Research Article

Convergence Theorems of Three-Step Iterative Scheme for a Finite Family of Uniformly Quasi-Lipschitzian Mappings in Convex Metric Spaces

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We consider a new Noor-type iterative procedure with errors for approximating the common fixed point of a finite family of uniformly quasi-Lipschitzian mappings in convex metric spaces. Under appropriate conditions, some convergence theorems are proved for such iterative sequences involving a finite family of uniformly quasi-Lipschitzian mappings. The results presented in this paper extend, improve and unify some main results in previous work.

1. Introduction and Preliminaries

Takahashi [1] introduced a notion of convex metric spaces and studied the fixed point theory for nonexpansive mappings in such setting. For the convex metric spaces, Kirk [2] and Goebel and Kirk [3] used the term "hyperbolic type space" when they studied the iteration processes for nonexpansive mappings in the abstract framework. For the Banach space, Petryshyn and Williamson [4] proved a sufficient and necessary condition for Picard iterative sequences and Mann iterative sequence to converge to fixed points for quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [5] extended the results of [4] and gave the sufficient and necessary condition for Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings. Liu [6–8] proved some sufficient and necessary conditions for Ishikawa iterative sequence and Ishikawa iterative sequence with errors to converge to fixed point for asymptotically quasi-nonexpansive mappings in Banach space and uniform convex Banach space. Tian [9] gave some sufficient and necessary conditions for an Ishikawa iteration sequence for an asymptotically quasi-nonexpansive mapping to converge to a fixed point in convex metric spaces. Very recently, Wang and Liu [10] gave some iteration sequence

with errors to approximate a fixed point of two uniformly quasi-Lipschitzian mappings in convex metric spaces. The purpose of this paper is to give some sufficient and necessary conditions for a new Noor-type iterative sequence with errors to approximate a common fixed point for a finite family of uniformly quasi-Lipschitzian mappings in convex metric spaces. The results presented in this paper generalize, improve, and unify some main results of [1–14].

First of all, let us list some definitions and notations.

Let T be a given self mapping of a nonempty convex subset C of an arbitrary real normed space.The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0 \in C,$$
$$x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 0,$$
$$y_n = a_n x_n + b_n T z_n + c_n v_n,$$
$$z_n = d_n x_n + e_n T x_n + f_n w_n,$$

(1.1)

is called the Noor iterative procedure with errors [11], where $\alpha_n, \beta_n, \gamma_n, a_n, b_n, c_n, d_n, e_n$, and f_n are appropriate sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1$, $n \geq 0$ and $\{u_n\}, \{v_n\}$, and $\{w_n\}$ are bounded sequences in C. If $d_n = 1$ ($e_n = f_n = 0$), $n \geq 0$ then (1.1) reduces to the Ishikawa iterative procedure with errors [15] defined as follows:

$$x_0 \in C,$$
$$x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 0,$$
$$y_n = a_n x_n + b_n T x_n + c_n v_n.$$

(1.2)

If $a_n = 1$ ($b_n = c_n = 0$) then (1.2) reduces to the following Mann type iterative procedure with errors [15]:

$$x_0 \in C,$$
$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n u_n, \quad n \geq 0.$$

(1.3)

Let (E, d) be a metric space. A mapping $T : E \to E$ is said to be asymptotically nonexpansive, if there exists a sequence $\{K_n\} \in [1, \infty], \lim_{n \to \infty} K_n = 1$, such that

$$d(T^n x, T^n y) \leq K_n d(x, y), \quad \forall x, y \in E, \quad n \geq 0.$$

(1.4)

Let $F(T)$ be the set of fixed points of T in E and $F(T) \neq \emptyset$, a mapping T is said to be asymptotically quasi-nonexpansive, if there exists $\{K_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} K_n = 1$ such that

$$d(T^n x, p) \leq K_n d(x, p), \quad \forall x \in E, \ p \in F(T), \ n \geq 0.$$

(1.5)

Moreover, T is said to be uniformly quasi-Lipschitzian, if there exists $L > 0$ such that

$$d(T^n x, p) \le L d(x, p), \qquad \forall x \in E, \ p \in F(T), \ n \ge 0. \tag{1.6}$$

Remark 1.1. If $F(T)$ is nonempty, then it follows from the above definitions that an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive, and an asymptotically quasi-nonexpansive mapping must be a uniformly quasi-Lipschitzian with $L = \sup_{n \ge 0}\{K_n\} < \infty$. However, the inverse is not true in general.

Definition 1.2 (see [9]). Let (E, d) be a metric space, and let $I = [0,1], \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$. A mapping $W : E^3 \times I^3 \to X$ is said to be a convex structure on E if, for any $(x, y, z, \alpha_n, \beta_n, \gamma_n) \in E^3 \times I^3$ and $u \in E$,

$$d(W(x, y, z, \alpha_n, \beta_n, \gamma_n)u) \le \alpha_n d(x, u) + \beta_n d(y, u) + \gamma_n d(z, u). \tag{1.7}$$

If (E, d) is a metric space with a convex structure W, then (E, d) is called a convex metric space. Let (E, d) be a convex metric space, a nonempty subset C of E is said to be convex if

$$W(x, y, z, \lambda_1, \lambda_2, \lambda_3) \in C, \quad \forall (x, y, z, \lambda_1, \lambda_2, \lambda_3) \in C^3 \times I^3. \tag{1.8}$$

Definition 1.3. Let (E, d) be a convex metric space with a convex structure $W : E^3 \times I^3$ and $T_i : E \to E$ be a finite family of uniformly quasi-Lipschitzian mappings with $i = 1, 2, \ldots, N$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}$, and $\{f_n\}$ be nine sequences in $[0, 1]$ with

$$\alpha_n + \beta_n + \eta_n = a_n + b_n + c_n = d_n + e_n + f_n = 1, \qquad n = 0, 1, 2, \ldots. \tag{1.9}$$

For a given $x_0 \in E$, define a sequence $\{x_n\}$ as follows:

$$\begin{aligned}
x_{n+1} &= W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \ge 0, \\
y_n &= W(f(x_n), T_n^n z_n, v_n; a_n, b_n, c_n), \\
z_n &= W(f(x_n), T_n^n x_n, w_n; d_n, e_n, f_n),
\end{aligned} \tag{1.10}$$

where $T_n^n = T_{n(\mathrm{mod}\, N)}^n$, $f : E \to E$ is a Lipschitz continuous mapping with a Lipschitz constant $\xi > 0$ and $\{u_n\}, \{v_n\}, \{w_n\}$ are any given three sequences in E. Then $\{x_n\}$ is called the Noor-type iterative sequence with errors for a finite family of uniformly quasi-Lipschitzian mappings $\{T_i\}_{i=1}^N$. If $f = I$ (the identity mapping on E) in (1.10), then the sequence $\{x_n\}$ defined by (1.10) can be written as follows:

$$\begin{aligned}
x_{n+1} &= W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \ge 0; \\
y_n &= W(x_n, T_n^n z_n, v_n; a_n, b_n, c_n), \\
z_n &= W(x_n, T_n^n x_n, w_n; d_n, e_n, f_n).
\end{aligned} \tag{1.11}$$

If $d_n = 1$ for all $n \geq 0$ in (1.10), then $z_n = x_n$ for all $n \geq 0$ and the sequence $\{x_n\}$ defined by (1.10) can be written as follows:

$$
\begin{aligned}
x_{n+1} &= W\left(f(x_n), T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n\right), \quad n \geq 0, \\
y_n &= W\left(f(x_n), T_n^n x_n, v_n; a_n, b_n, c_n\right).
\end{aligned}
\tag{1.12}
$$

If $f = I$ and $d_n = 1$ for all $n \geq 0$, then the sequence $\{x_n\}$ defined by (1.10) can be written as follows:

$$
\begin{aligned}
x_{n+1} &= W\left(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n\right), \quad n \geq 0, \\
y_n &= W\left(x_n, T_n^n x_n, v_n; a_n, b_n, c_n\right),
\end{aligned}
\tag{1.13}
$$

which is the Ishikawa type iterative sequence with errors considered in [9]. Further, if $f = I$ and $d_n = a_n = 1$ for all $n \geq 0$, then $z_n = y_n = x_n$ for all $n \geq 0$ and (1.10) reduces to the following Mann type iterative sequence with errors [9]:

$$
x_{n+1} \equiv W\left(x_n, T_n^n x_n, u_n; \alpha_n, \beta_n, \gamma_n\right), \quad n \geq 0.
\tag{1.14}
$$

In order to prove our main results, the following lemmas will be needed.

Lemma 1.4. *Let (E, d) be a convex metric space, $T_i : E \to E$ be a uniformly quasi-Lipschitzian mapping for $i = 1, 2, \ldots, N$ such that $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Then there exists a constant $L \geq 1$ such that, for all $i = 1, 2, \ldots, N$,*

$$
d\left(T_i^n x, p\right) \leq L d(x, p), \quad \forall x \in X, \ p \in F, \ n \geq 0.
\tag{1.15}
$$

Proof. In fact, for each $i = 1, 2, \ldots, N$, since $T_i : E \to E$ is a uniformly quasi-Lipschitzian mapping, we have

$$
d\left(T_i^n x, p\right) \leq L_i d(x, p) \leq L d(x, p), \quad \forall x \in E, \ p \in F, \ n \geq 0,
\tag{1.16}
$$

where

$$
L = \max_{i=1,2,\ldots,N} \{\max\{L_i, 1\}\}.
\tag{1.17}
$$

This completes the proof. $\qquad\square$

Lemma 1.5 (see [7]). *Let $\{p_n\}, \{q_n\}, \{r_n\}$ be three nonexpansive squences satisfying the following conditions:*

$$p_{n+1} \le (1 + q_n)p_n + r_n, \quad \forall n \ge 0, \quad \sum_{n=0}^{\infty} q_n < \infty, \quad \sum_{n=0}^{\infty} r_n < \infty. \tag{1.18}$$

Then

(1) $\lim_{n \to \infty} p_n$ exists;

(2) In addition, if $\liminf_{n \to \infty} p_n = 0$, then $\lim_{n \to \infty} p_n = 0$.

Lemma 1.6. *Let (E, d) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i : C \to C$ be a finite family of uniformly quasi-Lipschitzian mapping for $i = 1, 2, \ldots, N$ such that $F := \bigcap_{i=1}^{N} F(T_i) \ne \emptyset$ and $f : C \to C$ be a contractive mapping with a contractive constant $\xi \in (0, 1)$. Let $\{x_n\}$ be the iterative sequence with errors defined by (1.10) and $\{u_n\}, \{v_n\}, \{w_n\}$ be three bounded sequences in C. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}, \{f_n\}$ be sequences in [0,1] satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1, \quad \forall n \ge 0;$

(ii) $\sum_{n=0}^{\infty} (\beta_n + \gamma_n) < \infty;$

(iii) $M_0 = \operatorname{Sup}_{p \in F, n \ge 0} \{d(u_n, p) + d(v_n, p) + d(w_n, p) + d(f(p), p)\} < \infty.$

Then the following conclusions hold:

(1) *for all $p \in F$ and $n \ge 0$,*

$$d(x_{n+1}, p) \le \left[1 + \beta_n L \left(1 + L + L^2\right)\right] d(x_n, p) + M\eta_n, \tag{1.19}$$

where $L = \max_{i=1,2,\ldots,N} \{L_i\}$, $\eta_n = \beta_n + \gamma_n$ for all $n \ge 0$ and

$$M = L(1 + L) \left[d(u_n, p) + d(v_n, p) + d(w_n, p) + d(f(p), p)\right]. \tag{1.20}$$

(2) *there exists a constant $M_i > 0$ such that*

$$d(x_{n+m}, p) \le M_1 d(x_n, p) + MM_1 \sum_{k=n}^{n+m-1} \eta_k, \quad \forall p \in F, \tag{1.21}$$

for all $n, m \ge 0$.

Proof. (1) It follows from (1.7),(1.10), and Lemma 1.4 that

$$
\begin{aligned}
d(x_{n+1}, p) &= d\big(W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), p\big) \\
&\leq \alpha_n d(x_n, p) + \beta_n d(T_n^n y_n, p) + \gamma_n d(u_n, p) \\
&\leq \alpha_n d(x_n, p) + \beta_n L d(y_n, p) + \gamma_n d(u_n, p),
\end{aligned}
\tag{1.22}
$$

$$
\begin{aligned}
d(y_n, p) &= d\big(W(f(x_n), T_n^n z_n, v_n; a_n, b_n, c_n), p\big) \\
&\leq a_n d(f(x_n), p) + b_n d(T_n^n z_n, p) + c_n d(v_n, p) \\
&\leq a_n d(f(x_n), f(p)) + a_n d(f(p), p) + b_n L d(z_n, p) + c_n d(v_n, p) \\
&\leq a_n \xi d(x_n, p) + a_n d(f(p), p) + b_n L d(z_n, p) + c_n d(v_n, p),
\end{aligned}
\tag{1.23}
$$

$$
\begin{aligned}
d(z_n, p) &= d\big(W(f(x_n), T_n^n x_n, w_n; d_n, e_n, f_n), p\big) \\
&\leq d_n d(f(x_n), p) + e_n d(T_n^n x_n, p) + f_n d(w_n, p) \\
&\leq d_n d(f(x_n), f(p)) + d_n d(f(p), p) + e_n L d(x_n, p) + f_n d(w_n, p) \\
&\leq d_n \xi d(x_n, p) + d_n d(f(p), p) + e_n L d(x_n, p) + f_n d(w_n, p) \\
&\leq (d_n \xi + e_n L) d(x_n, p) + d_n d(f(p), p) + f_n d(w_n, p).
\end{aligned}
\tag{1.24}
$$

Substituting (1.23) into (1.22) and simplifying it, we have

$$
\begin{aligned}
d(x_{n+1}, p) &\leq \alpha_n d(x_n, p) \\
&\quad + \beta_n L\big[a_n \xi d(x_n, p) + a_n d(f(p), p) + b_n L d(z_n, p) + c_n d(v_n, p)\big] + \gamma_n d(u_n, p) \\
&\leq (\alpha_n + \beta_n L \xi a_n) d(x_n, p) + \beta_n L a_n d(f(p), p) \\
&\quad + \beta_n L^2 b_n d(z_n, p) + \beta_n L c_n d(v_n, p) + \gamma_n d(u_n, p).
\end{aligned}
\tag{1.25}
$$

Substituting (1.24) into (1.25) and simplifying it, we get

$$
\begin{aligned}
d(x_{n+1}, p) &\leq (\alpha_n + \beta_n L a_n \xi) d(x_n, p) \\
&\quad + \beta_n L^2 b_n\big[(d_n \xi + e_n L) d_n(x_n, p) + d_n d(f(p), p) + f_n d(w_n, p)\big] \\
&\quad + \beta_n L a_n d(f(p), p) + \beta_n L c_n d(v_n, p) + \gamma_n d(u_n, p) \\
&= \big\{\alpha_n + \beta_n L[a_n \xi + L b_n(d_n \xi + e_n L)]\big\} d(x_n, p) + \beta_n L^2 b_n d_n d(f(p), p) \\
&\quad + \beta_n L a_n d(f(p), p) + \beta_n L^2 b_n f_n d(w_n, p) + \beta_n L c_n d(v_n, p) + \gamma_n d(u_n, p)
\end{aligned}
$$

$$\leq \left[1 + \beta_n L\left(1 + L + L^2\right)\right] d(x_n, p) + \left[\beta_n L^2 b_n d_n + \beta_n L a_n\right] d(f(p), p) + \gamma_n d(u_n, p)$$

$$+ \beta_n L c_n d(v_n, p) + \beta_n L^2 b_n f_n d(w_n, p)$$

$$\leq \left[1 + \beta_n L\left(1 + L + L^2\right)\right] d(x_n, p) + \beta_n L(1 + L) d(f(p), p) + \gamma_n L(1 + L) d(f(p), p)$$

$$+ L(1 + L)(\beta_n + \gamma_n) d(u_n, p) + L(1 + L)(\beta_n + \gamma_n) d(v_n, p)$$

$$+ L(1 + L)(\beta_n + \gamma_n) d(w_n, p)$$

$$= \left[1 + \beta_n L\left(1 + L + L^2\right)\right] d(x_n, p)$$

$$+ L(1 + L)(\beta_n + \gamma_n)\left[d(u_n, p) + d(v_n, p) + d(w_n, p) + d(f(p), p)\right]$$

$$= \left[1 + \beta_n L\left(1 + L + L^2\right)\right] d(x_n, p) + M\eta_n, \quad \forall n \geq 0, \ p \in F,$$

$$(1.26)$$

where

$$M = L(1 + L)\left[d(u_n, p) + d(v_n, p) + d(w_n, p) + d(f(p), p)\right], \quad \eta_n = \beta_n + \gamma_n. \qquad (1.27)$$

(2) Since $1 + x \leq e^x$ for all $x \geq 0$, it follows from (1.26) that, for $n, m \geq 0$ and $p \in F$,

$$d(x_{n+m}, p) \leq \left[1 + \beta_{n+m-1} L\left(1 + L + L^2\right)\right] d(x_{n+m-1}, p) + M\eta_{n+m-1}$$

$$\leq e^{\beta_{n+m-1} L\left(1 + L + L^2\right)} d(x_{n+m-1}, p) + M\eta_{n+m-1}$$

$$\leq e^{\beta_{n+m-1} L\left(1 + L + L^2\right)} \left\{\left[1 + \beta_{n+m-2} L\left(1 + L + L^2\right)\right] d(x_{n+m-2}, p) + M\eta_{n+m-2}\right\} + M\eta_{n+m-1}$$

$$\leq e^{L\left(1 + L + L^2\right)(\beta_{n+m-1} + \beta_{n+m-2})} d(x_{n+m-1}, p) + M\left[e^{\beta_{n+m-1} L\left(1 + L + L^2\right)}\eta_{n+m-2} + \eta_{n+m-1}\right]$$

$$\leq \cdots$$

$$\leq M_1 d(x_n, p) + M_1 M \sum_{k=n}^{n+m-1} \eta_k,$$

$$(1.28)$$

where

$$M_1 = e^{L\left(1 + L + L^2\right)\sum_{k=0}^{\infty} \beta_k}. \qquad (1.29)$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

2. Main Results

Theorem 2.1. *Let (E, d) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i : C \to C$ be a finite family of uniformly quasi-Lipschitzian mapping for $i = 1, 2, \ldots, N$ such that $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $f : C \to C$ be a contractive mapping with a contractive constant $\xi \in (0, 1)$. Let $\{x_n\}$ be the iterative sequence with errors defined by (1.10) and $\{u_n\}, \{v_n\}, \{w_n\}$ be three bounded sequence in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}$ and $\{f_n\}$ be nine sequences in $[0,1]$ satisfying the following conditions:*

    (i) $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1, \forall n \geq 0$,

    (ii) $\sum_{n=0}^{\infty}(\beta_n + \gamma_n) < \infty$,

    (iii) $M_0 = \mathrm{Sup}_{p \in F, n \geq 0}\{d(u_n, p) + d(v_n, p) + d(w_n, p) + d(f(p), p)\} < \infty$.

Then the sequence $\{x_n\}$ converges to a common fixed point $p \in F$ if and only if $\liminf_{n \to \infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{d(x, F), p \in F\}$.

Proof. The necessity is obvious. Now prove the sufficiency. In fact, from **Lemma 1.6**, we have

$$d(x_{n+1}, F) \leq \left[1 + \beta_n L\left(1 + L + L^2\right)\right] d(x_n, F) + M\eta_n, \quad \forall n \geq 0, \tag{2.1}$$

where $\eta_n = \beta_n + \gamma_n$. By conditions (i) and (ii), we know that

$$\sum_{n=0}^{\infty} \eta_n < \infty, \qquad \sum_{n=0}^{\infty} \beta_n < \infty. \tag{2.2}$$

It follows from **Lemma 1.5** that $\lim_{n \to \infty} d(x_n, F)$ exists. Since $\liminf_{n \to \infty} d(x_n, F) = 0$, we have

$$\lim_{n \to \infty} d(x_n, F) = 0. \tag{2.3}$$

    Next prove that $\{x_n\}$ is a Cauchy sequence in C. In fact, for any given $\varepsilon > 0$, there exists a positive integer N_0 such that

$$d(x_n, F) \leq \frac{\varepsilon}{8M_1}, \qquad \sum_{n=N_0}^{\infty} \eta_n \leq \frac{\varepsilon}{4M_1 M}, \quad \forall n \geq 0. \tag{2.4}$$

From (2.4), there exist $p_1 \in F$ and positive integer $N_1 > N_0$ such that

$$d(x_{N_1}, p_1) < \frac{\varepsilon}{4M_1}. \tag{2.5}$$

Thus Lemma 1.6 implies that, for any positive integers n, m with $n > N_1$,

$$
\begin{aligned}
d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_1) + d(p_1, x_n) \\
&\leq M_1 d(x_{N_1}, P_1) + M_1 M \sum_{k=N_1}^{n+m-1} \eta_k + M_1 d(x_{N_1}, p_1) + M_1 M \sum_{k=N_1}^{n-1} \eta_k \\
&\leq 2M_1 \frac{\varepsilon}{4M_1} + 2M_1 M \frac{\varepsilon}{4M_1 M} \\
&= \varepsilon.
\end{aligned}
\tag{2.6}
$$

This shows that $\{x_n\}$ is a Cauchy sequence in a nonempty closed convex subset C of a complete convex metric space E. Without loss of generality, we can assume that $\lim_{n \to \infty} x_n = p^* \in E$. Next prove that $p^* \in F$. In fact, for any given $\varepsilon' > 0$, there exists a positive integer N_2 such that for all $n \geq N_2$,

$$
d(x_n, p^*) \leq \frac{\varepsilon'}{4L}, \quad d(x_n, F) \leq \frac{\varepsilon'}{8L}.
\tag{2.7}
$$

Again from (2.7), there exist $p_2 \in F$ and positive integer $N_3 > N$ such that

$$
d(x_{N_3}, P_2) \leq \frac{\varepsilon'}{4L}.
\tag{2.8}
$$

Thus, for any $i = 1, 2, \ldots, N$, from (2.7) and (2.8), we have

$$
\begin{aligned}
d(T_i P^*, P^*) &\leq d(T_i P^*, P_2) + d(P_2, T_i x_{N_3}) + d(T_i x_{N_3}, P^*) \\
&\leq L d(P^*, p_2) + L d(p_2, x_{N_3}) + L d(x_{N_3}, P^*) \\
&\leq L\{d(P^*, x_{N_3}) + d(x_{N_3}, p_2)\} + L d(p_2, x_{N_3}) + L d(x_{N_3}, P^*) \\
&= 2L d(P^*, x_{N_3}) + 2L d(x_{N_3}, p_2) \\
&< \frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} = \varepsilon'.
\end{aligned}
\tag{2.9}
$$

By the arbitrariness of $\varepsilon' > 0$, we know that $T_i P^* = P^*$ for all $i = 1, 2, \ldots, N$, that is, $p^* \in F$. This completes the proof of Theorem 2.1. $\qquad\square$

Taking $f = I$ in Theorem 2.1, then we have the following theorem.

Theorem 2.2. *Let (E, d) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i : C \rightarrow C$ be a finite family of uniformly quasi-Lipschitzian mapping for $i = 1, 2, \ldots, N$ such that $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the iterative sequence with errors defined by (1.11) and $\{u_n\}, \{v_n\}, \{w_n\}$ be three bounded sequence in C, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}, and \{f_n\}$ be nine sequence in $[0,1]$ satisfying the*

conditions (i)–(iii) of Theorem 2.1. Then the sequence $\{x_n\}$ converges to a common fixed point $p \in F$ if and only if

$$\liminf_{n \to \infty} d(x_n, F) = 0, \tag{2.10}$$

where $d(x, F) = \inf \{d(x, F), p \in F\}$.

Taking $d_n = 1$ in Theorem 2.1, then we have the following theorem.

Theorem 2.3. *Let (E, d) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i : C \to C$ be a finite family of uniformly quasi-Lipschitzian mapping for $i = 1, 2, \ldots, N$ such that $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $f : C \to C$ be a contractive mapping with a contractive constant $\xi \in (0, 1)$. Let $\{x_n\}$ be the iterative sequence with errors defined by (1.12) and $\{u_n\}, \{v_n\}$ be two bounded sequences in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\}$ be nine sequences in $[0, 1]$ satisfying the conditions (ii) and (iii) of Theorem 2.1 and $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = 1$ for all $n \geq 0$. Then the sequence $\{x_n\}$ converges to a common fixed point $p \in F$ if and only if*

$$\liminf_{n \to \infty} d(x_n, F) = 0, \tag{2.11}$$

where $d(x, F) = \inf \{d(x, p), p \in F\}$.

Remark 2.4. Theorems 2.1–2.3 generalize, improve, and unify some corresponding results in [1–14].

Similarly, we can obtain the following results.

Theorem 2.5. *Let (E, d) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i : C \to C$ be a finite family of asymptotically quasi-nonexpansive mapping for $i = 1, 2, \ldots, N$ such that $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $f : C \to C$ be a contractive mapping with a contractive constant $\xi \in (0, 1)$. Let $\{x_n\}$ be the iterative sequence with errors defined by (1.10) and $\{u_n\}, \{v_n\}, \{w_n\}$ be three bounded sequences in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\},$ and $\{f_n\}$ be nine sequences in $[0, 1]$ satisfying the conditions (i)–(iii) of Theorem 2.1. Then the sequence $\{x_n\}$ converges to a common fixed point $p \in F$ if and only if*

$$\liminf_{n \to \infty} d(x_n, F) = 0, \tag{2.12}$$

where $d(x, F) = \inf f\{d(x, p), p \in F\}$.

Proof. From Remark 1.1, we know that each asymptotically quasi-nonexpansive mapping $T_i : C \to C, i = 1, 2, \ldots, N$ must be a uniformly quasi-Lipschitzian with

$$L_i = \operatorname*{Sup}_{n \geq 0} \left\{ k_n^{(i)} \right\} < \infty, \tag{2.13}$$

where $\{k_n^{(i)}\} \subset [1, \infty)$ is the sequence appeared in (1.5). Hence the conclusion of Theorem 2.5 can be obtained from Theorem 2.1 immediately. This completes the proof. $\qquad\square$

Theorem 2.6. *Let (E, d) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i : C \to C$ be a finite family of asymptotically quasi-nonexpansive mapping for, $i = 1, 2, \ldots, N$ such that $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the iterative sequence with errors defined by (1.11) and $\{u_n\}, \{v_n\}, \{w_n\}$ be three bounded sequence in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}$, and $\{f_n\}$ be nine sequence in $[0, 1]$ satisfying the conditions (i)–(iii) of Theorem 2.1. Then the sequence $\{x_n\}$ converges to a common fixed point $p \in F$ if and only if*

$$\liminf_{n \to \infty} d(x_n, F) = 0, \tag{2.14}$$

where $d(x, F) = \inf \{d(x, p), p \in F\}$.

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