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Research Article

Strong Convergence of Iterative Schemes for Zeros of Accretive Operators in Reflexive Banach Spaces

Jong Soo Jung

Department of Mathematics, Dong-A University, Busan 604-714, South Korea

Correspondence should be addressed to Jong Soo Jung, jungjs@mail.donga.ac.kr

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We introduce composite iterative schemes by the viscosity iteration method for finding a zero of an accretive operator in reflexive Banach spaces. Then, under certain differen control conditions, we establish strong convergence theorems on the composite iterative schemes. The main theorems improve and develop the recent corresponding results of Aoyama et al. (2007), Chen and Zhu (2006, 2008), Jung (2010), Kim and Xu (2005), Qin and Su (2007) and Xu (2006) as well as Benavides et al. (2003), Kamimura and Takahashi (2000), Maingé (2006), and Nakajo (2006).

1. Introduction

Let E be a real Banach space and C a nonempty closed convex subset of E. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0,1)$ such that $\|f(x)-f(y)\| \le k\|x-y\|$, $x, y \in C$. We use Σ_C to denote the collection of mappings f verifying the above inequality. That is, $\Sigma_C = \{f : C \to C \quad f \text{ is a contraction with constant } k\}$. A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$, $x, y \in C$, and $F(T)$ denote the set of fixed points of T; that is, $F(T) = \{x \in C : x = Tx\}$.

Recall that a (possibly multivalued) operator $A \subset E \times E$ with the domain $D(A)$ and the range $R(A)$ in E is *accretive* if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists a $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \ge 0$. (Here J is the normalized duality mapping.) An accretive operator A is said to satisfy the *range condition* if $\overline{D(A)} \subset R(I + rA)$ for all $r > 0$. An accretive operator A is m-accretive if $R(I + rA) = E$ for each $r > 0$. If A is an accretive operator which satisfies the range condition, then we can define, for each $r > 0$ a mapping $J_r : R(I+rA) \to D(A)$ defined by $J_r = (I+rA)^{-1}$, which is called the *resolvent* of A. We know that J_r is nonexpansive and $F(J_r) = A^{-1}0$ for all $r > 0$, where $A^{-1}0 = \{z \in D(A) : 0 \in Az\}$ is the set of zeros of A. If $A^{-1}0 \neq \emptyset$, then the inclusion $0 \in Az$ is solvable.

We consider an iterative scheme: for resolvent J_{r_n} of m-accretive operator A,

$$x_{n+1} = J_{r_n} x_n, \quad n \geq 0, \tag{1.1}$$

where the initial guess $x_0 \in E$ is chosen arbitrarily. The iterative scheme (1.1) has extensively been studied over the last forty years for constructions of zeros of accretive operators (see, e.g., [1–11] and the references contained therein).

Kim and Xu [12] in 2005 and Xu [13] in 2006 provided a simpler modification of Mann iterative scheme in either a uniformly smooth Banach space ([12]) or a reflexive Banach space having a weakly sequentially continuous duality mapping ([13]) for finding a zero of an m-accretive operator A as follows: for resolvent J_{r_n} of A, $u \in \overline{D(A)}$ and $\{\alpha_n\} \subset [0,1]$,

$$\begin{aligned} x_0 &= x \in E, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n) J_{r_n} x_n, \quad n \geq 0, \end{aligned} \tag{1.2}$$

(see also [14, 15]). They proved that the sequence $\{x_n\}$ generated by (1.2) converges to a zero of an m-accretive operator A under the control conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$, or, equivalently, $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$;

(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(R1) $r_n \geq \varepsilon$ for some $\varepsilon > 0$ and for all $n \geq 0$ and $\sum_{n=0}^{\infty} |1 - r_n/r_{n+1}| < \infty$; or

(R2) $r_n \geq \varepsilon$ for some $\varepsilon > 0$ and for all $n \geq 0$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

In 2007, Aoyama et al. [16] studied the following iterative scheme in a uniformly convex Banach space having a uniformly Gâteaux differentiable norm: for resolvent J_{r_n} of an accretive operator A such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$ and $\{\alpha_n\} \subset [0,1]$,

$$\begin{aligned} x_0 &= x \in C, \\ x_{n+1} &= \alpha_n x + (1-\alpha_n) J_{r_n} x_n, \quad n \geq 0. \end{aligned} \tag{1.3}$$

They proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to a zero of A under the conditions (C1), (C2), and (C3) and the condition (R2) on $\{r_n\}$. In 2006, under the conditions (C1), (C2), and (C3) on $\{\alpha_n\}$ and the condition (R2)$'\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$ on $\{r_n\}$, Nakajo [17] also studied the strong convergence of iterative scheme (1.3) in the same Banach space. In case that C is a compact convex subset of a Banach space having a uniformly Gâteaux differentiable norm, Miyake and Takahashi [18] proved the convergence of the sequence $\{x_n\}$ generated by (1.3) to a zero of an accretive operator A such that $D(A) \subset C \subset \bigcap_{r>0} R(I + rA)$ under conditions (C1) and (C2) and $\lim_{n\to\infty} r_n = \infty$.

In 2007, Qin and Su [19] also considered the following iterative scheme in either a uniformly smooth Banach space or a reflexive Banach space having a weakly sequentially continuous duality mapping, which is a simpler modification of the iterative scheme (1.2): for resolvent J_{r_n} of an m-accretive operator A, $u \in \overline{D(A)}$ and $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$,

$$\begin{aligned} x_0 &= x \in E, \\ y_n &= \beta_n x_n + (1-\beta_n) J_{r_n} x_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n) y_n, \quad n \geq 0. \end{aligned} \tag{1.4}$$

They proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a zero of an m-accretive operator A under the conditions (C1), (C2), and (C3) on $\{\alpha_n\}$ and the condition

(B1) for $n \geq 0$, $\beta_n \in [0,a)$ for some $a \in (0,1)$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$

on $\{\beta_n\}$, and the condition (R2) on $\{r_n\}$.

On the other hand, as the viscosity iteration method ([20, 21]), in 2006 and 2008, Chen and Zhu [22, 23] considered the following iterative scheme: for resolvent J_{r_n} of an m-accretive operator A, $f \in \Sigma_C$ $(C = \overline{D(A)})$ and $\{\alpha_n\} \subset [0,1]$,

$$\begin{aligned} x_0 &= x \in C, \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n) J_{r_n} x_n, \quad n \geq 0. \end{aligned} \tag{1.5}$$

Under conditions (C1), (C2), and (C3) on $\{\alpha_n\}$ and (R2) on $\{r_n\}$, they showed in either a reflexive Banach space having a weakly sequentially continuous duality mapping [22] or a uniformly smooth Banach space [23] that the sequence $\{x_n\}$ generated by (1.5) converges strongly to a zero of A, which is a solution of a certain variational inequality. By using the following conditions:

(P1) $\lim_{n\to\infty}(\alpha_n/\alpha_{n-1}) = 1$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty}(1/\alpha_n)(1 - r_{n-1}/r_n) = 0$,

(P2) $r_n \geq \varepsilon$ (for some positive ε),

in 2006, Maingé [24] also studied in a uniformly smooth Banach space the strong convergence of the sequence $\{x_n\}$ generated by (1.5) to the unique fixed point of $Q \circ f$, where $Q: E \to A^{-1}0$ is the sunny nonexpansive retraction.

Very recently, Jung [25] also studied the following iterative scheme as the viscosity iteration method: for resolvent J_{r_n} of an accretive operator A such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$, $f \in \Sigma_C$ and $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$,

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \beta_n x_n + (1-\beta_n) J_{r_n} x_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n) y_n, \quad n \geq 0, \end{aligned} \tag{1.6}$$

and proved under the conditions (C1), (C2), and (C3) (or

(C4) $|\alpha_{n+1} - \alpha_n| \leq o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ (the perturbed control condition)) on $\{\alpha_n\}$, (B2) on $\{\beta_n\}$, and (R2) on $\{r_n\}$ that the sequence $\{x_n\}$ generated by (1.6) converges strongly to a zero of A, which is a solution of a certain variational inequality, in a reflexive Banach space having a uniformly Gâteaux differentiable norm such that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings.

Question. Are conditions (C1) $\lim_{n\to\infty}\alpha_n = 0$ and (C2) $\sum_{n=0}^{\infty}\alpha_n = \infty$ sufficient for the strong convergence of iterative schemes (1.2)–(1.6) for all resolvent J_{r_n} with different condition from (R1) or (R2) on $\{r_n\}$?

In this paper, motivated by the above-mentioned results, we consider the composite iterative scheme (1.6) as the viscosity iteration method and prove under certain different control conditions on $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ in reflexive Banach spaces that the sequence $\{x_n\}$ generated by (1.6) converges strongly to a zero of A, which is a solution of a certain variational inequality. Moreover, we study the strong convergence of the iterative scheme (1.6) with the weakly contractive mapping instead of the contraction f. By removing the condition (C3) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ (or (C4) $|\alpha_{n+1}-\alpha_n| \le o(\alpha_{n+1})+\sigma_n$, $\sum_{n=0}^{\infty}\sigma_n < \infty$) on $\{\alpha_n\}$, the condition $\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n| < \infty$ in (B1) on $\{\beta_n\}$, the condition $\sum_{n=0}^{\infty}|1-r_n/r_{n+1}| < \infty$ in (R1) and the condition $\sum_{n=0}^{\infty}|r_{n+1}-r_n| < \infty$ in (R2), the main results improve and develop the corresponding results of Aoyama et al. [16], Chen and Zhu [22, 23], Jung [25], Kim, and Xu [12], Maingé [24], Nakajo [17], Qin and Su [19] and Xu [13]. Consequently, we give an affirmative answer to the above question. Our results also complement the corresponding results of Benavides et al. [14] and Kamimura and Takahashi [15].

2. Preliminaries and Lemmas

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$) will denote strong (resp., weak) convergence of the sequence $\{x_n\}$ to x.

By a gauge function we mean a continuous strictly increasing function φ defined on $\mathbb{R}^+ := [0,\infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty}\varphi(r) = \infty$. The mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \left\{f \in E^* : \langle x, f\rangle = \|x\|\|f\|,\ \|f\| = \varphi(\|x\|)\right\} \quad \forall\, x \in E \tag{2.1}$$

is called the *duality mapping* with gauge function φ. In particular, the duality mapping with gauge function $\varphi(t) = t$ denoted by J is referred to as the *normalized duality mapping*. The following property of duality mapping is well known ([26]):

$$J_\varphi(\lambda x) = \operatorname{sign}\lambda\left(\frac{\varphi(|\lambda|\cdot\|x\|)}{\|x\|}\right)J(x) \quad \forall\, x \in E\setminus\{0\},\ \lambda \in \mathbb{R}, \tag{2.2}$$

where $\mathbb{R}$ is the set of all real numbers; in particular, $J(-x) = -J(x)$ for all $x \in E$.

We say that a Banach space E has a weakly sequential continuous duality mapping if there exists a gauge function φ such that the duality mapping J_φ is single valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $J_\varphi(x_n) \overset{*}{\rightharpoonup} J_\varphi(x)$. For example, every l^p space $(1 < p < \infty)$ has a weakly sequentially continuous duality mapping with gauge function $\varphi(t) = t^{p-1}$.

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{2.3}$$

exists for each $x,\ y$ in its unit sphere $U=\{x\in E:\|x\|=1\}$. The norm is said to be *uniformly Gâteaux differentiable* if for $y\in U$, the limit is attained uniformly for $x\in U$. The space E is said to have a *uniformly Fréchet differentiable norm* (and E is said to be *uniformly smooth*) if the limit in (2.3) is attained uniformly for $(x,y)\in U\times U$. It is known that E is smooth if and only if the normalized duality mapping J is single-valued. Also, it is well known that if E has a uniformly Gâteaux differentiable norm, then J is norm to weak* uniformly continuous on each bounded subsets of E.

Let C be a nonempty closed convex subset of E. C is said to have the *fixed point property* for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C has a fixed point in D. Let D be a subset of C. Then $Q:C\to D$ is called a *retraction* from C onto D if $Qx=x$ for all $x\in D$. A retraction $Q:C\to D$ is said to be *sunny* if $Q(Qx+t(x-Qx))=Qx$ for all $x\in C$ and $t\geq 0$ whenever $x+t(x-Qx)\in C$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D, for more details, see [27]. In a smooth Banach space E, it is known [27, page 48] that $Q:C\to D$ is a sunny nonexpansive retraction if and only if the following condition holds:

$$\langle x-Qx, J(z-Qx)\rangle \leq 0,\quad x\in C,\ z\in D. \tag{2.4}$$

(Note that this fact still holds by (2.2) if the normalized duality mapping J is replaced by a general duality mapping J_φ with gauge function φ.)

We need the following lemmas for the proof of our main results. We refer [26] for Lemma 1. Lemma 2 was found in [28] and Lemma 3 is essentially Lemma 2 of [29].

Lemma 1. *Let E be a real Banach space and φ a continuous strictly increasing function on $\mathbb{R}^+$ such that $\varphi(0)=0$ and $\lim_{r\to\infty}\varphi(r)=\infty$. Define*

$$\Phi(t)=\int_0^t\varphi(\tau)d\tau\quad \forall\ t\in\mathbb{R}^+. \tag{2.5}$$

Then the following inequality holds:

$$\Phi(\|x+y\|)\leq\Phi(\|x\|)+\langle y, j_\varphi(x+y)\rangle\quad \forall\ x,y\in E, \tag{2.6}$$

where $j_\varphi(x+y)\in J_\varphi(x+y)$. In particular, if E is smooth, then one has

$$\|x+y\|^2\leq\|x\|^2+2\langle y, J(x+y)\rangle\quad \forall\, x,y\in E. \tag{2.7}$$

Lemma 2. *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and $\{\gamma_n\}$ a sequence in $[0,1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1. \tag{2.8}$$

Suppose that $x_{n+1} = \gamma_n x_n + (1-\gamma_n) z_n$, $n \geq 0$, and

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{2.9}$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 3. *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-\lambda_n) s_n + \lambda_n \delta_n + \gamma_n, \quad n \geq 0, \tag{2.10}$$

where $\{\lambda_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} \lambda_n \delta_n < \infty$;

(iii) $\gamma_n \geq 0 \quad (n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 4 (demiclosedness principle). *Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping, C a nonempty closed convex subset of E, and $T : C \to E$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping; that is, $x_n \rightharpoonup x$ in E and $(I-T)x_n \to y$ imply that $x \in C$ and $(I-T)x = y$.*

We need the resolvent identity (see [26], where more details on accretive operators can be founded).

Lemma 5 (resolvent identity). *For $\lambda > 0$, $\mu > 0$ and $x \in E$,*

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda x \right). \tag{2.11}$$

Recall that a mapping $g : C \to C$ is said to be *weakly contractive* [30, 31] if

$$\|g(x) - g(y)\| \leq \|x - y\| - \psi(\|x - y\|), \quad \forall\, x,\ y \in C, \tag{2.12}$$

where $\psi : [0,+\infty) \to [0,+\infty)$ is a continuous and strictly increasing function such that ψ is positive on $(0,\infty)$ and $\psi(0) = 0$. As a special case, if $\psi(t) = (1-k)t$ for $t \in [0,+\infty)$, where $k \in (0,1)$, then the weakly contractive mapping g is a contraction with constant k. Rhoades [32] obtained the following result for weakly contractive mapping (see also [31]).

Lemma 6. *Let (X, d) be a complete metric space and g a weakly contractive mapping on X. Then g has a unique fixed point p in X.*

The following lemma was given in [33, 34].

Lemma 7. *Let $\{s_n\}$ and $\{\gamma_n\}$ be two sequences of nonnegative real numbers and $\{\lambda_n\}$ a sequence of positive numbers satisfying the conditions:*

(i) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(ii) $\lim_{n\to\infty}(\gamma_n/\lambda_n) = 0$.

Let the recursive inequality

$$s_{n+1} \leq s_n - \lambda_n \psi(s_n) + \gamma_n, \quad n \geq 0, \tag{2.13}$$

be given, where $\psi(t)$ is a continuous and strict increasing function on $[0, +\infty)$ with $\psi(0) = 0$. Then $\lim_{n\to\infty} s_n = 0$.

3. Main Results

Now, we study the strong convergence results for the composite iterative scheme in reflexive Banach spaces.

For $T : C \to C$ a nonexpansive mapping, $t \in (0,1)$ and $f \in \Sigma_C$, $tf + (1-t)T : C \to C$ defines a contraction. Thus, by the Banach contraction principle, there exists a unique fixed point x_t^f satisfying

$$x_t^f = tf(x_t) + (1-t)Tx_t^f. \tag{3.1}$$

For simplicity we will write x_t for x_t^f provided that no confusion occurs.

The following result for the existence of $\overline{q}$ which is a solution of a variational inequality

$$\langle (I-f)(\overline{q}), J(\overline{q}-p) \rangle \leq 0, \quad f \in \Sigma_C,\ p \in F(T) \tag{3.2}$$

was obtained by Jung [35–37] (see also [21, 22]).

Theorem J. *Let E be a Banach space, C a nonempty closed convex subset of E, and T nonexpansive mapping from C into itself with $F(T) \neq \emptyset$. If one of the following assumptions holds:*

(H1)*E is a reflexive Banach space, the norm of E is uniformly Gâteaux differentiable, and every weakly compact convex subset of E has the fixed point property for nonexpansive mappings;*

(H2)*E is a reflexive and strictly convex Banach space and the norm of E is uniformly Gâteaux differentiable;*

(H3)*E is a reflexive Banach space having a weakly sequentially continuous duality mapping J_φ with gauge function φ;*
then $\{x_t\}$ defined by (3.1) *converges strongly to a point in $F(T)$. If one defines $Q : \Sigma_C \to F(T)$ by*

$$Q(f) = \overline{q} := \lim_{t\to 0} x_t, \quad f \in \Sigma_C, \tag{3.3}$$

then $\overline{q}$ is the unique solution of the variational inequality

$$\langle (I-f)(\overline{q}), J(\overline{q}-p)\rangle \le 0, \quad f \in \Sigma_C,\ p \in F(T). \tag{3.4}$$

Remark 8. (1) In the case when assumption (H3) in Theorem J holds, (3.4) still holds by (2.2) if the normalized duality mapping J is replaced by a general duality mapping J_φ with gauge function φ, that is,

$$\langle (I-f)(\overline{q}), J_\varphi(\overline{q}-p)\rangle \le 0, \quad f \in \Sigma_C,\ p \in F(T). \tag{3.5}$$

(2) In Theorem J, if $f(x) = u$, $x \in C$, is a constant, then it follows from (2.4) that (3.3) is reduced to a sunny nonexpansive retraction from C onto $F(T)$,

$$\langle Qu - u, J(Qu-p)\rangle \le 0, \quad u \in C,\ p \in F(T), \tag{3.6}$$

(see also [38]). Namely, $F(T)$ is a sunny nonexpansive retract of C.

Using Theorem J, we have the following result.

Theorem 1. *Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex subset of E and $A \subset E \times E$ an accretive operator in E such that $F := A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Let $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset \mathbb{R}^+$ be sequences which satisfy the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(B) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n \le a < 1$ *for some constant* $a \in (0,1)$;

(R) $r_n \ge \varepsilon > 0$ *for* $n \ge 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Let $f \in \Sigma_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by

$$\begin{gathered} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n) J_{r_n} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \quad n \ge 0. \end{gathered} \tag{IS}$$

Then $\{x_n\}$ converges strongly to $\overline{q} \in F$, where $\overline{q}$ is the unique solution of the variational inequality

$$\langle (I-f)(\overline{q}), J_\varphi(\overline{q}-p)\rangle \le 0, \quad f \in \Sigma_C,\ p \in F. \tag{3.7}$$

Proof. Note that the definition of the weak sequential continuity of duality mapping J_φ with gauge function φ implies that E is smooth. First, we notice that by Theorem J and Remark 8(1), there exists the unique solution $\overline{q}$ of the variational inequality

$$\langle (I-f)(\overline{q}), J_\varphi(\overline{q}-p)\rangle \le 0, \quad f \in \Sigma_C,\ p \in F, \tag{3.8}$$

where $\overline{q} = \lim_{t\to 0} x_t$ and x_t is defined by $x_t = tf(x_t) + (1-t)J_r x_t$ for each $r > 0$ and $0 < t < 1$.

We divide the proof into several steps.

Step 1. We show that $\|x_n - p\| \le \max\{\|x_0 - p\|, (1/(1-k))\|f(p)-p\|\}$ for all $n \ge 0$ and all $p \in F$. Indeed, let $p \in F$ and $d = \max\{\|x_0 - p\|, (1/(1-k))\|f(p)-p\|\}$. Noting that

$$\|y_n - p\| \le \beta_n\|x_n - p\| + (1-\beta_n)\|J_{r_n}x_n - p\| \le \|x_n - p\|, \tag{3.9}$$

we have

$$\begin{aligned}
\|x_1 - p\| &\le (1-\alpha_0)\|y_0 - p\| + \alpha_0\|f(x_0) - p\| \\
&\le (1-\alpha_0)\|x_0 - p\| + \alpha_0(\|f(x_0) - f(p)\| + \|f(p) - p\|) \\
&\le (1-(1-k)\alpha_0)\|x_0 - p\| + \alpha_0\|f(p) - p\| \\
&\le (1-(1-k)\alpha_0)d + \alpha_0(1-k)d = d.
\end{aligned} \tag{3.10}$$

Using an induction, we obtain $\|x_{n+1} - p\| \le d$. Hence $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{J_{r_n}x_n\}$, and $\{f(x_n)\}$.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. To this end, set $\gamma_n = (1-\alpha_n)\beta_n,\ n \ge 0$. Then it follow from (C1) and (B) that

$$0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1. \tag{3.11}$$

Define

$$x_{n+1} = \gamma_n x_n + (1-\gamma_n)z_n. \tag{3.12}$$

Observe with the resolvent identity (2.11) in Lemma 5 that

$$\begin{aligned}
z_{n+1} - z_n &= \frac{x_{n+2} - \gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} - \frac{x_{n+1} - \gamma_n x_n}{1-\gamma_n} \\
&= \frac{\alpha_{n+1}f(x_{n+1}) + (1-\alpha_{n+1})y_{n+1} - \gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} - \frac{\alpha_n f(x_n) + (1-\alpha_n)y_n - \gamma_n x_n}{1-\gamma_n}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\alpha_{n+1} f(x_{n+1})}{1-\gamma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\gamma_n}\right) - \frac{(1-\alpha_n)\left[\beta_n x_n + (1-\beta_n) J_{r_n} x_n\right] - \gamma_n x_n}{1-\gamma_n} \\
&\quad + \frac{(1-\alpha_{n+1})\left[\beta_{n+1} x_{n+1} + (1-\beta_{n+1}) J_{r_{n+1}} x_{n+1}\right] - \gamma_{n+1} x_{n+1}}{1-\gamma_{n+1}} \\
&= \left(\frac{\alpha_{n+1} f(x_{n+1})}{1-\gamma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\gamma_n}\right) + \frac{(1-\alpha_{n+1})(1-\beta_{n+1}) J_{r_{n+1}} x_{n+1}}{1-\gamma_{n+1}} \\
&\quad - \frac{(1-\alpha_n)(1-\beta_n) J_{r_n} x_n}{1-\gamma_n} \\
&= \left(\frac{\alpha_{n+1} f(x_{n+1})}{1-\gamma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\gamma_n}\right) + \left(J_{r_{n+1}} x_{n+1} - J_{r_n} x_n\right) - \frac{\alpha_{n+1}}{1-\gamma_{n+1}} J_{r_{n+1}} x_{n+1} + \frac{\alpha_n}{1-\gamma_n} J_{r_n} x_n \\
&= \left(\frac{\alpha_{n+1} f(x_{n+1})}{1-\gamma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\gamma_n}\right) + \left(J_{r_{n+1}} x_{n+1} - J_{r_{n+1}} x_n\right) \\
&\quad + \left(J_{r_n}\left(\frac{r_n}{r_{n+1}} x_n + \left(1 - \frac{r_n}{r_{n+1}}\right) J_{r_{n+1}} x_n\right) - J_{r_n} x_n\right) - \frac{\alpha_{n+1}}{1-\gamma_{n+1}} J_{r_{n+1}} x_{n+1} + \frac{\alpha_n}{1-\gamma_n} J_{r_n} x_n.
\end{aligned}
\tag{3.13}
$$

It follows from (3.13) that

$$
\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\le \frac{\alpha_{n+1}}{1-\gamma_{n+1}} \left(\|f(x_{n+1})\| + \|J_{r_{n+1}} x_{n+1}\|\right) + \frac{\alpha_n}{1-\gamma_n} \left(\|f(x_n)\| + \|J_{r_n} x_n\|\right) \\
&\quad + \left\|\left(\frac{r_n}{r_{n+1}} x_n + \left(1 - \frac{r_n}{r_{n+1}}\right) J_{r_{n+1}} x_n\right) - x_n\right\| \\
&= \frac{\alpha_{n+1}}{1-\gamma_{n+1}} \left(\|f(x_{n+1})\| + \|J_{r_{n+1}} x_{n+1}\|\right) + \frac{\alpha_n}{1-\gamma_n} \left(\|f(x_n)\| + \|J_{r_n} x_n\|\right) \\
&\quad + \left|1 - \frac{r_n}{r_{n+1}}\right| L,
\end{aligned}
\tag{3.14}
$$

where $\|J_{r_{n+1}} x_n - x_n\| \le L$. Since $\{f(x_n)\}$ and $\{J_{r_n} x_n\}$ are bounded, by (C1), (R), (3.11), and (3.14) we obtain that

$$
\limsup_{n\to\infty} \left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\right) \le 0. \tag{3.15}
$$

Hence by Lemma 2, we have

$$
\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.16}
$$

It then follows from (3.11), (3.12), and (3.16) that

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\gamma_n) \|z_n - x_n\| = 0. \tag{3.17}
$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0$. Indeed, by (IS) $\|y_n - J_{r_n}x_n\| = \beta_n\|x_n - J_{r_n}x_n\|$, and hence we have

$$\begin{aligned}\|x_n - J_{r_n}x_n\| &\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - y_n\right\| + \left\|y_n - J_{r_n}x_n\right\| \\ &= \|x_n - x_{n+1}\| + \left\|x_{n+1} - y_n\right\| + \beta_n\|x_n - J_{r_n}x_n\|.\end{aligned} \tag{3.18}$$

Simplifying it and using Step 2, we have

$$\begin{aligned}(1-a)\|x_n - J_{r_n}x_n\| &\le \left(1-\beta_n\right)\|x_n - J_{r_n}x_n\| \\ &\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - y_n\right\| \\ &= \|x_n - x_{n+1}\| + \alpha_n\left\|f(x_n) - y_n\right\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty).\end{aligned} \tag{3.19}$$

This implies that

$$\|x_n - J_{r_n}x_n\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{3.20}$$

Step 4. We show that $\|x_n - J_r x_n\| \to 0$ for a fixed number r such that $\varepsilon > r > 0$. Indeed, from the resolvent identity (2.11) in Lemma 5, we obtain

$$\begin{aligned}\|J_{r_n}x_n - J_r x_n\| &= \left\|J_r\left(\frac{r}{r_n}x_n + \left(1 - \frac{r}{r_n}\right)J_{r_n}x_n\right) - J_r x_n\right\| \\ &\le \left(1 - \frac{r}{r_n}\right)\|x_n - J_{r_n}x_n\| \le \|x_n - J_{r_n}x_n\|.\end{aligned} \tag{3.21}$$

Therefore, from (3.21) we have

$$\begin{aligned}\|x_n - J_r x_n\| &\le \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_r x_n\| \\ &\le \|x_n - J_{r_n}x_n\| + \|x_n - J_{r_n}x_n\| = 2\|x_n - J_{r_n}x_n\|.\end{aligned} \tag{3.22}$$

Hence by Step 3 we obtain $\|x_n - J_{r_n}x_n\| \to 0$.

Step 5. We show that $\limsup_{n\to\infty}\langle (I-f)(\overline{q}), J_\varphi(\overline{q} - x_n)\rangle \le 0$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup p \in E$ and

$$\limsup_{n\to\infty}\left\langle (I-f)(\overline{q}), J_\varphi(\overline{q} - x_n)\right\rangle = \lim_{j\to\infty}\left\langle (I-f)(\overline{q}), J_\varphi\left(\overline{q} - x_{n_j}\right)\right\rangle. \tag{3.23}$$

From Step 4. it follows that

$$\left\|J_r x_{n_j} - x_{n_i}\right\| \longrightarrow 0 \quad (\text{as } j \longrightarrow \infty). \tag{3.24}$$

By Lemma 4, we have $p = J_r p$ for each $r \in \mathbb{R}^+$ and so $p \in F$. Thus by the weakly sequentially continuity of the duality mapping J_φ and (3.4), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle (I-f)(\overline{q}), J_\varphi(\overline{q}-x_n)\rangle &= \lim_{j\to\infty}\left\langle (I-f)(\overline{q}), J_\varphi\left(\overline{q}-x_{n_j}\right)\right\rangle \\ &= \langle (I-f)(\overline{q}), J_\varphi(\overline{q}-p)\rangle \le 0.\end{aligned} \tag{3.25}$$

Step 6. We show that $\lim_{n\to\infty}\|x_n-\overline{q}\| = 0$. By using (IS), we have

$$\begin{aligned}x_{n+1}-\overline{q} &= \alpha_n(f(x_n)-\overline{q}) + (1-\alpha_n)(y_n-\overline{q}) \\ &= \alpha_n(f(x_n)-f(\overline{q})) + (1-\alpha_n)(y_n-\overline{q}) + \alpha_n(f(\overline{q})-\overline{q}).\end{aligned} \tag{3.26}$$

As a consequence, since Φ is an increasing convex function with $\Phi(0) = 0$, by applying Lemma 1, we obtain from (3.9)

$$\begin{aligned}\Phi(\|x_{n+1}-\overline{q}\|) &\le \Phi(\|\alpha_n(f(x_n)-f(\overline{q})) + (1-\alpha_n)(y_n-\overline{q})\|) + \alpha_n\langle f(\overline{q})-\overline{q}, J_\varphi(x_{n+1}-\overline{q})\rangle \\ &\le \Phi(k\alpha_n\|x_n-\overline{q}\| + (1-\alpha_n)\|y_n-\overline{q}\|) + \alpha_n\langle f(\overline{q})-\overline{q}, J_\varphi(x_{n+1}-\overline{q})\rangle \\ &\le \Phi((1-(1-k)\alpha_n)\|x_n-\overline{q}\|) + \alpha_n\langle f(\overline{q})-\overline{q}, J_\varphi(x_{n+1}-\overline{q})\rangle \\ &\le (1-(1-k)\alpha_n)\Phi(\|x_n-\overline{q}\|) + \alpha_n\langle (I-f)(\overline{q}), J_\varphi(\overline{q}-x_{n+1})\rangle.\end{aligned} \tag{3.27}$$

Put

$$\lambda_n = (1-k)\alpha_n \qquad \delta_n = \frac{1}{1-k}\langle (I-f)(\overline{q}), J_\varphi(\overline{q}-x_{n+1})\rangle. \tag{3.28}$$

From (C1), (C2), and Step 5, it follows that $\lambda_n \to 0$, $\sum_{n=0}^{\infty}\lambda_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Since (3.27) reduces to

$$\Phi(\|x_{n+1}-\overline{q}\|) \le (1-\lambda_n)\Phi(\|x_n-\overline{q}\|) + \lambda_n\delta_n, \tag{3.29}$$

from Lemma 3, we conclude that $\lim_{n\to\infty}\Phi(\|x_n-\overline{q}\|) = 0$, and thus $\lim_{n\to\infty}x_n = \overline{q}$. This completes the proof. □

Theorem 2. *Let E be a Banach space, C a nonempty closed convex subset of E, and $A \subset E\times E$ be an accretive operator in E such that $F := A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0}R(I+rA)$. Let $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset \mathbb{R}^+$ be sequences which satisfy the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(B) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n \le a < 1$ *for some constant* $a \in (0,1)$;

(R) $r_n \ge \varepsilon > 0$ *for* $n \ge 0$ *and* $\lim_{n\to\infty}|r_{n+1}-r_n| = 0$.

Let $f \in \Sigma_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by (IS). If one of the following assumptions holds:

(H1)*E is a reflexive Banach space, the norm of E is uniformly Gâteaux differentiable, and every weakly compact convex subset of E has the fixed point property for nonexpansive mappings;*

(H2)*E is a reflexive and strictly convex Banach space and the norm of E is uniformly Gâteaux differentiable;*

then $\{x_n\}$ converges strongly to $\overline{q} \in F$, where $\overline{q}$ is the unique solution of the variational inequality

$$\langle (I-f)(\overline{q}), J(\overline{q}-p)\rangle \le 0, \quad f \in \Sigma_C,\ p \in F. \tag{3.30}$$

Proof. We also notice that by Theorem J, there exists the unique solution $\overline{q}$ of the variational inequality

$$\langle (I-f)(\overline{q}), J(\overline{q}-p)\rangle \le 0, \quad f \in \Sigma_C,\ p \in F, \tag{3.31}$$

where $\overline{q} = \lim_{t\to 0} x_t$ and x_t is defined by $x_t = tf(x_t) + (1-t)J_r x_t$ for each $r > 0$ and $0 < t < 1$. We only give proofs of differences.

Now, by the proof of Theorem 1, we also have the following.

Step 1. $\|x_n - p\| \le \max\{\|x_0 - p\|, (1/(1-k))\|f(p) - p\|\}$ for all $n \ge 0$ and all $p \in F$ and so $\{x_n\}$, $\{y_n\}$, $\{J_{r_n} x_n\}$, and $\{f(x_n)\}$ are bounded.

Step 2. $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 3. $\lim_{n\to\infty}\|x_n - J_{r_n} x_n\| = 0$.

Step 4. $\|x_n - J_r x_n\| \to 0$ for a fixed number r such that $\varepsilon > r > 0$.

Step 5. We show that $\limsup_{n\to\infty}\langle (I-f)(\overline{q}), J(\overline{q}-x_n)\rangle \le 0$. To prove this, let a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ be such that $x_{n_j} \rightharpoonup p$ for some $p \in E$ and

$$\limsup_{n\to\infty}\langle (I-f)(\overline{q}), J(\overline{q}-x_n)\rangle = \lim_{j\to\infty}\Big\langle (I-f)(\overline{q}), J\Big(\overline{q}-x_{n_j}\Big)\Big\rangle. \tag{3.32}$$

Now let x_t be defined by $x_t = tf(x_t) + (1-t)J_r x_t$ for each $r > 0$ and $0 < t < 1$. Then

$$x_t - x_n = (1-t)(J_r x_t - x_n) + t\big(f(x_t) - x_n\big). \tag{3.33}$$

Applying Lemma 1, we have

$$\|x_t - x_n\|^2 \le (1-t)^2\|J_r x_t - x_n\|^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle. \tag{3.34}$$

Putting

$$a_j(t) = (1-t)^2 \left\| J_r x_{n_j} - x_{n_j} \right\| \left(2\left\| x_t - x_{n_j} \right\| + \left\| J_r x_{n_j} - x_{n_j} \right\| \right) \longrightarrow 0 \quad (j \to \infty) \tag{3.35}$$

by Step 4 and using Lemma 1, we obtain

$$\begin{aligned} \left\| x_t - x_{n_j} \right\|^2 &\le (1-t)^2 \left\| J_r x_t - x_{n_j} \right\|^2 + 2t\left\langle f(x_t) - x_{n_j}, J\left(x_t - x_{n_j}\right) \right\rangle \\ &\le (1-t)^2 \left(\left\| J_r x_t - J_r x_{n_j} \right\| + \left\| J_r x_{n_j} - x_{n_j} \right\| \right)^2 \\ &\quad + 2t\left\langle f(x_t) - x_t, J\left(x_t - x_{n_j}\right) \right\rangle + 2t\left\| x_t - x_{n_j} \right\|^2 \\ &\le (1-t)^2 \left\| x_t - x_{n_j} \right\|^2 + a_j(t) \\ &\quad + 2t\left\langle f(x_t) - x_t, J\left(x_t - x_{n_j}\right) \right\rangle + 2t\left\| x_t - x_{n_j} \right\|^2. \end{aligned} \tag{3.36}$$

The last inequality implies

$$\left\langle x_t - f(x_t), J\left(x_t - x_{n_j}\right) \right\rangle \le \frac{t}{2} \left\| x_t - x_{n_j} \right\|^2 + \frac{1}{2t} a_j(t). \tag{3.37}$$

It follows that

$$\lim_{j \to \infty} \left\langle x_t - f(x_t), J\left(x_t - x_{n_j}\right) \right\rangle \le \frac{t}{2} M, \tag{3.38}$$

where $M > 0$ is a constant such that $M \ge \|x_t - x_n\|^2$ for all $n \ge 0$ and $t \in (0,1)$. Taking the $\limsup$ as $t \to 0$ in (3.38) and noticing the fact that the two limits are interchangeable due to the fact that J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak* topology of E^*, we have

$$\lim_{j \to \infty} \left\langle (I - f)(\overline{q}), J\left(\overline{q} - x_{n_j}\right) \right\rangle \le 0. \tag{3.39}$$

Step 6. We show that $\lim_{n \to \infty} \|x_n - \overline{q}\| = 0$. By using (IS), we have

$$x_{n+1} - \overline{q} = \alpha_n \left(f(x_n) - \overline{q} \right) + (1 - \alpha_n)\left(y_n - \overline{q} \right). \tag{3.40}$$

Applying Lemma 1, we obtain

$$\begin{aligned}
\|x_{n+1}-\overline{q}\|^2 &\le (1-\alpha_n)^2\|y_n-\overline{q}\|^2+2\alpha_n\langle f(x_n)-\overline{q},J(x_{n+1}-\overline{q})\rangle \\
&\le (1-\alpha_n)^2\|x_n-\overline{q}\|^2+2\alpha_n\langle f(x_n)-f(\overline{q}),J(x_{n+1}-\overline{q})\rangle \\
&\quad +2\alpha_n\langle f(\overline{q})-\overline{q},J(x_{n+1}-\overline{q})\rangle \\
&\le (1-\alpha_n)^2\|x_n-\overline{q}\|^2+2k\alpha_n\|x_n-\overline{q}\|\|x_{n+1}-\overline{q}\| \\
&\quad +2\alpha_n\langle f(\overline{q})-\overline{q},J(x_{n+1}-\overline{q})\rangle \\
&\le (1-\alpha_n)^2\|x_n-\overline{q}\|^2+k\alpha_n\left(\|x_n-\overline{q}\|^2+\|x_{n+1}-\overline{q}\|^2\right) \\
&\quad +2\alpha_n\langle f(\overline{q})-q,J(x_{n+1}-\overline{q})\rangle.
\end{aligned} \tag{3.41}$$

It then follows that

$$\begin{aligned}
\|x_{n+1}-\overline{q}\|^2 &\le \frac{1-(2-k)\alpha_n+\alpha_n^2}{1-k\alpha_n}\|x_n-\overline{q}\|^2+\frac{2\alpha_n}{1-k\alpha_n}\langle (I-f)(\overline{q}),J(\overline{q}-x_{n+1})\rangle \\
&\le \frac{1-(2-k)\alpha_n}{1-k\alpha_n}\|x_n-\overline{q}\|^2+\frac{\alpha_n^2}{1-k\alpha_n}M \\
&\quad +\frac{2\alpha_n}{1-k\alpha_n}\langle (I-f)(\overline{q}),J(\overline{q}-x_{n+1})\rangle,
\end{aligned} \tag{3.42}$$

where $M=\sup_{n\ge 0}\|x_n-\overline{q}\|^2$. Put

$$\lambda_n=\frac{2(1-k)\alpha_n}{1-k\alpha_n} \qquad \delta_n=\frac{M\alpha_n}{2(1-k)}+\frac{1}{1-k}\langle (I-f)(\overline{q}),J(\overline{q}-x_{n+1})\rangle. \tag{3.43}$$

From (C1), (C2), and Step 5, it follows that $\lambda_n \to 0$, $\sum_{n=0}^{\infty}\lambda_n=\infty$ and $\limsup_{n\to\infty}\delta_n\le 0$. Since (3.42) reduces to

$$\|x_{n+1}-\overline{q}\|^2\le(1-\lambda_n)\|x_n-\overline{q}\|^2+\lambda_n\delta_n, \tag{3.44}$$

from Lemma 3, we conclude that $\lim_{n\to\infty}\|x_n-\overline{q}\|=0$. This completes the proof. □

Corollary 1. *Let E be a uniformly smooth Banach space. Let A be an m-accretive operator in E such that $C=\overline{D(A)}$ is convex. Let $F:=A^{-1}(0)\neq\emptyset$. Let $f\in\Sigma_C$ and $x_0\in C$ be chosen arbitrarily. Let the sequences $\{\alpha_n\}$, $\{\beta_n\}\subset[0,1]$ and $\{r_n\}\subset\mathbb{R}^+$ satisfy the conditions (C1), (C2), (B), and (R) in Theorem 2. Let $\{x_n\}$ be the sequence generated by (IS). Then $\{x_n\}$ converges strongly to $\overline{q}\in F$, where $\overline{q}$ is the unique solution of the variational inequality*

$$\langle (I-f)(\overline{q}),J(\overline{q}-p)\rangle\le 0,\quad f\in\Sigma_C,\ p\in F. \tag{3.45}$$

Remark 9. (1) By removing the condition $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ on $\{\alpha_n\}$, the condition $\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n|<\infty$ on $\{\beta_n\}$, and the condition $\sum_{n=0}^{\infty}|1-r_n/r_{n+1}|<\infty$ and the condition $\sum_{n=0}^{\infty}|r_{n+1}-r_n|<\infty$ on $\{r_n\}$, Theorems 1 and 2 improve the corresponding results of Aoyama et al. [16], Chen and Zhu [22, 23], Kim and Xu [12], Nakajo [17], Qin and Su [19], and Xu [13] (i.e., Theorem 4.3 in [16], Theorem 3.4 in [22], Theorem 3.2 of [23], Theorem 2 in [12], Theorem 4.2 in [17], Theorems 2.1 and 2.2 in [19] and Theorem 4.2 in [13]) in several aspects.

(2) Corollary 1 generalizes Corollary 3.2 of Jung [25]. Also Corollary 1 develops Theorem 2 of Kim and Xu [12], Theorems 2.1 and 2.2 of Qin and Su [19], and Theorem 4.2 of Xu [13] to the viscosity iteration method.

Next, we consider the viscosity iteration method with the weakly contractive mapping for zeros of accretive operators.

Theorem 3. *Let E be a Banach space, C a nonempty closed convex subset of E, and $A\subset E\times E$ an accretive operator in E such that $F:=A^{-1}0\neq\emptyset$ and $\overline{D(A)}\subset C\subset\bigcap_{r>0}R(I+rA)$. Let the sequences $\{\alpha_n\}$, $\{\beta_n\}\subset[0,1]$ and $\{r_n\}\subset\mathbb{R}^+$ satisfy the conditions (C1) (C2), (B), (R) in Theorem 1. Let $g:C\to C$ be a weakly contractive mapping with the function ψ and let $x_0\in C$ be chosen arbitrarily. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_0 &= x\in C,\\ y_n &= \beta_n x_n+(1-\beta_n)J_{r_n}x_n,\\ x_{n+1} &= \alpha_n g(x_n)+(1-\alpha_n)y_n,\quad n\geq 0. \end{aligned}\tag{3.46}$$

If one of the following assumptions holds:

(H1)*E is a reflexive Banach space, the norm of E is uniformly Gâteaux differentiable, and every weakly compact convex subset of E has the fixed point property for nonexpansive mappings;*

(H2)*E is a reflexive and strictly convex Banach space and the norm of E is uniformly Gâteaux differentiable;*

(H3)*E is a reflexive Banach space having a weakly sequentially continuous duality mapping J_φ with gauge function φ;*

then $\{x_n\}$ converges strongly to $Q(g(x^*))=x^*\in F$, where Q is the sunny nonexpansive retraction from C onto F.

Proof. It follows from Remark 8(2) that F is the sunny nonexpansive retract of C. Denote by Q the sunny nonexpansive retraction of C onto F. Then $Q\circ g$ is a weakly contractive mapping of C into itself. Indeed, for all $x,y\in C$,

$$\|Q(g(x))-Q(g(y))\|\leq\|g(x)-g(y)\|\leq\|x-y\|-\psi(\|x-y\|).\tag{3.47}$$

Lemma 6 assures, that there exists a unique element $x^*\in C$ such that $x^*=Q(g(x^*))$. Such a $x^*\in C$ is an element of F.

Now we define a iterative scheme as follows:

$$\begin{aligned} z_n &= \beta_n w_n+(1-\beta_n)J_{r_n}w_n,\\ w_{n+1} &= \alpha_n g(x^*)+(1-\alpha_n)z_n,\quad n\geq 0. \end{aligned}\tag{3.48}$$

Let $\{w_n\}$ be the sequence generated by (3.48). Then Theorems 1 and 2 with $f = g(x^*)$ a constant assures that $\{w_n\}$ converges strongly to $Q(g(x^*)) = x^*$ as $n \to \infty$. For any n, we have

$$\begin{aligned}
\|x_{n+1} - w_{n+1}\| &\le \alpha_n \left\| g(x_n) - g(x^*) \right\| + (1-\alpha_n)\|J_{r_n}x_n - J_{r_n}w_n\| \\
&\le \alpha_n \left(\left\| g(x_n) - g(w_n) \right\| + \left\| g(w_n) - g(x^*) \right\| \right) + (1-\alpha_n)\|x_n - w_n\| \\
&\le \|x_n - w_n\| - \alpha_n \psi(\|x_n - w_n\|) + \alpha_n \left(\|w_n - x^*\| - \psi(\|w_n - x^*\|) \right) \\
&\le \|x_n - w_n\| - \alpha_n \psi(\|x_n - w_n\|) + \alpha_n \|w_n - x^*\|.
\end{aligned} \tag{3.49}$$

Thus, we obtain for $s_n = \|x_n - w_n\|$ the following recursive inequality:

$$s_{n+1} \le s_n - \alpha_n \psi(s_n) + \alpha_n \|w_n - x^*\|. \tag{3.50}$$

Since $\|w_n - x^*\| \to 0$, it follows from Lemma 7 that $\lim_{n\to\infty}\|x_n - w_n\| = 0$. Hence

$$\lim_{n\to\infty} \|x_n - x^*\| \le \lim_{n\to\infty} (\|x_n - w_n\| + \|w_n - x^*\|) = 0. \tag{3.51}$$

This completes the proof. □

Remark 10. Theorem 3 also improves and develops the corresponding results of Aoyama et al. [16], Benavides et al. [14], Chen and Zhu [22, 23], Jung [25], Kim and Xu [12], Maingé [24], Nakajo [17], and Xu [13].

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