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Research Article

Mann Type Implicit Iteration Approximation for Multivalued Mappings in Banach Spaces

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Let K be a nonempty compact convex subset of a uniformly convex Banach space E and let T be a multivalued nonexpansive mapping. For the implicit iterates $x_0 \in K$, $x_n = \alpha_n x_{n-1} + (1-\alpha_n)y_n$, $y_n \in Tx_n$, $n \geq 1$. We proved that $\{x_n\}$ converges strongly to a fixed point of T under some suitable conditions. Our results extended corresponding ones and revised a gap in the work of Panyanak (2007) .

1. Introduction

Let K be a nonempty subset of a Banach space E. We will denote 2^E by the family of all subsets of E, $CB(E)$ the family of nonempty closed and bounded subsets of E, $C(E)$ the family of nonempty compact subsets of E. Let $CK(E)$ symbolize the family of nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be Hausdorff metric on $CB(E)$; that is,

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\right\}, \quad \forall A,B \in CB(E), \tag{1.1}$$

where $d(x,B) = \inf\{\|x-y\| : y \in B\}$. A multivalued mapping $T : K \to CB(E)$ is called nonexpansive (resp., contractive), if for any $x, y \in K$, there holds

$$\begin{gathered} H(Tx,Ty) \leq \|x-y\|, \\ \left(\text{resp., } H(Tx,Ty) \leq k\|x-y\|, \text{ for some } k \in (0,1)\right). \end{gathered} \tag{1.2}$$

A point x is called a fixed point of T if $x \in Tx$. In this paper, $F(T)$ stands for the fixed point set of a mapping T.

The fixed point theory of multivalued nonexpansive mappings is much more complicated and difficult than the corresponding theory of single-valued nonexpansive mappings. However, some classical fixed point theorems for single-valued nonexpansive mappings have already been extended to multivalued mappings.

In 1968, Markin [1] firstly established the nonexpansive multivalued convergence results in Hilbert space. Banach's Contraction Principle was extended to a multivalued contraction in 1969. (Below is stated in a Banach space setting.)

Theorem 1.1 (see [2]). *Let K be a nonempty closed subset of a Banach space E and $T : K \to CB(K)$ a multivalued contraction. Then T has a fixed point.*

In 1974, one breakthrough was achieved by Lim using Edelstein's method of asymptotic centers [3].

Theorem 1.2 (see Lim [3]). *Let K be a nonempty closed bounded convex subset of a uniformly convex Banach space E and $T : K \to C(E)$ a multivalued nonexpansive mapping. Then T has a fixed point.*

In 1990, Kirk and Massa [4] obtained another important result for multivalued nonexpansive mappings.

Theorem 1.3 (see Kirk and Massa [4]). *Let K be a nonempty closed bounded convex subset of a Banach space E and $T : K \to CK(E)$ a multivalued nonexpansive mapping. Suppose that the asymptotic center in E of each bounded sequence of X is nonempty and compact. Then T has a fixed point.*

In 1999, Sahu [5] obtained the strong convergence theorems of the nonexpansive type and nonself multivalued mappings for the following (1.3) iteration process:

$$x_n = t_n u + (1 - t_n) y_n, \quad n \geq 0, \tag{1.3}$$

where $y_n \in Tx_n$, $u \in K$, $t_n \in (0, 1)$ and $\lim_{n\to\infty} t_n = 0$. He proved that $\{x_n\}$ converges strongly to some fixed points of T. Xu [6] extended Theorem 1.3 to a multivalued nonexpansive nonself mapping and obtained the fixed theorem in 2001. The recent fixed point results for nonexpansive mappings can be found in [7–12] and references therein.

Recently, Panyanak [13] studied the Mann iteration (1.4) and Ishikawa iterative processes (1.5) for $x_0 \in K$ as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n, \quad n \geq 0, \tag{1.4}$$

where $\alpha_n \in [0, 1]$, $y_n \in Tx_n$, and fixed $p \in F(T)$ are such that $\|y_n - p\| \leq d(p, Tx_n)$,

$$\begin{aligned} y_n &= (1 - \beta_n) x_n + \beta_n z_n, \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n z'_n, \quad n \geq 0, \end{aligned} \tag{1.5}$$

where $\alpha_n \in [0,1]$, $\beta_n \in [0,1]$, $z_n \in Tx_n$, $z'_n \in Ty_n$, and fixed $p \in F(T)$ are such that $\|z_n - p\| \le d(p, Tx_n)$ and $\|z'_n - p\| \le d(p, Ty_n)$ and proved the strong convergence theorems for multivalued nonexpansive mappings T in Banach spaces.

In this paper, motivated by Panyanak [13] and the previous results, we will study the following iteration process (1.6). Let K be a nonempty convex subset of E, $\alpha_n \in [0,1]$,

$$\begin{gathered} x_0 \in K, \\ x_n = \alpha_n x_{n-1} + (1-\alpha_n) y_n, \quad y_n \in Tx_n,\ n \ge 1, \end{gathered} \tag{1.6}$$

and we prove some strong convergence theorems of the sequence $\{x_n\}$ defined by (1.6) for nonexpansive multivalued mappings in Banach spaces. The results presented in this paper establish a new type iteration convergence theorems for multivalued nonexpansive mappings in Banach spaces and extend the corresponding results of Panyanak [13].

2. Preliminaries

Let E be a real Banach space and let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{ f \in E^*, \langle x, f \rangle = \|x\| \|f\|, \|x\| = \|f\| \right\}, \quad \forall x \in E, \tag{2.1}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pair. It is well known that if E^* is strictly convex, then J is single valued. And if Banach space E admits sequentially continuous duality mapping J from weak topology to weak star topology, then, by [14, Lemma 1], we know that the duality mapping J is also single valued. In this case, the duality mapping J is also said to be weakly sequentially continuous; that is, if $\{x_n\}$ is a subject of E with $x_n \rightharpoonup x$, then $J(x_n) \overset{*}{\rightharpoonup} J(x)$. By Theorem 1 of [14], we know that if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition, and E is smooth; for the details, see [14]. In the sequel, we will denote the single-valued duality mapping by j.

Throughout this paper, we write $x_n \rightharpoonup x$ (resp., $x_n \overset{*}{\rightharpoonup} x$) to indicate that the sequence x_n weakly (resp., weak $*$) converges to x, as usual $x_n \to x$ will symbolize strong convergence. In order to show our main results, the following concepts and lemmas are needed.

A Banach space E is called uniformly convex if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \le 1$ and $\|x - y\| \ge \epsilon$, $\|x + y\| \le 2(1-\delta)$ holds. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf\left\{ 1 - \left\| \frac{1}{2}(x+y) \right\| : \|x\|, \|y\| \le 1,\ \|x - y\| \ge \epsilon \right\}, \tag{2.2}$$

for all $\epsilon \in [0,2]$. E is said to be uniformly convex if $\delta_E(0) = 0$, and $\delta(\epsilon) > 0$ for all $0 < \epsilon \le 2$.

Lemma 2.1 (see [10]). *In Banach space E, the following result (subdifferential inequality) is well known: for all $x, y \in E$, for all $j(x) \in J(x)$, for all $j(x+y) \in J(x+y)$,*

$$\|x\|^2 + 2\langle y, j(x)\rangle \le \|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle. \tag{2.3}$$

Definition 2.2. A Banach space E is said to satisfy *Opial's condition* if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ $(n \to \infty)$ implies

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|, \quad \forall y \in E \text{ with } x \ne y. \tag{2.4}$$

We know that Hilbert spaces, l^p $(1 < p < \infty)$, and Banach space with weakly sequentially continuous duality mappings satisfy Opial's condition; for the details, see [14, 15].

Definition 2.3. A multivalued mapping $T : K \to CB(K)$ is said to satisfy *Condition I* if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$ such that

$$d(x, Tx) \ge f(d(x, F(T))), \quad \forall x \in K. \tag{2.5}$$

Example of mappings that satisfy *Condition I* can be founded in [13].

3. Main Results

Now, we prove our results.

Theorem 3.1. *Let K be a nonempty compact convex subset of a uniformly convex Banach space E and let $T : K \to CB(K)$ be a multivalued nonexpansive mapping, where $\alpha_n \in (0,1)$ and $\lim_{n\to\infty}\alpha_n = 0$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated by (1.6).*

Then,

(i) *by the compactness of K, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to p$ for some $p \in K$. In addition if $\|y_n - p\| \le d(p, Tx_n)$, then*

(ii) *p is a fixed point of T and the sequence $\{x_n\}$ converges strongly to p.*

Proof. Part (i) is trivial. And part (ii) remains to be proved.

Due to the compactness of K and boundness of $CB(K)$, there exists a real number $M > 0$ such that

$$\|x_{n-1} - y_n\| \le M. \tag{3.1}$$

It follows from (1.6), that

$$\|x_n - y_n\| = \alpha_n\|x_{n-1} - y_n\| \le \alpha_n M, \tag{3.2}$$

thus

$$\|x_n - y_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{3.3}$$

therefore

$$\begin{aligned} d(p,Tp) &\le \|p - x_n\| + d(x_n,Tx_n) + H(Tx_n,Tp) \\ &\le 2\|p - x_n\| + d(x_n,Tx_n) \\ &\le 2\|p - x_n\| + \|x_n - y_n\|, \end{aligned} \tag{3.4}$$

so

$$d(p,Tp) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.5}$$

Hence, p is a fixed point of T.

Next we show that $\lim_{n\to\infty}\|x_n - p\|$ exists.

For all $n \ge 1$, there exist $j(x_n - p) \in J(x_n - p)$, using Lemma 2.1, we obtain

$$\begin{aligned} \|x_n - p\|^2 &= \langle \alpha_n x_{n-1} + (1-\alpha_n)y_n - p, j(x_n - p)\rangle \\ &= (1-\alpha_n)\langle y_n - p, j(x_n - p)\rangle + \alpha_n\langle x_{n-1} - p, j(x_n - p)\rangle \\ &\le (1-\alpha_n)\|y_n - p\| \cdot \|x_n - p\| + \alpha_n\|x_{n-1} - p\| \cdot \|x_n - p\| \\ &\le (1-\alpha_n)H(Tx_n,Tp) \cdot \|x_n - p\| + \alpha_n\|x_{n-1} - p\| \cdot \|x_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|x_{n-1} - p\| \cdot \|x_n - p\|, \end{aligned} \tag{3.6}$$

so

$$\|x_n - p\|^2 \le \|x_{n-1} - p\| \cdot \|x_n - p\|. \tag{3.7}$$

If $\|x_n - p\| = 0$, then $\lim_{n\to\infty}\|x_n - p\| = 0$ apparently holds.

Let $\|x_n - p\| > 0$, from (3.7), we have

$$\|x_n - p\| \le \|x_{n-1} - p\|. \tag{3.8}$$

We get that $\{\|x_n - p\|\}$ is a decreasing sequence, so

$$\lim_{n\to\infty}\|x_n - p\| \text{ exists.} \tag{3.9}$$

So the desired conclusion follows.

The proof is completed. □

Remark 3.2. The above result modified the gap in the proof of Theorem 3.1 in [13] by a new method; the gap discovered by Song and Wang [16] is as follows.

Panyanak [13] introduced the Ishikawa iterates (1.5) of a multivalued mapping T. It is obvious that x_n depends on p and T. For $p \in F(T)$, we have

$$\begin{aligned} \|z_n - p\| &= d(p, Tx_n) \leq H(Tp, Tx_n) \leq \|x_n - p\|, \\ \|z'_n - p\| &= d(p, Ty_n) \leq H(Tp, Ty_n) \leq \|y_n - p\|. \end{aligned} \tag{3.10}$$

Clearly, if $q \in F(T)$ and $q \neq p$, then the above inequalities cannot be assured. Indeed, from the monotone decreasing sequence of $\{\|x_n - p\|\}$ in the proof of (Theorem 3.1 [13]), we cannot obtain that $\{\|x_n - q\|\}$ is a decreasing sequence. Hence, the conclusion of Theorem 3.1 in [13] cannot be achieved.

Theorem 3.3. *Let E be a Banach space satisfying Opial's condition and let K be a nonempty weakly compact convex subset of E. Suppose that $T : K \to CB(K)$ is a multivalued nonexpansive mapping, where $\alpha_n \in (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated by* (1.6).

Then,

(i) *by the weak compactness of K, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$ for some $p \in K$. In addition if, $\|y_n - p\| \leq d(p, Tx_n)$, then*

(ii) *p is a fixed point of T and the sequence $\{x_n\}$ converges weakly to p.*

Proof. Part (i) is trivial. Now we prove part (ii).

It follows from (3.3) of Theorem 3.1 that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.11}$$

Since K is weakly compact, from part (i), there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} \rightharpoonup p, \quad \text{for some } p \in K. \tag{3.12}$$

Suppose that p does not belong to Tp. By the compactness of Tp, for any given x_{n_i}, there exist $z_i \in Tp$ such that $\|x_{n_i} - z_i\| = d(x_{n_i}, Tp)$ and $z_i \to z \in Tp$.

Thus $p \neq z$, from

$$\begin{aligned} \limsup_{i\to\infty} \|x_{n_i} - z\| &\leq \limsup_{i\to\infty} [\|x_{n_i} - z_i\| + \|z_i - z\|] \\ &= \limsup_{i\to\infty} \|x_{n_i} - z_i\| \\ &\leq \limsup_{i\to\infty} \left[d(x_{n_i}, Tx_{n_i}) + H(Tx_{n_i}, Tp)\right] \\ &\leq \limsup_{i\to\infty} \|x_{n_i} - p\| < \limsup_{i\to\infty} \|x_{n_i} - z\|. \end{aligned} \tag{3.13}$$

This is a contradiction by satisfying Opial's condition.

Hence, p is a fixed point of T.

It follows from (3.7) of Theorem 3.1 that

$$\lim_{n\to\infty}\|x_n - p\| \text{ exists.} \tag{3.14}$$

Next we show $x_n \rightharpoonup p$. Suppose not. There exists another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q \neq p$.

Then, we also obtain $q \in Tq$. From Opial's condition, we have

$$\begin{aligned}\lim_{i\to\infty}\|x_n - p\| &= \limsup_{i\to\infty}\|x_{n_i} - p\| \\ &< \limsup_{i\to\infty}\|x_{n_i} - q\| = \limsup_{k\to\infty}\|x_{n_k} - q\| \\ &< \limsup_{k\to\infty}\|x_{n_k} - p\| = \lim_{i\to\infty}\|x_n - p\|.\end{aligned} \tag{3.15}$$

Which is a contradiction, so the conclusion of the theorem follows.

The proof is completed. □

Corollary 3.4. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*, and let K be a nonempty weakly compact convex subset of E. Suppose that $T : K \to CB(K)$ is a multivalued nonexpansive mapping, where $\alpha_n \in (0,1)$ and $\lim_{n\to\infty}\alpha_n = 0$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated by* (1.6).

Then,

(i) *by the weak compactness of K, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$ for some $p \in K$. In addition if, $\|y_n - p\| \leq d(p, Tx_n)$, then*

(ii) *p is a fixed point of T and the sequence $\{x_n\}$ converges weakly to p.*

Proposition 3.5. *Let K be a nonempty compact convex subset of a uniformly convex Banach space E and let $T : K \to CB(K)$ be a multivalued nonexpansive mapping. Then $F(T)$ is a closed subset of K.*

Proof. Suppose $q_n \subset F(T)$, $n \geq 1$, such that $\lim_{n\to\infty} q_n = q$, then we have

$$\begin{aligned}d(q, Tq) &\leq \|p - q_n\| + d(q_n, Tq_n) + H(Tq_n, Tq) \\ &\leq 2\|q - q_n\| + d(q_n, Tq_n),\end{aligned} \tag{3.16}$$

so

$$d(q, Tq) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.17}$$

Hence, q is a fixed point of T.

Thus, $F(T)$ is a closed subset of K.

The proof is completed. □

Theorem 3.6. *Let K be a nonempty compact convex subset of a uniformly convex Banach space E and let $T : K \to CB(K)$ be a multivalued nonexpansive mapping satisfying Condition I, where $\alpha_n \in (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$, then the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.6) converges strongly to a fixed point.*

Proof. It follows from (3.3) of Theorem 3.1 that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.18}$$

The proof of remained part is omitted because it is similar to the proof of Theorem 3.8 in [13]. □

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