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Research Article

Fixed Points and Stability for Functional Equations in Probabilistic Metric and Random Normed Spaces

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We prove a general Ulam-Hyers stability theorem for a nonlinear equation in probabilistic metric spaces, which is then used to obtain stability properties for different kinds of functional equations (linear functional equations, generalized equation of the square root, spiral generalized gamma equations) in random normed spaces. As direct and natural consequences of our results, we obtain general stability properties for the corresponding functional equations in (deterministic) metric and normed spaces.

1. Introduction

Hyers ([1], 1941) has given an affirmative answer to a question of Ulam by proving the stability of additive Cauchy equations in Banach spaces. Then Aoki, Bourgin, Rassias, Forti and Gajda considered the stability problem with unbounded Cauchy differences. Their results include the following theorem.

Theorem 1.1 (see Hyers [1], Aoki [2], and Gajda [3]). *Suppose that E is a real normed space, F is a real Banach space and $f : E \to F$ is a given function, such that the following condition holds:*

$$\left\| f(x+y) - f(x) - f(y) \right\|_F \le \theta\left(\|x\|_E^p + \|y\|_E^p \right), \quad \forall x, y \in E, \tag{1.1}$$

for some $p \in [0, \infty) \setminus \{1\}$. Then there exists a unique additive function $a : E \to F$ such that

$$\left\| f(x) - a(x) \right\|_F \le \frac{2\theta}{|2 - 2^p|} \|x\|_E^p, \quad \forall x \in E, \tag{1.2}$$

This phenomenon is called *generalized Ulam-Hyers stability* and has been extensively investigated for different functional equations. Almost all proofs used the idea conceived by Hyers. Namely, the additive function $a : E \to F$ is constructed, starting from the given function f, by the following formulae:

$$a(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x), \quad \text{if } p < 1,$$
$$a(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right), \quad \text{if } p > 1. \tag{1.3}$$

This method is called *the direct method* or *Hyers method*. It is often used to construct a solution of a given functional equation and is seen to be a powerful tool for studying the stability of many functional equations (see [4–6] for details). It is worth noting that in 1978 Rassias [7] proved that the additive mapping a, obtained by Hyers or Aoki, is linear if, in addition, for each $x \in E$ the mapping $f(tx)$ is continous in $t \in \mathbb{R}$. Subsequently, the above results were extended by replacing the control mapping $\theta(\|x\|_E^p + \|y\|_E^p)$ in (1.1) by functions φ with suitable properties (see [8–10]).

On the other hand, in [11–13] a *fixed point method* was proposed, by showing that many theorems concerning the stability of Cauchy and Jensen equations are consequences of the fixed point alternative. The method has also been used in [14–16] or [17].

Our aim is to highlight generalized Ulam-Hyers stability results for functional equations defined by mappings with values in probabilistic metric spaces and in random normed spaces, obtained by using the fixed point alternative, which is now recalled, for the sake of convenience (cf. [18], see also [19, chapter 5]):

Theorem 1.2. *Suppose that one is given a complete generalized metric space $(\mathcal{E}, d)$-that is, one for which the metric d may assume infinite values—and a strictly contractive mapping $A : \mathcal{E} \to \mathcal{E}$, with the Lipschitz constant L. Then, for each given element $f \in \mathcal{E}$, either*

(A_1) $d(A^n f, A^{n+1} f) = +\infty$,*for all* $n \geq 0$, *or;*

(A_2) *there exists a natural number* n_0 *such that;*

(A_{20}) $d(A^n f, A^{n+1} f) < +\infty$, *for all* $n \geq n_0$;

(A_{21}) *the sequence* $(A^n f)$ *is convergent to a fixed point* f^* *of* A;

(A_{22}) f^* *is the unique fixed point of* A *in the set* $\mathcal{E}^\star = \{g \in \mathcal{E}, d(A^{n_0} f, g) < +\infty\}$;

(A_{23}) $d(g, f^*) \leq (1/(1-L)) d(g, Ag)$, *for all* $g \in \mathcal{E}^\star$.

Remark 1.3. The fixed points of A, if any, need not be uniquely determined *in the whole space* $\mathcal{E}$ and do depend on the initial guess f.

We recall (see, e.g., Schweizer and Sklar [20]) that a *distance distribution function F* is a mapping from $(-\infty, \infty)$ into $[0, 1]$, nondecreasing, and left-continuous, with $F(0) = 0$. The class of all distribution functions F, with $\lim_{x\to\infty} F(x) = 1$, is denoted by D_+. For any

$$a \in [0, \infty), \quad \varepsilon_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a \end{cases} \tag{1.4}$$

is an element of D_+. For any $\lambda > 0$ and $F \in D_+$, the distribution function $\lambda \circ F$ is defined by

$$(\lambda \circ F)(t) := F\left(\frac{t}{\lambda}\right), \quad \text{for all } t > 0. \tag{1.5}$$

Let us consider X a real vector space, $\mathcal{F}$ a mapping from X into D_+ (for any x in X, $\mathcal{F}(x)$ is denoted by F_x) and T a t-norm. The triple $(X, \mathcal{F}, T)$ is called *a random normed space* (briefly RN-space) iff the following conditions are satisfied:

(RN-1): $F_x = \varepsilon_0$ if and only if $x = \theta$, the null vector;

(RN-2): $F_{ax} = |a| \circ F_x, \forall t > 0, \forall a \in \mathbb{R}, a \neq 0$;

(RN-3): $F_{x+y}(t_1 + t_2) \geq T(F_x(t_1), F_y(t_2)), \forall x, y \in X$ and $t_1, t_2 > 0$.

Notice that a *triangular norm (t-norm)* is a mapping $T : [0,1] \times [0,1] \to [0,1]$, which is associative, commutative and increasing in each variable, with $T(a,1) = 1$, for all $a \in [0,1]$. The most important t-norms are $T_M(a,b) = \min\{a,b\}$, $\mathrm{Prod}(a,b) = a \cdot b$, $T_1(a,b) = \max\{a + b - 1, 0\}$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space $(X, \mathcal{F}, T_M)$, with $F_x \in D_+$ and

$$F_x(t) = \frac{t}{t + \|x\|}, \quad \forall t > 0. \tag{1.6}$$

If the t-norm T is such that $\sup_{0<a<1} T(a,a) = 1$, then every random normed space $(X, \mathcal{F}, T)$ is a *metrizable linear topological space* with the (ε, λ)-topology, induced by the base of neighborhoods $U(\varepsilon, \lambda) = \{x \in X, F_x(\varepsilon) > 1 - \lambda\}$. If $T = T_M$, then $(X, \mathcal{F}, T)$ is locally convex ([20, Theorems 12.1.6 and 15.1.2]).

A sequence (x_n) in a random normed space $(X, \mathcal{F}, T)$ *converges* to $x \in X$ in the (ε, λ)-topology if $\lim_{n\to\infty} F_{x_n - x}(t) = 1$, for all $t > 0$. A sequence (x_n) is called *Cauchy sequence* if $\lim_{m,n\to\infty} F_{x_n - x_m}(t) = 1$, for all $t > 0$. The random normed space $(X, \mathcal{F}, T)$ is *complete* if every Cauchy sequence in X is convergent.

It is worth notice that a random norm induces a probabilistic metric by the formula $F_{uv}(t) := F_{u-v}(t), \forall t$ (see, e.g., [20, Theorem 15.1.2]). For more details on probabilistic metric spaces and random normed spaces, see (Schweizer and Sklar [20, Chapters 8, 12 and 15]), Radu [21] or Chang et al. [22].

In the present paper we prove a very general Ulam-Hyers stability theorem for a nonlinear equation in probabilistic metric spaces, which is then used to obtain stability properties for different kinds of functional equations (linear functional equations, generalized equation of the square root spiral, generalized gamma equations) in random normed spaces. As direct and natural consequences of our results, we obtain general stability properties for the corresponding functional equations in (deterministic) metric and normed spaces.

2. The Generalized Ulam-Hyers Stability of a Nonlinear Equation

The Ulam-Hyers stability for the nonlinear equation

$$f(x) = E(x, f(\eta(x))) \tag{2.1}$$

was discussed by Baker [23]. Recently, we extended his result in [15], by proving the generalized stability in Ulam-Hyers sense for (2.1), by means of the fixed point alternative. In [24] the stability of additive Cauchy equation in random normed spaces is proved.

In the next theorem we prove a generalized Ulam-Hyers stability result for the nonlinear equation (2.1), where the unknown f is mapping a nonempty set X into a *complete probabilistic metric space* $(Y, \mathcal{F}, T_M)$.

Theorem 2.1. *Let X be a nonempty set and let $(Y, \mathcal{F}, T_M)$ be a complete probabilistic metric space. Let us consider the mappings $\eta : X \to X$, $g : X \to \mathbb{R} \setminus \{0\}$ and $E : X \times Y \to Y$. Suppose that*

$$F_{E(x,u)E(x,v)} \geq |g(x)| \circ F_{uv}, \quad \forall x \in X,\ \forall u, v \in Y. \tag{2.2}$$

If $f : X \to Y$ satisfies

$$F_{f(x)E(x,f(\eta(x)))} \geq \psi_x, \quad \forall x \in X, \tag{$\mathbf{C}_\psi$}$$

where $\psi : X \to D_+$ is a mapping for which there exists $L < 1$ such that

$$|g(x)| \circ \psi_{\eta(x)}(Lt) \geq \psi_x(t), \quad \forall x \in X,\ \forall t > 0, \tag{$\mathbf{H}_\psi$}$$

then there exists a unique mapping $c : X \to Y$ which satisfies both the equation

$$c(x) = E(x, c(\eta(x))), \quad \forall x \in X, \tag{2.3}$$

and the estimation

$$F_{c(x)f(x)}(t) \geq \psi_x((1-L)t), \quad \forall x \in X, \tag{$\mathbf{Est}_\psi$}$$

for almost all $t > 0$. Moreover, the solution mapping c has the form

$$c(x) = \lim_{n \to \infty} F(x, F(F(\eta(x), \ldots, F(\eta(x), (f \circ \eta^n)(\eta(x)))))), \quad \forall x \in X. \tag{2.4}$$

Proof. Let us consider the mapping $G(x,t) := \psi_x(t)$, and the set $\mathcal{E} := \{h : X \to Y, h(0) = 0\}$. We introduce a generalized metric on $\mathcal{E}$ (as usual, $\inf \emptyset = \infty$):

$$d(h_1, h_2) = d_G(h_1, h_2) = \inf\{K \in \mathbb{R}_+, F_{h_1(x)h_2(x)}(Kt) \geq G(x,t),\ \forall x \in X,\ \forall t > 0\}. \tag{2.5}$$

The proof of the fact that $(\mathcal{E}, d_G)$ is a *complete generalized metric space* can be found e.g. in (Radu [25], Hadžić et al. [26] or Miheţ and Radu [27]).

Now, define the operator

$$J : \mathcal{E} \longrightarrow \mathcal{E}, \qquad Jh(x) := E(x, h(\eta(x))). \tag{$\mathbf{OP}$}$$

Step 1. Using our hypotheses, it follows that J *is strictly contractive* on $\mathcal{E}$. Indeed, for any $h_1, h_2 \in \mathcal{E}$ we have:

$$d(h_1, h_2) < K \Longrightarrow F_{h_1(x)h_2(x)}(Kt) \geq G(x,t) = \varphi_x(t), \quad \forall x \in X,\ \forall t > 0,$$

$$\begin{aligned} F_{Jh_1(x)Jh_2(x)}(LKt) &= F_{E(x,h_1(\eta(x)))E(x,h_2(\eta(x)))}(LKt) \geq F_{h_1(\eta(x))h_2(\eta(x))}\left(\frac{LKt}{|g(x)|}\right) \\ &\geq \varphi_{\eta(x)}\left(\frac{Lt}{|g(x)|}\right) \geq \varphi_x(t). \end{aligned} \tag{2.6}$$

Therefore $d(Jh_1, Jh_2) \leq LK$, which implies

$$d(Jh_1, Jh_2) \leq Ld(h_1, h_2), \quad \forall g, h \in \mathcal{E}. \tag{$\mathbf{CC_L}$}$$

This says that J is a *strictly contractive* self-mapping of $\mathcal{E}$, with the Lipschitz constant $L < 1$.

Step 2. Obviously, $d(f, Jf) < \infty$. In fact, using the relation (1.6) it results that $d(f, Jf) \leq 1$.

Step 3. Now we can apply the fixed point alternative to obtain the existence of a mapping $c : X \to Y$ such that,

(i) c is a fixed point of J, that is

$$c(x) = E(x, c(\eta(x))), \quad \forall x \in X. \tag{2.7}$$

The mapping c is the unique fixed point of J in the set

$$\mathcal{F} = \{h \in \mathcal{E},\ d_G(f, h) < \infty\}. \tag{2.8}$$

This says that c is the unique mapping verifying *both* the above equation (2.7) and the next condition:

$$\exists K < \infty \quad \text{such that } F_{c(x)-f(x)}(Kt) \geq \varphi(x)(t), \quad \forall x \in X,\ \forall t > 0. \tag{2.9}$$

Moreover,

(ii) $d(J^n f, c) \underset{n\to\infty}{\longrightarrow} 0$, which implies $\lim_{n\to\infty} F_{c(x)J^n f(x)}(t) = 1, \forall t > 0$ *and* $\forall x \in X$, whence

$$c(x) = \lim_{n\to\infty} F(x, F(F(\eta(x), \ldots, F(\eta(x), (f \circ \eta^n)(\eta(x)))))), \quad \forall x \in X. \tag{2.10}$$

(iii) $d(f,c) \le (1/(1-L))d(f,Jf)$, which implies the inequality

$$d(f,c) \le \frac{1}{1-L}, \tag{2.11}$$

hence

$$F_{c(x)f(x)}\left(\frac{t}{1-L}\right) \ge \varphi_x(t), \quad \forall x \in X, \tag{2.12}$$

for almost all $t > 0$. It results that

$$F_{c(x)f(x)}(t) \ge \varphi_x((1-L)t), \quad \forall x \in X, \tag{2.13}$$

for almost all $t > 0$, that is (2.3) is seen to be true. □

As a direct consequence of Theorem 2.1, the following Ulam-Hyers stability result for the nonlinear equation (2.1) is obtained.

Corollary 2.2. *Let X be a nonempty set and let $(Y, \mathcal{F}, T_M)$ be a complete probabilistic metric space. Consider the mappings $\eta : X \to X$, $E : X \times Y \to Y$ and $0 \le L < 1$.*

Suppose that

$$F_{E(x,u)E(x,v)}(Lt) \ge F_{uv}(t), \quad \forall x \in X,\ \forall u, v \in Y,\ \forall t > 0. \tag{2.14}$$

If $f : X \to Y$ satisfies

$$F_{f(x)E(x,f(\eta(x)))} \ge \varepsilon_\delta, \quad \forall x \in X, \tag{2.15}$$

where

$$\varepsilon_\delta(t) = \begin{cases} 0, & \text{if } t \le \delta, \\ 1, & \text{if } t > \delta \end{cases} \tag{2.16}$$

(an element of D_+ for $\delta > 0$), then there exists a unique mapping $c : X \to Y$ which satisfies both the equation

$$c(x) = E(x, c(\eta(x))), \quad \forall x \in X, \tag{2.17}$$

and the estimation

$$F_{c(x)f(x)}(t) \ge \varepsilon_\delta((1-L)t), \quad \forall x \in X, \tag{2.18}$$

for almost all $t > 0$. Moreover, the solution mapping c has the form

$$c(x) = \lim_{n\to\infty} F\big(x, F\big(F\big(\eta(x), \dots, F\big(\eta(x), (f \circ \eta^n)(\eta(x))\big)\big)\big)\big), \quad \forall x \in X. \tag{2.19}$$

Proof. It follows from Theorem 2.1, by choosing $\psi_x(t) := \varepsilon_\delta(t), \forall t > 0$, where $\delta > 0$ is fixed. □

2.1. A Consequence for Metric Spaces

By using Theorem 2.1, we can immediately obtain the following generalized stability result of Ulam-Hyers type for the nonlinear equation

$$f(x) = F(x, f(\eta(x))) \tag{2.20}$$

in *complete metric spaces*, firstly proved by us in ([15, Theorem 4.1]).

Corollary 2.3. *Consider a nonempty set X, a complete metric space (Y, d) and the mappings $\eta : X \to X$, $g : X \to \mathbb{R} \setminus \{0\}$ and $F : X \times Y \to Y$. Suppose that*

$$d(F(x,u), F(x,v)) \le |g(x)| \cdot d(u,v), \quad \forall x \in X,\ \forall u, v \in Y. \tag{2.21}$$

If $f : X \to Y$ satisfies

$$d\big(f(x), F\big(x, f(\eta(x))\big)\big) \le \varphi(x), \quad \forall x \in X, \tag{2.22}$$

with a mapping $\varphi : X \to [0,\infty)$ for which there exists $L < 1$ such that

$$|g(x)| (\varphi \circ \eta)(x) \le L\varphi(x), \quad \forall x \in X, \tag{2.23}$$

then there exists a unique mapping $c : X \to Y$ which satisfies both the equation

$$c(x) = F\big(x, c(\eta(x))\big), \quad \forall x \in X \tag{2.24}$$

and the estimation

$$d\big(f(x), c(x)\big) \le \frac{\varphi(x)}{1-L}, \quad \forall x \in X. \tag{2.25}$$

Moreover,

$$c(x) = \lim_{n\to\infty} F\big(x, F\big(F\big(\eta(x), \dots, F\big(\eta(x), (f \circ \eta^n)(\eta(x))\big)\big)\big)\big), \quad \forall x \in X. \tag{2.26}$$

Proof. Let us consider the probabilistic metric $Y \times Y \ni (u,v) \to F_{uv} \in D_+$, where

$$F_{uv}(t) = 1 - \exp\left(-\frac{t}{d(u,v)}\right), \quad \forall t > 0, \tag{2.27}$$

and the mapping

$$\psi : X \to D_+, \qquad \psi_x(t) = 1 - \exp\left(-\frac{t}{\varphi(x)}\right). \tag{2.28}$$

Then Y is a complete probabilistic metric space under $T_M = \text{Min}$ (see [20, Theorem 8.4.2]). The hypothesis relations (2.21), (2.22) and (2.23) are equivalent to (2.2), (1.6) and ($\mathbf{C}_\psi$), respectively. Therefore we can apply our Theorem 2.1 to obtain the above result. □

Remark 2.4. Corollary 2.3 can also be obtained by using the probabilistic metric

$$Y \times Y \ni (u, v) \longrightarrow F_{uv} \in D_+, \qquad F_{uv}(t) = \frac{t}{t + d(u, v)}, \quad \forall t > 0, \tag{2.29}$$

and the mapping

$$\psi : X \longrightarrow D_+, \qquad \psi_x(t) = \frac{t}{t + \varphi(x)}, \quad \forall t > 0. \tag{2.30}$$

Again by using Theorem 2.1, the following Ulam-Hyers stability result (cf. (see [23, Theorem 2]) or ([28, Theorem 13]) for the nonlinear equation (2.1) in *complete metric spaces* can be proved.

Corollary 2.5. *Let X be a nonempty set and (Y, d) be a complete metric space. Let $\eta : X \to X$, $F : X \times Y \to Y$ and $0 \leq L < 1$. Suppose that*

$$d(F(x, u), F(x, v)) \leq L \cdot d(u, v), \quad \forall x \in X, \ \forall u, v \in Y. \tag{2.31}$$

If $f : X \to Y$ satisfies

$$d(f(x), F(x, f(\eta(x)))) \leq \delta, \quad \forall x \in X, \tag{2.32}$$

with a fixed constant $\delta > 0$, then there exists a unique mapping $c : X \to Y$ which satisfies both the equation

$$c(x) = F(x, c(\eta(x))), \quad \forall x \in X \tag{2.33}$$

and the estimation

$$d(f(x), c(x)) \leq \frac{\delta}{1 - L}, \quad \forall x \in X. \tag{2.34}$$

Moreover,

$$c(x) = \lim_{n \to \infty} F(x, F(F(\eta(x), \ldots, F(\eta(x), (f \circ \eta^n)(\eta(x)))))), \quad \forall x \in X. \tag{2.35}$$

Proof. It follows, by choosing in Theorem 2.1 the probabilistic metric

$$Y \times Y \ni (u,v) \longrightarrow F_{uv} \in D_+, \qquad F_{uv}(t) = \frac{t}{t + d(u,v)}, \quad \forall t > 0, \tag{2.36}$$

and the mapping

$$\psi : X \longrightarrow D_+, \qquad \psi_x(t) = \frac{t}{t+\delta}, \quad \forall t > 0. \tag{2.37}$$

□

Remark 2.6. The above result can also be obtained by Corollary 2.2 and the probabilistic metric

$$Y \times Y \ni (u,v) \longrightarrow F_{uv} \in D_+, \qquad F_{uv}(t) = \varepsilon_1\left(\frac{t}{d(u,v)}\right), \quad \forall t > 0,\ d(u,v) > 0. \tag{2.38}$$

3. The Generalized Ulam-Hyers Stability of a Linear Functional Equation

If we consider

$$E(x, f(\eta(x))) = g(x) \cdot f(\eta(x)) + h(x), \tag{3.1}$$

then (2.1) becomes

$$f(x) = g(x) \cdot f(\eta(x)) + h(x), \tag{3.2}$$

where g, η, h are given mappings and f is an unknown function. The above equation is called *linear functional equation*. A lot of results concerning monotonic solutions, regular solutions and convex solutions of (3.2) were given by Kuczma et al. [29].

In this section we prove a generalized Ulam-Hyers stability result for the linear functional equation (3.2), as a particular case of Theorem 2.1. The unknown is a mapping f from the nonempty set X into a *complete random normed space* $(Y, \mathcal{F}, T_M)$.

Theorem 3.1. *Let X be a nonempty set and let $(Y, \mathcal{F}, T_M)$ be a complete random normed space. Suppose that $\eta : X \to X$, $g : X \to \mathbb{R} \setminus \{0\}$. If $f : X \to Y$ satisfies*

$$F_{g(x)\cdot f(\eta(x)) + h(x) - f(x)} \geq \psi_x, \quad \forall x \in X, \tag{3.3}$$

where $\psi : X \to D_+$ is a mapping for which there exists $L < 1$ such that

$$|g(x)| \circ \psi_{\eta(x)}(Lt) \geq \psi_x(t), \quad \forall x \in X,\ \forall t > 0, \tag{3.4}$$

then there exists a unique mapping $c : X \to Y$,

$$c(x) = h(x) + \lim_{n\to\infty}\left(f(\eta^n(x)) \cdot \prod_{i=0}^{n-1} g\left(\eta^i(x)\right) + \sum_{j=0}^{n-2}\left(h\left(\eta^{j+1}(x)\right) \cdot \prod_{i=0}^{j} g\left(\eta^i(x)\right)\right)\right), \quad (3.5)$$

for all $x \in X$, *which satisfies both the equation*

$$c(x) = g(x) \cdot c(\eta(x)) + h(x), \quad \forall x \in X, \quad (3.6)$$

and the estimation

$$F_{c(x)-f(x)}(t) \geq \psi_x((1-L)t), \quad \forall x \in X, \quad (3.7)$$

for almost all $t > 0$.

Proof. We consider in Theorem 2.1 the probabilistic metric F_{uv} on $Y \times Y$, induced by the random norm, namely, $F_{uv}(t) := F_{u-v}(t), \forall t > 0$ and the function $E(x, f(\eta(x))) := g(x)f(\eta(x)) + h(x), \forall x \in X$, with g, η, h as in hypothesis of Theorem 3.1. The relation (2.2) holds with equality. Applying Theorem 2.1, there exists a unique mapping c which satisfies the equation (3.2) and the estimation (3.7). Moreover,

$$c(x) = \lim_{n\to\infty} J^n f(x), \quad \forall x \in X, \quad (3.8)$$

where

$$\begin{aligned}(J^n f)(x) &= g(x) \cdot \left(J^{n-1}f\right)(\eta(x)) + h(x) \\ &= g(x) \cdot g(\eta(x)) \cdot \left(J^{n-2}f\right)\left(\eta^2(x)\right) + g(x) \cdot h(\eta(x)) + h(x), \quad \forall x \in X,\end{aligned} \quad (3.9)$$

whence, for all $x \in X$,

$$J^n f(x) := h(x) + f(\eta^n(x)) \cdot \prod_{i=0}^{n-1} g\left(\eta^i(x)\right) + \sum_{j=0}^{n-2}\left(h\left(\eta^{j+1}(x)\right) \cdot \prod_{i=0}^{j} g\left(\eta^i(x)\right)\right). \quad (3.10)$$

□

As in Section 2.1, we can obtain Ulam-Hyers stability result for (3.2) in complete random normed space, by taking $\psi_x(t) := \varepsilon_\delta(t), \forall t > 0$ in Theorem 3.1.

3.1. A Consequence for Normed Spaces

As a particular case of Theorem 3.1, we can prove immediately the generalized stability result of Ulam-Hyers type for the linear functional equation

$$f(x) = g(x) \cdot f(\eta(x)) + h(x), \quad (3.11)$$

in Banach spaces, obtained by us in ([15, Theorem 5.1]), by using the fixed point alternative.

Corollary 3.2. *Consider X a nonempty set and Y a real (or complex) Banach space. Suppose that $\eta : X \to X$, $g : X \to \mathbb{R} \setminus \{0\}$. If $f : X \to Y$ satisfies*

$$\|f(x) - g(x)f(\eta(x)) - h(x)\|_Y \le \varphi(x), \quad \forall x \in X, \tag{3.12}$$

with some fixed mapping $\varphi : X \to [0,\infty)$ and there exists $L < 1$ such that

$$|g(x)|(\varphi \circ \eta)(x) \le L\varphi(x), \quad \forall x \in X, \tag{3.13}$$

then there exists a unique mapping $c : X \to Y$

$$c(x) = h(x) + \lim_{n\to\infty}\left(f(\eta^n(x)) \cdot \prod_{i=0}^{n-1} g\left(\eta^i(x)\right) + \sum_{j=0}^{n-2}\left(h\left(\eta^{j+1}(x)\right) \cdot \prod_{i=0}^{j} g\left(\eta^i(x)\right)\right)\right), \tag{3.14}$$

for all $x \in X$, which satisfies both the equation

$$c(x) = g(x) \cdot c(\eta(x)) + h(x), \quad \forall x \in X \tag{3.15}$$

and the estimation

$$\|f(x) - c(x)\|_Y \le \frac{\varphi(x)}{1-L}, \quad \forall x \in X. \tag{3.16}$$

Proof. Let us consider the random norm $Y \ni u \to F_u \in D_+$, where

$$F_u(t) = 1 - \exp\left(-\frac{t}{\|u\|}\right), \quad \forall t > 0, \tag{3.17}$$

and the mapping

$$\psi : X \to D_+, \qquad \psi_x(t) = 1 - \exp\left(-\frac{t}{\varphi(x)}\right). \tag{3.18}$$

Then Y is a complete random normed space under $T_M = \mathrm{Min}$. The hypothesis relations (3.12), (3.13) and (3.16) are equivalent to (3.3), (3.4) and (3.7), respectively. Therefore we can apply our Theorem 3.1 to obtain the above result. □

Remark 3.3. Corollary 3.2 can also be obtained by using the random norm

$$Y \ni u \to F_u \in D_+, \qquad F_u(t) = \frac{t}{t + \|u\|}, \quad \forall t > 0, \tag{3.19}$$

and the mapping

$$\psi : X \to D_+, \qquad \psi_x(t) = \frac{t}{t + \varphi(x)}. \tag{3.20}$$

4. Applications to the Generalized Equation of the Square Root Spiral

Wang et al. proved in [30], by the direct method, a generalized Ulam-Hyers stability result for the *generalized equation of the square root spiral*

$$f\left(p^{-1}(p(x) + k)\right) = f(x) + h_1(x), \tag{4.1}$$

in Banach spaces.

We consider this equation for mappings f from an Abelian semigroup X into a *complete random normed space* $(Y, \mathcal{F}, T_M)$, with $p : X \to X$ and $h_1 : X \to Y$ some given functions (p^{-1} is the inverse of p and k is a fixed constant, which is not an identity element in X).

We can prove, as a consequence of Theorem 3.1, the following generalized Ulam-Hyers stability result for (4.1):

Theorem 4.1. *Let X be an Abelian semigroup and let $(Y, \mathcal{F}, T_M)$ be a complete random normed space. If $f : X \to Y$ satisfies*

$$F_{f(p^{-1}(p(x)+k)) - h_1(x) - f(x)} \geq \psi_x, \quad \forall x \in X, \tag{4.2}$$

where $\psi : X \to D_+$ is a mapping for which there exists $L < 1$ such that

$$\psi_{p^{-1}(p(x)+k)}(Lt) \geq \psi_x(t), \quad \forall x \in X,\ \forall t > 0, \tag{4.3}$$

then there exists a unique mapping $c : X \to Y$,

$$c(x) = \lim_{n \to \infty} \left(f\left(p^{-1}(p(x) + nk)\right) - \sum_{i=0}^{n-1} h_1\left(p^{-1}(p(x) + ik)\right) \right), \quad \forall x \in X, \tag{4.4}$$

which satisfies both the equation

$$c\left(p^{-1}(p(x) + k)\right) = c(x) + h_1(x), \quad \forall x \in X, \tag{4.5}$$

and the estimation

$$F_{c(x) - f(x)}(t) \geq \psi_x((1 - L)t), \quad \forall x \in X, \tag{4.6}$$

for almost all $t > 0$.

Proof. It is easy to see that (4.1) is a particular case of (3.2). In fact, we can consider $g \equiv 1$, $h := -h_1$, $\eta(x) := p^{-1}(p(x)+k), \forall x \in X$, with p bijective on X and k a fixed constant, which is not an identity element in X. By using the above notations, Theorem 4.1 can be obtained as a consequence of Theorem 3.1, with

$$\begin{aligned} c(x) &= h(x) + \lim_{n\to\infty}\left(f(\eta^n(x)) \cdot \prod_{i=0}^{n-1} g\left(\eta^i(x)\right) + \sum_{j=0}^{n-2}\left(h\left(\eta^{j+1}(x)\right) \cdot \prod_{i=0}^{j} g\left(\eta^i(x)\right)\right)\right) \\ &= -h_1(x) + \lim_{n\to\infty}\left(f\left(p^{-1}(p(x)+nk)\right) - \sum_{i=1}^{n-1} h_1\left(p^{-1}(p(x)+ik)\right)\right) \\ &= \lim_{n\to\infty}\left(f\left(p^{-1}(p(x)+nk)\right) - \sum_{i=0}^{n-1} h_1\left(p^{-1}(p(x)+ik)\right)\right), \quad \forall x \in X. \end{aligned} \tag{4.7}$$

□

As a consequence of Theorem 4.1, we can prove the generalized stability result of Ulam-Hyers type for the generalized equation of the square root spiral (4.1) in Banach spaces, obtained by us in ([15, Theorem 3.1]), by using the fixed point alternative.

Corollary 4.2. *Let X be an Abelian semigroup, let Y be a Banach space and let $p : X \to X$ and $h_1 : X \to Y$ some given functions, with p bijective. If k is a fixed constant, which is not an identity element in X and $f : X \to Y$ satisfies*

$$\left\| f(p^{-1}(p(x)+k)) - f(x) - h_1(x)\right\|_Y \le \psi(x), \quad \forall x \in X \tag{4.8}$$

with a mapping $\psi : X \to [0,\infty)$ for which there exists $L < 1$ such that

$$\psi\left(p^{-1}(p(x)+k)\right)(x) \le L\psi(x), \quad \forall x \in X, \tag{4.9}$$

then there exists a unique mapping $c : X \to Y$ which satisfies both the equation

$$c\left(p^{-1}(p(x)+k)\right) = c(x) + h_1(x), \quad \forall x \in X \tag{4.10}$$

and the estimation

$$\| f(x) - c(x)\|_Y \le \frac{\psi(x)}{1-L}, \quad \forall x \in X. \tag{4.11}$$

Moreover,

$$c(x) = \lim_{n\to\infty}\left(f\left(p^{-1}(p(x)+nk)\right) - \sum_{i=0}^{n-1} h_1\left(p^{-1}(p(x)+ik)\right)\right), \quad \forall x \in X. \tag{4.12}$$

The proof is similar to that of the Corollary 3.2, and therefore will be omitted.

A *special case* of (4.1) is obtained on $(0,\infty)$, for $k = 1$, $p(x) = x^n$, $n \geq 2$ and $h_1(x) = \arctan(1/x)$. It is the so-called "nth root spiral equation"

$$f\left(\sqrt[n]{x^n+1}\right) = f(x) + \arctan\frac{1}{x}. \tag{4.13}$$

If we take in Theorem 4.1 $X = \mathbb{R}_+ = (0,\infty)$ and $Y = \mathbb{R}$, we obtain the following generalized stability result for the functional equation (4.13):

Corollary 4.3. *Let* $(\mathbb{R}, \mathcal{F}, T_M)$ *be a complete random normed space. If* $f : \mathbb{R}_+ \to \mathbb{R}$ *satisfies*

$$F_{f(\sqrt[n]{x^n+1})-\arctan(1/x)-f(x)} \geq \psi(x), \quad \forall x \in \mathbb{R}_+, \tag{4.14}$$

where $\psi : \mathbb{R}_+ \to [0,\infty)$ *is a mapping for which there exists* $L < 1$ *such that*

$$\psi_{\sqrt[n]{x^n+1}}(Lt) \geq \psi_x(t), \quad \forall x \in \mathbb{R}_+,\ \forall t > 0, \tag{4.15}$$

then there exists a unique mapping $c : \mathbb{R}_+ \to \mathbb{R}$,

$$c(x) = \lim_{m\to\infty}\left(\sqrt[n]{x^n+m} - \sum_{i=0}^{m-1}\arctan\frac{1}{\sqrt[n]{x^n+i}}\right), \quad \forall x \in \mathbb{R}_+, \tag{4.16}$$

which satisfies both (4.13) *and the estimation*

$$F_{c(x)-f(x)}(t) \geq \psi_x((1-L)t), \quad \forall x \in \mathbb{R}_+, \tag{4.17}$$

for almost all $t > 0$.

Moreover, as in Corollary 3.2, it is obtained a generalized Ulam-Hyers stability result for (4.13), in $\mathbb{R}$:

Corollary 4.4 (see [15, Theorem 3.2]). *If* $f : \mathbb{R}_+ \to \mathbb{R}$ *satisfies*

$$\left| f\left(\sqrt[n]{x^n+1}\right) - f(x) - \arctan\frac{1}{x}\right| \leq \varphi(x), \quad \forall x \in \mathbb{R}_+, \tag{4.18}$$

with some fixed mapping $\varphi : \mathbb{R}_+ \to [0,\infty)$ *and there exists* $L < 1$ *such that*

$$\varphi\left(\sqrt[n]{x^n+1}\right) \leq L\varphi(x), \quad \forall x \in \mathbb{R}_+, \tag{4.19}$$

then there exists a unique mapping $c : \mathbb{R}_+ \to \mathbb{R}$,

$$c(x) = \lim_{m\to\infty}\left(\sqrt[n]{x^n+m} - \sum_{i=0}^{m-1}\arctan\frac{1}{\sqrt[n]{x^n+i}}\right), \quad \forall x \in \mathbb{R}_+, \tag{4.20}$$

which satisfies both (4.13) *and the estimation*

$$\left|f(x) - c(x)\right| \leq \frac{\varphi(x)}{1-L}, \quad \forall x \in \mathbb{R}_+. \tag{4.21}$$

5. Applications to the Generalized Gamma Functional Equation

As a consequence of Theorem 3.1, we obtain the following slight extension of the result in (Kim [31]) for the *generalized gamma functional equation*.

$$f(x+k) = \widetilde{g}(x) f(x). \tag{5.1}$$

Theorem 5.1. *Let X be an Abelian group and let $(Y, \mathcal{F}, T_M)$ be a complete random normed space. Suppose that $\widetilde{g} : X \to \mathbb{R} \setminus \{0\}$ and k is a fixed constant in X, different from the identity element. If $f : X \to Y$ satisfies*

$$F_{f(x+k)-\widetilde{g}(x)\cdot f(x)} \geq \psi_{x+k}, \quad \forall x \in X, \tag{5.2}$$

where $\psi : X \to D_+$ is a mapping for which there exists $L < 1$ such that

$$\left|\widetilde{g}(x)\right| \circ \psi_x(Lt) \geq \psi_{x+k}(t), \quad \forall x \in X, \forall t > 0, \tag{5.3}$$

then there exists a unique mapping $c : X \to Y$,

$$c(x) = \lim_{n\to\infty} f(x-nk) \cdot \prod_{i=1}^{n} \widetilde{g}(x-ik), \tag{5.4}$$

for all $x \in X$, which satisfies both the equation

$$c(x+k) = \widetilde{g}(x) \cdot c(x), \quad \forall x \in X, \tag{5.5}$$

and the estimation

$$F_{c(x)-f(x)}(t) \geq \psi_x((1-L)t), \quad \forall x \in X, \tag{5.6}$$

for almost all $t > 0$.

Proof. It is easy to see that (5.1) is a particular case of the equation (3.2). In fact, we can consider $h \equiv 0$, $g(x) := \widetilde{g}(x-k)$, $\eta(x) := x-k, \forall x \in X$, with a fixed constant k in X, different from the identity element. By using the above notations, Theorem 5.1 can be obtained as a consequence of Theorem 3.1. □

Another generalized Ulam-Hyers stability result for (5.1) can be obtained by using our Theorem 3.1:

Theorem 5.2. *Let X be an Abelian semigroup and let $(Y, \mathcal{F}, T_M)$ be a complete random normed space. Suppose that $\widehat{g} : X \to \mathbb{R} \setminus \{0\}$ and k is a fixed constant, which is not an identity element in X. If $f : X \to Y$ satisfies*

$$F_{f(x+k)-\widehat{g}(x)\cdot f(x)} \geq \left|\widehat{g}(x)\right| \circ \psi_x, \quad \forall x \in X, \tag{5.7}$$

where $\psi : X \to D_+$ is a mapping for which there exists $L < 1$ such that

$$\psi_{x+k}(Lt) \geq \left|\widehat{g}(x)\right| \circ \psi_x(t), \quad \forall x \in X, \ \forall t > 0, \tag{5.8}$$

then there exists a unique mapping $c : X \to Y$,

$$c(x) = \lim_{n\to\infty}\left(f(x+nk) \cdot \prod_{i=0}^{n-1} \frac{1}{\widehat{g}(x+ik)}\right), \tag{5.9}$$

for all $x \in X$, which satisfies both the equation

$$c(x+k) = \widehat{g}(x) \cdot c(x), \quad \forall x \in X, \tag{5.10}$$

and the estimation

$$F_{c(x)-f(x)}(t) \geq \psi_x((1-L)t), \quad \forall x \in X, \tag{5.11}$$

for almost all $t > 0$.

Proof. Obviously, (5.1) is a particular case of the equation (3.2). In fact, we can consider $h \equiv 0$, $g(x) = 1/\widehat{g}(x)$, $\eta(x) := x + k, \forall x \in X$, with a fixed constant k in X, different from the identity element. With the above notations, Theorem 5.2 can be obtained by using Theorem 3.1. □

As a particular case of (5.1) we have, for $k = 1$ and $\widehat{g}(x) := x$, the *gamma functional equation*

$$f(x+1) = xf(x), \quad \forall x \in (0,\infty). \tag{5.12}$$

If we consider in Theorem 5.2: $X = (0,\infty)$, $Y = \mathbb{R}$ and k and $\widehat{g}$ as in the above, we obtain the following generalized Ulam-Hyers stability for the functional equation (5.12) in random normed spaces.

Corollary 5.3. *Let $(\mathbb{R}, \mathcal{F}, T_M)$ be a complete random normed space. If $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies*

$$F_{f(x+1)-x\cdot f(x)} \geq x \circ \psi_x, \quad \forall x \in (0,\infty), \tag{5.13}$$

where $\psi : (0,\infty) \to D_+$ is a mapping for which there exists $L < 1$ such that

$$\psi_{x+1}(Lt) \geq x \circ \psi_x(t), \quad \forall x \in (0,\infty), \ \forall t > 0, \tag{5.14}$$

then there exists a unique mapping $c : (0, \infty) \to \mathbb{R}$,

$$c(x) = \lim_{n\to\infty} f(x+nk) \cdot \prod_{i=0}^{n-1} \frac{1}{x+i}, \quad \forall x \in (0,\infty), \tag{5.15}$$

which satisfies both the equation

$$c(x+1) = xc(x), \quad \forall x \in (0,\infty), \tag{5.16}$$

and the estimation

$$F_{c(x)-f(x)}(t) \geq \psi_x((1-L)t), \quad \forall x \in (0,\infty), \tag{5.17}$$

for almost all $t > 0$.

Remark 5.4. The corresponding results of generalized Ulam-Hyers stability for (5.1) and (5.12) can be obtained, in the deterministic case, as in Corollary 3.2.

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