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Research Article

Common Fixed Points of Generalized Contractive Hybrid Pairs in Symmetric Spaces

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Several fixed point theorems for hybrid pairs of single-valued and multivalued occasionally weakly compatible maps satisfying generalized contractive conditions are established in a symmetric space.

1. Introduction and Preliminaries

In 1968, Kannan [1] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. This paper was a genesis for a multitude of fixed point papers over the next two decades. Sessa [2] coined the term weakly commuting maps. Jungck [3] generalized the notion of weak commutativity by introducing compatible maps and then weakly compatible maps [4]. Al-Thagafi and Shahzad [5] gave a definition which is proper generalization of nontrivial weakly compatible maps which have coincidence points. Jungck and Rhoades [6] studied fixed point results for occasionally weakly compatible (owc) maps. Recently, Zhang [7] obtained common fixed point theorems for some new generalized contractive type mappings. Abbas and Rhoades [8] obtained common fixed point theorems for hybrid pairs of single-valued and multivalued owc maps defined on a symmetric space (see also [9]). For other related fixed point results in symmetric spaces and their applications, we refer to [10–15]. The aim of this paper is to obtain fixed point theorems involving hybrid pairs of single-valued and multivalued owc maps satisfying a generalized contractive condition in the frame work of a symmetric space.

Definition 1.1. A symmetric on a set X is a mapping $d : X \times X \to [0, \infty)$ such that

$$\begin{aligned} &d(x, y) = 0 \quad \text{iff } x = y, \\ &d(x, y) = d(y, x). \end{aligned} \tag{1.1}$$

A set X together with a symmetric d is called a *symmetric space*.

We will use the following notations, throughout this paper, where (X, d) is a symmetric space, $x \in X$ and $A \subseteq X$, $d(x, A) = \inf\{d(x, a) : a \in A\}$, and $B(X)$ is the class of all nonempty bounded subsets of X. The diameter of $A, B \in B(X)$ is denoted and defined by

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}. \tag{1.2}$$

Clearly, $\delta(A, B) = \delta(B, A)$. For $\delta(\{a\}, B)$ and $\delta(\{a\}, \{b\})$ we write $\delta(a, B)$ and $d(a, b)$, respectively. We appeal to the fact that $\delta(A, B) = 0$ if and only if $A = B = \{x\}$ for $A, B \in B(X)$.

Recall that $x \in X$ is called a coincidence point (resp., common fixed point) of $f : X \to X$ and $T : X \to B(X)$ if $fx \in Tx$ (resp., $x = fx \in Tx$).

Definition 1.2. Maps $f : X \to X$ and $T : X \to B(X)$ are said to be *compatible* if $fTx \in B(X)$ for each $x \in X$ and $\delta(fTx_n, Tfx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $Tx_n \to \{t\}$ ($\delta(Tx_n, t) \to 0$) and $fx_n \to t$ for some $t \in X$ [21].

Definition 1.3. Maps $f : X \to X$ and $T : X \to B(X)$ are said to be *weakly compatible* if $fTx = Tfx$ whenever $fx \in Tx$.

Definition 1.4. Maps $f : X \to X$ and $T : X \to B(X)$ are said to be *owc* if and only if there exists some point x in X such that $fx \in Tx$ and $fTx \subseteq Tfx$.

Example 1.5. Consider $X = [0, \infty)$ with usual metric.

(a) Define $f : X \to X$ and $T : X \to B(X)$ as: $f(x) = x^2$ and

$$T(x) = \begin{cases} \left(0, \dfrac{1}{x}\right], & \text{when } x \neq 0, \\ \{0\}, & \text{when } x = 0, \end{cases} \tag{1.3}$$

then f and T are weakly compatible.

(b) Define $f : X \to X$, $T : X \to B(X)$ by

$$\begin{aligned} fx &= \begin{cases} 0, & 0 \le x < 1, \\ x + 1, & 1 \le x < \infty, \end{cases} \\ Tx &= \begin{cases} \{x\}, & 0 \le x < 1, \\ [1, x + 2], & 1 \le x < \infty, \end{cases} \end{aligned} \tag{1.4}$$

It can be easily verified that $x = 1$ is coincidence point of f and T, but f and T are not weakly compatible there, as $Tf1 = [1,4] \neq fT1 = [2,4]$. Hence f and T are not compatible. However, the pair $\{f, T\}$ is occasionally weakly compatible, since the pair $\{f, T\}$ is weakly compatible at $x = 0$.

Assume that $F : [0, \infty) \to R$ satisfies the following.

(i) $F(0) = 0$ and $F(t) > 0$ for each $t \in (0, \infty)$.

(ii) F is nondecreasing on $[0, \infty)$.

Define, $\mathcal{F}[0, \infty) = \{F : F$ satisfies (i)-(ii) above$\}$.

Let $\psi : [0, \infty) \to R$ satisfy the following.

(iii) $\psi(t) < t$ for each $t \in (0, \infty)$.

(iv) ψ is nondecreasing on $[0, \infty)$.

Define, $\Psi[0, \infty) = \{\psi : \psi$ satisfies (iii)-(iv) above$\}$.

For some examples of mappings F which satisfy (i)-(ii), we refer to [7].

2. Common Fixed Point Theorems

In the sequel we shall consider, $F \in \mathcal{F}[0, \infty)$ which is defined on $[0, F(\infty - 0))$, where $\infty - 0$ stands for a real number to the left of ∞ and assume that the mapping ψ satisfies (iii)-(iv) above.

Theorem 2.1. *Let f, g be self maps of a symmetric space X, and let T, S be maps from X into $B(X)$ such that the pairs $\{f, T\}$ and $\{g, S\}$ are owc. If*

$$F(\delta(Tx, Sy)) \leq \psi F(M(x, y)), \tag{2.1}$$

for each $x, y \in X$ for which $fx \neq gy$, where

$$M(x, y) := \max\{d(fx, gy), d(fx, Tx), d(gy, Sy), \delta(fx, Sy), \delta(gy, Tx)\}, \tag{2.2}$$

then f, g, T, and S have a unique common fixed point.

Proof. By hypothesis there exist points x, y in X such that $fx \in Tx$, $gy \in Sy$, $fTx \subseteq Tfx$, and $gSy \subseteq Sgy$. Also, $d(f^2x, g^2y) \leq \delta(Tfx, Sgy)$. Therefore by (2.2) we have

$$\begin{aligned} M(fx, gy) &= \max\left\{d\left(f^2x, g^2y\right), d\left(f^2x, Tfx\right), d\left(g^2y, Sgy\right), \delta\left(f^2x, Sgy\right), \delta\left(g^2y, Tfx\right)\right\} \\ &\leq \delta(Tfx, Sgy). \end{aligned} \tag{2.3}$$

Now we claim that $gy = fx$. For, otherwise, by (2.1),

$$\begin{aligned} F(\delta(Tfx, Sgy)) &\leq \psi(F(M(fx, gy))) \\ &\leq \psi(F(\delta(Tfx, Sgy))) < F(\delta(Tfx, Sgy)), \end{aligned} \tag{2.4}$$

a contradiction and hence $gy = fx$. Obviously, $d(fx, g^2y) \leq \delta(Tx, Sfx)$. Thus (2.2) gives

$$M(x, fx) = \max\left\{d\left(fx, g^2y\right), d(fx, Tx), d\left(g^2y, Sgy\right), \delta(gy, Sgy), \delta\left(g^2y, Tx\right)\right\}$$
$$\leq \delta(Tx, Sfx). \tag{2.5}$$

Next we claim that $x = fx$. If not, then (2.1) implies

$$F(\delta(Tx, Sfx)) \leq \psi(F(M(x, fx))) \leq \psi(F(\delta(Tx, Sfx)))$$
$$< F(\delta(Tx, Sfx)), \tag{2.6}$$

which is a contradiction and the claim follows. Similarly, we obtain $y = gy$. Thus f, g, T, and S have a common fixed point. Uniqueness follows from (2.1). □

Corollary 2.2. *Let f, g be self maps of a symmetric space X and let T, S be maps from X into $B(X)$ such that the pairs $\{f, T\}$ and $\{g, S\}$ are owc. If*

$$F(\delta(Tx, Sy)) \leq \psi(F(m(x, y))) \tag{2.7}$$

for each $x, y \in X$, for which $fx \neq gy$, where

$$m(x, y) = h\max\left\{d(fx, gy), d(fx, Tx), d(gy, Sy), \frac{1}{2}[\delta(fx, Sy) + \delta(gy, Tx)]\right\} \tag{2.8}$$

and $0 \leq h < 1$, then f, g, S, T have a unique common fixed point.

Proof. Since (2.7) is a special case of (2.1), the result follows from Theorem 2.1. □

Corollary 2.3. *Let f, g be self maps of a symmetric space X and let T, S be maps from X into $B(X)$ such that the pairs $\{f, T\}$ and $\{g, S\}$ are owc. If*

$$F(\delta(Tx, Sy)) \leq \psi(F(M(x, y))) \tag{2.9}$$

for each $x, y \in X$ for which $fx \neq gy$, where

$$M(x, y) = \alpha d(fx, gy) + \beta\max\{d(fx, Tx), d(gy, Sy)\}$$
$$+ \gamma\max\{d(fx, gy), \delta(fx, Sy), \delta(gy, Tx)\}, \tag{2.10}$$

where $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$. Then f, g, T, and S have a unique common fixed point.

Proof. Note that

$$M(x,y) \le (\alpha+\beta+\gamma)\max\{d(fx,gy),d(fx,Tx),d(gy,Sy),\delta(fx,Sy),\delta(gy,Tx)\}. \tag{2.11}$$

So, (2.9) is a special case of (2.1) and hence the result follows from Theorem 2.1. □

Corollary 2.4. *Let f be a self map on a symmetric space X and let T be a map from X into $B(X)$ such that f and T are owc. If*

$$F(\delta(Tx,Ty)) \le \psi(F(m(x,y))) \tag{2.12}$$

for each $x,y \in X$, for which $fx \ne fy$, where

$$m(x,y) = \max\left\{d(fx,fy),\frac{1}{2}[d(fx,Tx)+d(fy,Ty)],\frac{1}{2}[\delta(fy,Tx)+\delta(fx,Ty)]\right\}. \tag{2.13}$$

Then f and T have a unique common fixed point.

Proof. Condition (2.12) is a special case of condition (2.1) with $f = g$ and $T = S$. Therefore the result follows from Theorem 2.1. □

Theorem 2.5. *Let f,g be self maps of a symmetric space X and let T,S be maps from X into $B(X)$ such that the pairs $\{f,T\}$ and $\{g,S\}$ are owc. If*

$$F((\delta(Tx,Sy))^p) \ \le \psi(F(M_p(x,y))) \tag{2.14}$$

for each $x,y \in X$ for which $fx \ne gy$,

$$\begin{aligned} M_p(x,y) = {} & \alpha(\delta(Tx,gy))^p \\ & + (1-\alpha)\max\Big\{(d(fx,Tx))^p,(d(gy,Sy))^p,(d(fx,Tx))^{p/2}(d(gy,Tx))^{p/2}, \\ & (\delta(gy,Tx))^{p/2}(\delta(fx,Sy))^{p/2}\Big\}, \end{aligned} \tag{2.15}$$

where $0 < a \le 1$, and $p \ge 1$, then f,g,T, and S have a unique common fixed point.

Proof. By hypothesis there exist points x, y in X such that $fx \in Tx, gy \in Sy, fTx \subseteq Tfx$ and $gSy \subseteq Sgy$. Therefore by (2.15) we have

$$\begin{aligned}
&M_p(fx,gy)\\
&\quad=\alpha\left(\delta\left(Tfx,g^2y\right)\right)^p\\
&\qquad+(1-\alpha)\max\left\{\left(d\left(f^2x,Tfx\right)\right)^p,\left(d\left(g^2y,Sgy\right)\right)^p,\left(d\left(f^2x,Tfx\right)\right)^{p/2}\left(d\left(g^2y,Tfx\right)\right)^{p/2},\right.\\
&\qquad\qquad\left.\left(\delta\left(g^2y,Tfx\right)\right)^{p/2}\left(\delta\left(f^2x,Sgy\right)\right)^{p/2}\right\}\\
&\quad=\alpha\left(\delta\left(g^2y,Tfx\right)\right)^p+(1-\alpha)\left(\delta\left(g^2y,Tfx\right)\right)^{p/2}\left(\delta\left(f^2x,Sgy\right)\right)^{p/2}\\
&\quad\le\alpha(\delta(Tfx,Sgy))^p+(1-\alpha)(\delta(Tfx,Sgy))^p\\
&\quad=(\delta(Tfx,Sgy))^p.
\end{aligned}\tag{2.16}$$

Now we show that $gy = fx$. Suppose not. Then condition (2.14) implies that

$$\begin{aligned}
F((\delta(Tfx,Sgy))^p)&\le\varphi(F(M_p(fx,gy)))\\
&\le\varphi(F((\delta(Tfx,Sgy))^p))<F((\delta(Tfx,Sgy))^p),
\end{aligned}\tag{2.17}$$

which is a contradiction and hence $gy = fx$. Note that, $d(fx, g^2y) \le \delta(Tx, Sfx)$. Thus (2.15) gives

$$\begin{aligned}
M_p(x,fx)&=\alpha(\delta(Tx,gfx))^p\\
&\quad+(1-\alpha)\max\left\{(d(fx,Tx))^p,(d(gfx,Sfx))^p,(d(fx,Tx))^{p/2}(d(gfx,Tx))^{p/2},\right.\\
&\qquad\left.(\delta(gfx,Tx))^{p/2}(\delta(fx,Sfx))^{p/2}\right\}\\
&=\alpha(\delta(gfx,Tx))^p+(1-\alpha)\left(\delta\left(g^2y,Tx\right)\right)^{p/2}(\delta(fx,Sgy))^{p/2}\\
&\le\alpha(\delta(Tx,Sgy))^p+(1-\alpha)(\delta(Tx,Sgy))^p\\
&=(\delta(Tx,Sgy))^p.
\end{aligned}\tag{2.18}$$

Now we claim that $x = fx$. If not, then condition (2.14) implies that

$$\begin{aligned}
F((\delta(Tx,Sfx))^p)&\le\varphi(F(M_p(x,fx)))\\
&\le\varphi(F((\delta(Tx,Sgy))^p))<F((\delta(Tfx,Sgy))^p),
\end{aligned}\tag{2.19}$$

which is a contradiction, and hence the claim follows. Similarly, we obtain $y = gy$. Thus f, g, T, and S have a common fixed point. Uniqueness follows easily from (2.14). □

Define $G = \{\dot{g} : \mathbb{R}^5 \to \mathbb{R}^5\}$ such that

(g_1) $\dot{g}$ is nondecreasing in the 4th and 5th variables,

(g_2) if $u \in \mathbb{R}^+$ is such that

$$u \leq \dot{g}(u,0,0,u,u) \text{ or } u \leq \dot{g}(0,u,0,u,u) \text{ or } u \leq \dot{g}(0,0,u,u,u), \tag{2.20}$$

then $u = 0$.

Theorem 2.6. *Let f, g be self maps of a symmetric space X and let T, S be maps from X into $B(X)$ such that the pairs $\{f, T\}$ and $\{g, S\}$ are owc. If*

$$\begin{aligned} &F(\delta(Tx,Sy)) \\ &\quad \leq \dot{g}(F(d(fx,gy)),F(d(fx,Tx)),F(d(gy,Sy)),F(\delta(fx,Sy)),F(\delta(gy,Tx))) \end{aligned} \tag{2.21}$$

for all $x, y \in$ X for which $fx \neq gy$, where $\dot{g} \in$ G, then f, g, T, and S have a unique common fixed point.

Proof. By hypothesis there exist points x, y in X such that $fx \in Tx$, $gy \in Sy$, $fTx \subseteq Tfx$, and $gSy \subseteq Sgy$. Also, $d(fx, gy) \leq \delta(Tx, Sy)$. First we show that $gy = fx$. Suppose not. Then condition (2.21) implies that

$$\begin{aligned} F(\delta(Tx,Sy)) &\leq \dot{g}(F(d(fx,gy)),0,0,F(\delta(fx,Sy)),F(\delta(gy,Tx))) \\ &\leq \dot{g}(F(\delta(Tx,Sy)),0,0,F(\delta(Tx,Sy)),F(\delta(Tx,Sy))), \end{aligned} \tag{2.22}$$

which, from (g_2), implies that $\delta(Tx, Sy) = 0$; this further implies that, $d(fx, gy) = 0$, a contradiction. Hence the claim follows. Also, $d(fx, f^2x) \leq \delta(Tfx, Sy)$. Next we claim that $fx = f^2x$. If not, then condition (2.21) implies that

$$\begin{aligned} F(\delta(Tfx,Sy)) &\leq \dot{g}\left(F\left(d\left(f^2x,gy\right)\right),0,0,F\left(\delta\left(f^2x,Sy\right)\right),F(\delta(gy,Tfx))\right) \\ &\leq \dot{g}(F(\delta(Tfx,Sy)),0,0,F(\delta(Tfx,Sy)),F(\delta(Tfx,Sy))), \end{aligned} \tag{2.23}$$

which, from (g_1) and (g_2), implies that $\delta(Tfx, Sy) = 0$; this further implies that $d(fx, f^2x) = 0$. Hence the claim follows. Similarly, it can be shown that $gy = g^2y$ which proves that fx is a common fixed point of f, g, S, and T. Uniqueness follows from (2.21) and (g_2). □

A control function $\Phi : R^+ \to R^+$ is a continuous monotonically increasing function that satisfies $\Phi(2t) \leq 2\Phi(t)$ and, $\Phi(0) = 0$ if and only if $t = 0$.

Let $\Psi : R^+ \to R^+$ be such that $\Psi(t) < t$ for each $t > 0$.

Theorem 2.7. *Let f, g be self maps of symmetric space X and let T, S be maps from X into $B(X)$ such that the pairs $\{f, T\}$ and $\{g, S\}$ are owc. If for a control function Φ, one has*

$$F(\Phi(\delta(Tx, Sy))) \leq \psi(F(M_\Phi(x, y))) \tag{2.24}$$

for each $x, y \in X$ for which right-hand side of (2.24) is not equal to zero, where

$$\begin{aligned} M_\Phi(x, y) = \max\Big\{\Big\{&\Phi(d(fx, gy)), \Phi(d(fx, Tx)), \Phi(d(gy, Sy)),\\ &\frac{1}{2}[\Phi(\delta(fx, Sy)) + \Phi(\delta(gy, Tx))]\Big\}\Big\}, \end{aligned} \tag{2.25}$$

then f, g, S, and T have a unique common fixed point.

Proof. By hypothesis there exist points x, y in X such that $fx \in Tx$, $gy \in Sy$, $fTx \subseteq Tfx$, and $gSy \subseteq Sgy$. Also, using the triangle inequality, we obtain $d(fx, gy) \leq \delta(Tx, Sy)$. Therefore by (2.25) we have

$$\begin{aligned} M_\Phi(x, y) &= \max\Big\{\Phi(d(fx, gy)), 0, 0, \frac{1}{2}\Phi(2\delta(Tx, Sy))\Big\}\\ &\leq \Phi(\delta(Tx, Sy)). \end{aligned} \tag{2.26}$$

Now we show that $\delta(Tx, Sy) = 0$. Suppose not. Then condition (2.24) implies that

$$\begin{aligned} F(\Phi(\delta(Tx, Sy))) &\leq \psi(F(M_\Phi(x, y)))\\ &= \psi(F(\Phi(\delta(Tx, Sy)))) < F(\Phi(\delta(Tx, Sy))), \end{aligned} \tag{2.27}$$

which is a contradiction. Therefore $\delta(Tx, Sy) = 0$, which further implies that, $d(fx, gy) = 0$. Hence the claim follows. Again, $d(f^2x, fx) \leq \delta(Tfx, Sy)$. Therefore by (2.25) we have

$$\begin{aligned} M_\Phi(fx, y) &= \max\Big\{\Phi\Big(d\Big(f^2x, gy\Big)\Big), 0, 0, \frac{1}{2}\Phi(2\delta(Tfx, Sy))\Big\}\\ &\leq \Phi(\delta(Tfx, Sy)). \end{aligned} \tag{2.28}$$

Next we claim that $\delta(Tfx, Sy) = 0$. If not, then condition (2.24) implies

$$\begin{aligned} F(\Phi(\delta(Tfx, Sy))) &\leq \psi(F(M_\Phi(fx, y)))\\ &\leq \psi(F(\Phi(\delta(Tfx, Sy)))) < F(\Phi(\delta(Tfx, Sy))), \end{aligned} \tag{2.29}$$

which is a contradiction. Therefore $\delta(Tfx, Sy) = 0$, which further implies that $d(fx, f^2x) = 0$. Hence the claim follows. Similarly, it can be shown that $gy = g^2y$ which proves the result. □

Set $G = \{\psi : [0,\infty) \to [0,\infty) : \psi$ is continuous and nondecreasing mapping with $\psi(t) = 0$ if and only if $t = 0\}$.

The following theorem generalizes [16, Theorem 2.1].

Theorem 2.8. *Let f, g be self maps of a symmetric space X, and let T, S be maps from X into $B(X)$ such that the pairs $\{f, T\}$ and $\{g, S\}$ are owc. If*

$$\psi(\delta(Tx, Sy)) \leq \psi(d(fx, gy)) - \varphi(d(fx, gy)) \tag{2.30}$$

for all $x, y \in X$, for which right-hand side of (2.30) *is not equal to zero, where $\psi, \varphi \in G$, then f, g, S, and T have a unique common fixed point.*

Proof. By hypothesis there exist points x, y in X such that $fx \in Tx$, $gy \in Sy$, $fTx \subseteq Tfx$, and $gSy \subseteq Sgy$. Also, using the triangle inequality, we obtain, $d(fx, gy) \leq \delta(Tx, Sy)$. Now we claim that $gy = fx$. For, otherwise, by (2.30),

$$\begin{aligned} \psi(\delta(Tx, Sy)) &\leq \psi(d(fx, gy)) - \varphi(d(fx, gy)) \\ &\leq \psi(\delta(Tx, Sy)) - \varphi(d(fx, gy)) \end{aligned} \tag{2.31}$$

which is a contradiction. Therefore $fx = gy$. Hence the claim follows. Again, $d(f^2x, fx) \leq \delta(Tfx, Sy)$. Now we claim that $f^2x = fx$. If not, then condition (2.30) implies that

$$\begin{aligned} \psi(\delta(Tfx, Sy)) &\leq \psi\left(d\left(f^2x, gy\right)\right) - \varphi\left(d\left(f^2x, gy\right)\right) \\ &= \psi\left(d\left(f^2x, fx\right)\right) - \varphi\left(d\left(f^2x, fx\right)\right) \\ &\leq \psi(\delta(Tfx, Sy)) - \varphi\left(d\left(f^2x, fx\right)\right), \end{aligned} \tag{2.32}$$

which is a contradiction, and hence the claim follows. Similarly, it can be shown that $gy = g^2y$ which, proves that fx is a common fixed point of f, g, S, and T. Uniqueness follows easily from (2.30). □

Example 2.9. Let $X = \{1, 2, 3\}$. Define $d : X \times X \to [0, \infty)$ by

$$\begin{aligned} &d(1,1) = d(2,2) = d(3,3) = 0, \qquad d(1,2) = d(2,1) = 2, \\ &d(1,3) = d(3,1) = 4, \qquad d(2,3) = d(3,2) = 1. \end{aligned} \tag{2.33}$$

Note that d is symmetric but not a metric on X.

Define $T, S : X \to B(X)$ by

$$\begin{aligned} &T(1) = \{1,3\}, \qquad T(2) = \{1,2,3\}, \qquad T(3) = \{1,3\}, \\ &S(1) = \{1,2\}, \qquad S(2) = \{1,3\}, \qquad S(3) = \{2,3\}, \end{aligned} \tag{2.34}$$

and $f, g : X \to X$ as follows:

$$\begin{aligned} &f(1) = 1, \quad f(2) = 3, \quad f(3) = 1, \\ &g(1) = 1, \quad g(2) = 1, \quad g(3) = 2. \end{aligned} \tag{2.35}$$

Clearly, $f(1) \in T(1)$ but $fT(1) \neq Tf(1)$, and $f(3) \in T(3)$ but $fT(3) \neq Tf(3)$; they show that $\{f, T\}$ is not weakly compatible. On the other hand, $f(2) \in T(2)$ gives that $fT(2) = Tf(2)$. Hence $\{f, T\}$ is occasionally weakly compatible. Note that $g(1) \in S(1)$, $gS(1) \neq Sg(1)$, $g(3) \in S(3)$, and $gS(3) \neq Sg(3)$; they imply that $\{g, S\}$ is not weakly compatible. Now $g(2) \in S(2)$ gives that $gS(2) = Sg(2)$. Hence $\{g, S\}$ is occasionally weakly compatible. As $f(1) = g(1) \in T(1)$ and $f(1) = g(1) \in S(1)$, so 1 is the unique common fixed point of f, g, S, and T.

Remarks 2.10. Weakly compatible maps are occasionally weakly compatible but converse is not true in general. The class of symmetric spaces is more general than that of metric spaces. Therefore the following results can be viewed as special cases of our results:

(a) ([17, Theorem 1] and [18, Theorem 1]) are special cases of Theorem 2.7.

(b) [19, Theorem 1], [20, Theorem 2.1], [21, Theorem 4.1], and [22, Theorem 2] are special cases of Corollary 2.2. Moreover, [23, Theorem 2] and [24, Theorem 1] also become special cases of Corollary 2.2.

(c) ([25, Theorem 2]) is a special case of Theorem 2.1. Theorem 2.1 also generalizes ([26, Theorem 1]) and ([27, Theorems 1 and 2]).

(d) [28, Theorem 3.1] becomes special case of Corollary 2.4.

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