping T. All of the conditions of Theorem 1.1 hold for the nonlinear mapping T and hence it is T-stable.

Acknowledgments

The authors would like to thank referees and area editor Professor Nan-jing Huang for giving useful comments and suggestions for the improvement of this paper. This paper is dedicated to Professor Mehdi Dehghan

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