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Research Article

The Methods of Hilbert Spaces and Structure of the Fixed-Point Set of Lipschitzian Mapping

Jarosław Górnicki

Department of Mathematics, Rzeszów University of Technology, P.O. Box 85, 35-595 Rzeszów, Poland

Correspondence should be addressed to Jarosław Górnicki, gornicki@prz.edu.pl

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The purpose of this paper is to prove, by asymptotic center techniques and the methods of Hilbert spaces, the following theorem. Let H be a Hilbert space, let C be a nonempty bounded closed convex subset of H, and let $M = [a_{n,k}]_{n,k\geq 1}$ be a strongly ergodic matrix. If $T : C \to C$ is a lipschitzian mapping such that $\liminf_{n\to\infty}\inf_{m=0,1,\ldots}\sum_{k=1}^{\infty} a_{n,k}\cdot \|T^{k+m}\|^2 < 2$, then the set of fixed points $Fix\,T = \{x \in C : Tx = x\}$ is a retract of C. This result extends and improves the corresponding results of [7, Corollary 9] and [8, Corollary 1].

1. Introduction

Let E be a Banach space and let C be a nonempty bounded closed convex subset of E. We say that a mapping $T : C \to C$ is nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\| \quad \text{for every } x, y \in C. \tag{1.1}$$

The result of Bruck [1] asserts that if a nonexpansive mapping $T : C \to C$ has a fixed point in every nonempty closed convex subset of C which is invariant under T and if C is convex and weakly compact, then Fix $T = \{x \in C : Tx = x\}$, the set of fixed points, is nonexpansive retract of C (i.e., there exists a nonexpansive mapping $R : C \to$ Fix T such that $R_{|\text{Fix } T} = I$). A few years ago, the Bruck results were extended by T. Domínguez Benavides and Lorenzo Ramírez [2] to the case of asymptotically nonexpansive mappings if the space E was sufficiently regular.

On the other hand it is known that, the set of fixed points of k-lipschitzian mapping can be very irregular for any $k > 1$.

Example 1.1 (Goebel [3, 4]). Let F be a nonempty closed subset of C. Fix $z \in F$, $0 < \varepsilon < 1$ and put

$$Tx = x + \varepsilon \cdot \operatorname{dist}(x, F) \cdot (z - x), \quad x \in C. \tag{1.2}$$

It is not difficult to see that $\operatorname{Fix} T = F$ and the Lipschitz constant of T tends to 1 if $\varepsilon \downarrow 0$.

For more information on the structure of fixed point sets see [4, 5] and references therein.

In 1973, Goebel and Kirk [3] introduced the class of uniformly k-lipschitzian mappings, recall that a mapping $T : C \to C$ is *uniformly k-lipschitzian*, $k \geqslant 1$, if

$$\|T^n x - T^n y\| \leqslant k\|x - y\| \quad \text{for every } x, y \in C, n \in \mathbb{N}, \tag{1.3}$$

and proved the following theorem.

Theorem 1.2. *Let E be a uniformly convex Banach space with modulus of convexity δ_E and let C be a nonempty bounded closed convex subset of E. Suppose that $T : C \to C$ is uniformly k-lipschitzian and*

$$k\left(1 - \delta_E\left(\frac{1}{k}\right)\right) < 1. \tag{1.4}$$

Then T has a fixed point in C. (Note that in a Hilbert space, $k < 1/2\sqrt{5}$.)

Recently Sędłak and Wiśnicki [6] proved that *under the assumptions of Theorem 1.2,* $\operatorname{Fix} T$ *is not only connected but even a retract of* C, and next the author proved the following theorem [7, Corollary 9].

Theorem 1.3. *Let H be a Hilbert space, C a nonempty bounded closed convex subset of H, and $T : C \to C$ a uniformly k-lipschitzian mapping with $k < \sqrt{2}$. Then T has a fixed point in C and* $\operatorname{Fix} T$ *is a retract of C.*

In this paper we shall continue this work. Precisely, by means of techniques of asymptotic centers and the methods of Hilbert spaces, we establish some result on the structure of fixed point sets for mappings with lipschitzian iterates in a Hilbert space. The class of mappings with lipschitzian iterates is importantly greater than the class of uniformly lipschitzian mappings; see [8, Example 1].

2. Asymptotic Center

Denote by $\|T\|$ the Lipschitz norm of T:

$$\|T\| = \sup\left\{\frac{\|Tx - Ty\|}{\|x - y\|} : x, y \in C, x \neq y\right\}. \tag{2.1}$$

Lifshitz [9] significantly extended Goebel and Kirk's result and found an example of a fixed point free uniformly $\pi/2$–lipschitzian mapping which leaves invariant a bounded closed convex subset of l^2. The validity of Lifshitz's Theorem in a Hilbert space for $\sqrt{2} \leqslant k < \pi/2$ remains open.

A more general approach was proposed by the present author using the methods of Hilbert spaces, asymptotic techniques, and strongly ergodic matrix. We recall that a matrix $M = [a_{n,k}]_{n,k\geqslant 1}$ is called *strongly ergodic* if

(i) for all n, k $a_{n,k} \geqslant 0$,

(ii) for all k $\lim_{n\to\infty} a_{n,k} = 0$,

(iii) for all n $\sum_{k=1}^{\infty} a_{n,k} = 1$,

(iv) $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k+1} - a_{n,k}| = 0$.

Then we have the following theorem.

Theorem 2.1 (see [8]). *Let C be a nonempty bounded closed convex subset of a Hilbert space and let $M = [a_{n,k}]_{n,k\geqslant 1}$ be a strongly ergodic matrix. If $T : C \to C$ is a mapping such that*

$$g = \liminf_{n\to\infty} \inf_{m=0,1,\ldots} \sum_{k=1}^{\infty} a_{n,k} \cdot \left\| T^{k+m} \right\|^2 < 2, \tag{2.2}$$

then T has a fixed point in C.

This result generalizes Lifshitz's Theorem (in case of a Hilbert space) and shows that the theorem admits certain perturbations in the behavior of the norm of successive iterations in infinite sets; see [8, Example 1].

Let E be a Banach space. Recall that *the modulus of convexity* δ_E is the function δ_E : $[0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{ 1 - \frac{1}{2}\|x+y\| : \|x\| \leqslant 1, \|y\| \leqslant 1, \|x-y\| \geqslant \varepsilon \right\} \tag{2.3}$$

and *uniform convexity* means $\delta_E(\varepsilon) > 0$ for $\varepsilon > 0$. A Hilbert space H is uniformly convex. This fact is a direct consequence of parallelogram identity.

Now we prove some version of Sędłak and Wiśnicki's result [6, Lemma 2.1]. Let C be a nonempty bounded closed convex subset of a real Hilbert space H, let $M = [a_{n,k}]_{n,k\geqslant 1}$ be a strongly ergodix matrix, and let $T : C \to C$ be a mapping such that $\|T^k\| \geqslant 1$ for all $k = 1, 2, \ldots$, and

$$\limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \cdot \left\| T^k \right\|^2 = B < \infty. \tag{2.4}$$

Let $x, y \in C$ we use

$$r\left(y, \left\{T^k x\right\}\right) = \limsup_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k} \cdot \left\| y - T^k x \right\|^2,$$
$$r\left(C, \left\{T^k x\right\}\right) = \inf_{y \in C} r\left(y, \left\{T^k x\right\}\right) \tag{2.5}$$

to denote the asymptotic radius of $\{T^k x\}$ at y and the asymptotic radius of $\{T^k x\}$ in C, respectively. It is well known in a Hilbert space [8] that the asymptotic center of $\{T^k x\}$ in C:

$$A\left(C, \left\{T^k x\right\}\right) = \left\{y \in C : r\left(y, \left\{T^k x\right\}\right) = r\left(C, \left\{T^k x\right\}\right)\right\} \tag{2.6}$$

is a singleton.

Let $A : C \to C$ denote a mapping which associates with a given $x \in C$ a unique $z \in A(C, \{T^k x\})$, that is, $z = Ax$. The following Lemma is a crucial tool to prove Theorem 4.1.

Lemma 2.2. *Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Then the mapping $A : C \to C$ is continuous.*

Proof. On the contrary, suppose that there exists $x_0 \in C$ and $\varepsilon_0 > 0$ such that for all $\eta > 0$ there exists $x_1 \in C$ such that $\|x_1 - x_0\| < \eta$ and $\|z_1 - z_0\| \geqslant \varepsilon_0$, where $\{z_0\} = A(C, \{T^k x_0\}), \{z_1\} = A(C, \{T^k x_1\})$.

Fix $\eta > 0$ and take $x_1 \in C$ such that

$$\|x_1 - x_0\| < \eta, \qquad \|z_1 - z_0\| \geqslant \varepsilon_0. \tag{2.7}$$

Let $R_0 = r(C, \{T^k x_0\})$, $R_1 = r(C, \{T^k x_1\})$ and $R = \overline{\lim}_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k} \cdot \|z_1 - T^k x_0\|^2$. Notice that

$$R_0 < R. \tag{2.8}$$

Choose $\varepsilon > 0$. Then

$$\begin{cases} \left\|z_1 - T^k x_0\right\| & < \sqrt{R + \varepsilon}, \\ \left\|z_0 - T^k x_0\right\| & < \sqrt{R_0 + \varepsilon} < \sqrt{R + \varepsilon}, \\ \|z_0 - z_1\| & \geq \varepsilon_0 \end{cases} \tag{2.9}$$

for all but finitely many k.

If, for example, $\|z_1 - T^k x_0\| \geqslant \sqrt{R + \varepsilon}$ for all everyone k, then

$$\left\|z_1 - T^k x_0\right\|^2 \geqslant R + \varepsilon. \tag{2.10}$$

Multiplying both sides of this inequality (for fixed k) by suitable element of the matrix M and summing up such obtained inequalities for $k \geqslant 1$, we have for $n = 1, 2, \ldots,$

$$\sum_{k=1}^{\infty} a_{n,k} \cdot \left\| z_1 - T^k x_0 \right\|^2 \geqslant R + \varepsilon. \tag{2.11}$$

Taking the limit superior as $n \to \infty$ on each side, we get

$$R = \limsup_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k} \cdot \left\| z_1 - T^k x_0 \right\|^2 \geqslant R + \varepsilon > R, \tag{2.12}$$

which is contradiction.

It follows by (2.9) and the properties of δ_H that

$$\begin{aligned} \left\| T^k x_0 - \frac{z_1 + z_0}{2} \right\| &\leqslant \left(1 - \delta_H \left(\frac{\varepsilon_0}{\sqrt{R + \varepsilon}} \right) \right) \sqrt{R + \varepsilon}, \\ \left\| T^k x_0 - \frac{z_1 + z_0}{2} \right\|^2 &\leqslant \left(1 - \delta_H \left(\frac{\varepsilon_0}{\sqrt{R + \varepsilon}} \right) \right)^2 (R + \varepsilon). \end{aligned} \tag{2.13}$$

Multiplying both sides of this inequality by suitable elements of the matrix M and summing up such obtained inequalities for $k \geqslant 1$, taking the limit superior as $n \to \infty$ on each side, we get

$$\begin{aligned} R_0 &< \limsup_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k} \cdot \left\| T^k x_0 - \frac{z_1 + z_0}{2} \right\|^2 \\ &\leqslant \left(1 - \delta_H \left(\frac{\varepsilon_0}{\sqrt{R + \varepsilon}} \right) \right)^2 (R + \varepsilon). \end{aligned} \tag{2.14}$$

Moreover,

$$\begin{aligned} \left\| T^k x_0 - z_1 \right\|^2 &\leqslant \left(\left\| T^k x_0 - T^k x_1 \right\| + \left\| T^k x_1 - z_1 \right\| \right)^2 \\ &\leqslant \left\| T^k x_0 - T^k x_1 \right\|^2 + 2 \left\| T^k x_0 - T^k x_1 \right\| \cdot \left\| T^k x_1 - z_1 \right\| + \left\| T^k x_1 - z_1 \right\|^2 \\ &\leqslant \left\| T^k \right\|^2 \cdot \| x_0 - x_1 \|^2 + 2 \left\| T^k \right\| \cdot \| x_0 - x_1 \| \cdot \left\| T^k x_1 - z_1 \right\| + \left\| T^k x_1 - z_1 \right\|^2 \\ &\leqslant \left(\left\| T^k \right\|^2 + 2 \left\| T^k \right\| \right) \cdot \| x_0 - x_1 \| \cdot \operatorname{diam} C + \left\| T^k x_1 - z_1 \right\|^2 \\ &\leqslant 3 \left\| T^k \right\|^2 \cdot \operatorname{diam} C \cdot \| x_0 - x_1 \| + \left\| T^k x_1 - z_1 \right\|^2. \end{aligned} \tag{2.15}$$

Multiplying both sides of this inequality by suitable elements of the matrix M and summing up such obtained inequalities for $k \geqslant 1$, taking the limit superior as $n \to \infty$ on each side, we get

$$\begin{aligned} R &= \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \cdot \left\| T^k x_0 - z_1 \right\|^2 \\ &\leqslant 3 \cdot \operatorname{diam} C \cdot \|x_0 - x_1\| \cdot \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \cdot \left\| T^k \right\|^2 + \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \cdot \left\| T^k x_1 - z_1 \right\|^2 \\ &\leqslant 3 \cdot B \cdot \operatorname{diam} C \cdot \eta + R_1 + \varepsilon. \end{aligned} \tag{2.16}$$

Similarly,

$$\begin{aligned} R_1 &< \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \cdot \left\| T^k x_1 - z_0 \right\|^2 \\ &\leqslant 3 \cdot \operatorname{diam} C \cdot \|x_1 - x_0\| \cdot \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \cdot \left\| T^k \right\|^2 + \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \cdot \left\| T^k x_0 - z_0 \right\|^2 \\ &\leqslant 3 \cdot B \cdot \operatorname{diam} C \cdot \eta + R_0 + \varepsilon. \end{aligned} \tag{2.17}$$

From (2.16) and (2.17), we have

$$R \leqslant 3 \cdot B \cdot \operatorname{diam} C \cdot \eta + R_1 + \varepsilon < 6 \cdot B \cdot \operatorname{diam} C \cdot \eta + 2 \cdot \varepsilon + R_0. \tag{2.18}$$

If $R_0 = 0$, then from (2.18) it follows $R = 0$. This is contradiction with (2.8). If $R_0 > 0$, then combining (2.18) with (2.14) and applying the monotonicity of δ_H, we obtain

$$R_0 < \left(1 - \delta_H\left(\frac{\varepsilon_0}{\sqrt{6 \cdot B \cdot \operatorname{diam} C \cdot \eta + 3 \cdot \varepsilon + R_0}}\right)\right)^2 (6 \cdot B \cdot \operatorname{diam} C \cdot \eta + 3 \cdot \varepsilon + R_0). \tag{2.19}$$

Letting $\eta,\ \varepsilon \downarrow 0$ and using the continuity of δ_H, we conclude that

$$1 \leqslant \left(1 - \delta_H\left(\frac{\varepsilon_0}{\sqrt{R_0}}\right)\right)^2 < 1. \tag{2.20}$$

This contradiction proves the continuity of mapping A. □

3. The Methods of Hilbert Spaces

Let M, T be as above. We define functionals

$$d(u) = \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \cdot \left\| u - T^k u \right\|^2,$$
$$r(x) = \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \cdot \left\| x - T^k u \right\|^2, \tag{3.1}$$

where $u, x \in C$. Let z in C be an asymptotic center of $\{T^k u\}_{k\geqslant 1}$ with respect to $r(\cdot)$ and C, which minimizes the functional $r(x)$ over x in C (for fix $u \in C$).

Lemma 3.1. *One has* $r(z) \leqslant d(u)$.

Proof. It is consequence of the above definitions. □

Lemma 3.2. *One has* $\|z - u\| \leqslant 2\sqrt{d(u)}$.

Proof. For any $k \in \mathbb{N}$, we have

$$\begin{aligned}\|z-u\|^2 &= 2\left(\left\| z - T^k u \right\|^2 + \left\| T^k u - u \right\|^2 \right) - \left\| z + u - 2T^k u \right\|^2 \\ &\leqslant 2\left\| z - T^k u \right\|^2 + 2\left\| T^k u - u \right\|^2.\end{aligned} \tag{3.2}$$

Multiplying both sides of this inequality by suitable elements of the matrix M and summing up such obtained inequalities for $k \geqslant 1$, taking the limit superior as $n \to \infty$ on each side, we get

$$\begin{aligned}\|z-u\|^2 &\leqslant 2\limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \cdot \left\| z - T^k u \right\|^2 + 2\limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \left\| T^k u - u \right\|^2 \\ &= 2(r(z) + d(u)) \leqslant 4d(u).\end{aligned} \tag{3.3}$$

□

Lemma 3.3. *One has* $r(T^s z) \leqslant \|T^s\|^2 \cdot r(z)$ *for all* $s \in \mathbb{N}$.

Proof. Fix $s \in \mathbb{N}$, then we have

$$\begin{aligned}&\sum_{k=1}^{\infty} a_{n,k} \left\| T^s z - T^k u \right\|^2 \\ &\quad\leqslant \sum_{k=1}^{s} a_{n,k} \left\| T^s z - T^k u \right\|^2 + \|T^s\|^2 \cdot \sum_{k=s+1}^{\infty} a_{n,k} \left\| z - T^{k-s} u \right\|^2 \\ &\quad= \sum_{k=1}^{s} a_{n,k} \left\| T^s z - T^k u \right\|^2 + \|T^s\|^2 \cdot \left(\sum_{k=1}^{\infty} a_{n,k} \left\| z - T^{k-s} u \right\|^2 - \sum_{k=1}^{s} a_{n,k} \left\| z - T^{k-s} u \right\|^2 \right).\end{aligned} \tag{3.4}$$

Since the matrix M is strongly ergodic,

$$\sum_{k=1}^{s} a_{n,k}\left\|T^{s}z - T^{k}u\right\|^{2} \longrightarrow 0,$$
$$\sum_{k=1}^{s} a_{n,k}\left\|z - T^{k-s}u\right\|^{2} \longrightarrow 0, \tag{3.5}$$

as $n \to \infty$, we get thesis. □

Lemma 3.4. *One has $r(z) + \|z - x\|^2 \leqslant r(x)$ for every $x \in C$.*

Proof. For $x \in C$ and $0 < t < 1$, we have

$$\left\|tx + (1-t)z - T^{k}u\right\|^{2} = t\left\|x - T^{k}u\right\|^{2} + (1-t)\left\|z - T^{k}u\right\|^{2} - t(1-t)\|x - z\|^{2}. \tag{3.6}$$

Multiplying both sides of this inequality by suitable elements of the matrix M and summing up such obtained inequalities for $k \geqslant 1$, taking the limit superior as $n \to \infty$ on each side, we get

$$\begin{aligned}
&\limsup_{n\to\infty}\sum_{k-1}^{\infty} a_{n,k}\left\|tx + (1-t)z - T^{k}u\right\|^{2} \\
&\quad = t\cdot\limsup_{n\to\infty}\sum_{k=1}^{\infty} a_{n,k}\left\|x - T^{k}u\right\|^{2} + (1-t)\cdot\limsup_{n\to\infty}\sum_{k=1}^{\infty} a_{n,k}\left\|z - T^{k}u\right\|^{2} - t(1-t)\|x - z\|^{2}.
\end{aligned} \tag{3.7}$$

Since $r(z) \leqslant r(tx + (1-t)z)$, we obtain

$$r(z) \leqslant t\cdot r(x) + (1-t)\cdot r(z) - t(1-t)\|x - z\|^{2},$$
$$r(z) \leqslant r(x) - (1-t)\|x - z\|^{2}. \tag{3.8}$$

Taking $t \downarrow 0_{+}$, we get, $r(z) + \|z - x\|^2 \leqslant r(x)$. □

4. Main Result

We are now in position to prove our main result.

Theorem 4.1. *Let C be a nonempty bounded closed convex subset of a Hilbert space and let $M = [a_{n,k}]_{n,k\geqslant 1}$ be a strongly ergodic matrix. If $T : C \to C$ is a mapping such that*

$$g = \liminf_{n\to\infty}\inf_{m=0,1,\dots}\sum_{k=1}^{\infty} a_{n,k}\cdot\left\|T^{k+m}\right\|^{2} < 2, \tag{4.1}$$

then Fix $T = \{x \in C : Tx = x\}$ *is a retract of C.*

Proof. Let $\{n_i\}$ and $\{m_i\}$ be sequences of natural numbers such that

$$g = \lim_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \left\| T^{k+m_i} \right\|^2 < 2. \tag{4.2}$$

By Theorem 2.1, Fix $T \neq \emptyset$. For any $x \in C$ we can inductively define a sequence $\{z_j\}$ in the following manner: z_1 is the unique point in C that minimizes the functional

$$\limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \left\| y - T^{k+m_i} x \right\|^2 \tag{4.3}$$

over $y \in C$, and z_{j+1} is the unique point in C that minimizes the functional

$$\limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \left\| y - T^{k+m_i} z_j \right\|^2 \tag{4.4}$$

over $y \in C$, that is, $z_j = A^j x$, $j = 1, 2, \ldots$. First we prove the following inequality:

$$\widehat{d}(z) \leqslant (g-1)\widehat{d}(u), \tag{4.5}$$

where

$$\widehat{d}(u) = \limsup_{i \to \infty} \sum_{k=1}^{\infty} \left\| u - T^{k+m_i} u \right\|^2, \tag{4.6}$$

and z is the asymptotic center in C which minimizes the functional

$$\widehat{r}(x) = \limsup_{i \to \infty} \sum_{k=1}^{\infty} \left\| x - T^{k+m_i} u \right\|^2 \tag{4.7}$$

over x in C.

In fact, we put in Lemma 3.4 $x = T^p z$. Then by Lemma 3.3, we get

$$\begin{gathered} \widehat{r}(z) + \|z - T^p z\|^2 \leqslant \widehat{r}(T^p z) \leqslant \|T^p\|^2 \cdot \widehat{r}(z), \\ \|z - T^p z\|^2 \leqslant \left(\|T^p\|^2 - 1 \right) \cdot \widehat{r}(z). \end{gathered} \tag{4.8}$$

For $p = m + k_i$ we have

$$\left\| z - T^{k+m_i} z \right\|^2 \leqslant \left(\left\| T^{k+m_i} \right\|^2 - 1 \right) \cdot \widehat{r}(z), \tag{4.9}$$

and hence

$$\begin{aligned}
\widehat{d}(z) &= \limsup_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \left\| z - T^{k+m_i} z \right\|^2 \\
&\leqslant \left(\lim_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \left\| T^{k+m_i} \right\|^2 - 1 \right) \cdot \widehat{r}(z) \\
&= (g-1) \cdot \widehat{r}(z) \quad (\text{by Lemma 3.1}) \\
&\leqslant (g-1) \cdot \widehat{d}(u).
\end{aligned} \tag{4.10}$$

Next by Lemma 3.2 and inequality (4.5), we have

$$\left\| z_{j+1} - z_j \right\| = \left\| A^{j+1} x - A^j x \right\| \leqslant 2\sqrt{(g-1)^j d(x)} \leqslant 2 \cdot \alpha^j \cdot \sqrt{\operatorname{diam} C}, \tag{4.11}$$

where $\alpha = \sqrt{g-1} < 1$ for $x \in C$, $j = 1, 2, \ldots$. Thus

$$\sup_{x\in C} \left\| A^p x - A^j x \right\| \leqslant \frac{\alpha^j}{1-\alpha} \cdot 2 \cdot \sqrt{\operatorname{diam} C} \longrightarrow 0 \quad \text{if } p, j \longrightarrow \infty, \tag{4.12}$$

which implies that the sequence $\{A^j x\}$ converges uniformly to a function

$$Rx = \lim_{j\to\infty} A^j x, \qquad x \in C. \tag{4.13}$$

It follows from Lemma 2.2 that $R : C \to C$ is continuous. Moreover,

$$\begin{aligned}
\left\| Rx - T^{k+m_i} Rx \right\|^2 &= 2\left(\left\| Rx - A^j x \right\|^2 + \left\| A^j x - T^{k+m_i} Rx \right\|^2 \right) - \left\| Rx + T^{k+m_i} Rx - 2A^j x \right\|^2 \\
&\leqslant 2\left\| Rx - A^j x \right\|^2 + 2\left\| A^j x - T^{k+m_i} Rx \right\|^2 \\
&\leqslant 2\left\| Rx - A^j x \right\|^2 + 2\left(2\left\| A^j x - T^{k+m_i} A^j x \right\|^2 + 2\left\| T^{k+m_i} A^j x - T^{k+m_i} Rx \right\|^2 \right) \\
&\leqslant \left(2 + 4\left\| T^{k+m_i} \right\|^2 \right) \cdot \left\| Rx - A^j x \right\|^2 + 4\left\| A^j x - T^{k+m_i} A^j x \right\|^2.
\end{aligned} \tag{4.14}$$

Multiplying both sides of this inequalities by suitable elements of the matrix M and summing up such obtained inequalities for $k \geqslant 1$, taking the limit superior as $i \to \infty$ on each side, we get

$$\begin{aligned}\widehat{d}(Rx) &= \limsup_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \left\| Rx - T^{k+m_i} Rx \right\|^2 \\ &\leqslant \left(2 + 4 \lim_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \left\| T^{k+m_i} \right\|^2 \right) \left\| Rx - A^j x \right\|^2 \\ &\quad + 4 \limsup_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \left\| A^j x - T^{k+m_i} A^j x \right\|^2 \\ &= (2+4g) \left\| Rx - A^j x \right\|^2 + 4\widehat{d}(A^j x) \quad (\text{by } (4.5)) \\ &\leqslant (2+4g) \left\| Rx - A^j x \right\|^2 + 4(g-1)^j \cdot \widehat{d}(x) \to 0 \quad \text{if } j \to \infty. \end{aligned} \tag{4.15}$$

Thus, $\widehat{d}(Rx) = 0$. This implies that $Rx = TRx$; see [8] for details. Thus $Rx = TRx$ for every $x \in C$ and R is a retraction of C onto Fix T. □

If $M = [a_{n,k}]_{n,k\geqslant 1}$ is the Cesaro matrix, that is, for $n = 1, 2, \ldots,$

$$a_{n,k} = \begin{cases} \dfrac{1}{n} & \text{for } k = 1, 2, \ldots, n, \\ 0 & \text{for } k \geqslant n+1, \end{cases} \tag{4.16}$$

then we have the following corollary.

Corollary 4.2. *Let C be a nonempty bounded closed convex subset of a Hilbert space. If $T : C \to C$ is a mapping such that*

$$g = \liminf_{n\to\infty} \inf_{m=0,1,\ldots} \frac{1}{n} \sum_{k=1}^{n} \left\| T^{k+m} \right\|^2 < 2, \tag{4.17}$$

then Fix $T = \{x \in C : Tx = x\}$ *is a retract of C.*

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