Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 165098, 18 pages
doi:10.1155/2010/165098

Research Article

A New Iterative Method for Solving Equilibrium Problems and Fixed Point Problems for Infinite Family of Nonexpansive Mappings

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Received 7 January 2010; Revised 21 May 2010; Accepted 11 July 2010

Academic Editor: Simeon Reich

We introduce a new iterative scheme for finding a common element of the solutions sets of a finite family of equilibrium problems and fixed points sets of an infinite family of nonexpansive mappings in a Hilbert space. As an application, we solve a multiobjective optimization problem using the result of this paper.

1. Introduction

Let H be a Hilbert space and C be a nonempty, closed, and convex subset of H. Let Φ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for the bifunction $\Phi : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$\Phi(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of the above inequality is denoted by $\mathrm{EP}(\Phi)$. Many problems arising from physics, optimization, and economics can reduce to finding a solution of an equilibrium problem.

In 2007, S. Takahashi and W. Takahashi [1] first introduced an iterative scheme by the viscosity approximation method for finding a common element of the solutions set of equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space

H and proved a strong convergence theorem which is based on Combettes and Hirstoaga's result [2] and Wittmann's result [3]. More precisely, they obtained the following theorem.

Theorem 1.1 (see [1]). *Let C be a nonempty closed and convex subset of H. Let $\Phi : C \times C \to \mathbb{R}$ be a bifunction which satisfies the following conditions:*

(A1) $\Phi(x, x) = 0$ *for all* $x \in C$;

(A2) Φ *is monotone, that is,* $\Phi(x, y) + \Phi(y, x) \leq 0$ *for all* $x, y \in C$;

(A3) *For all* $x, y, z \in C$,

$$\lim_{t \downarrow 0} \Phi(tz + (1-t)x, y) \leq \Phi(x, y); \tag{1.2}$$

(A4) *For each* $x \in C$, $y \mapsto \Phi(x, y)$ *is convex and lower semicontinuous.*

Let $S : C \to H$ be a nonexpansive mapping with $\mathrm{Fix}(S) \cap \mathrm{EP}(\Phi) \neq \emptyset$, *where* $\mathrm{Fix}(S)$ *denotes the set of fixed points of the mapping S, and let $f : H \to H$ be a contraction, if there exists a constant $\lambda \in (0, 1)$ such that $\|fx - fy\| \leq \lambda \|x - y\|$ for all $x, y \in H$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1 \in H$ and*

$$\begin{gathered} \Phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S u_n, \quad \forall n \geq 1, \end{gathered} \tag{1.3}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

$$\begin{gathered} \lim_{n \to \infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \\ \liminf_{n \to \infty} r_n > 0, \qquad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty. \end{gathered} \tag{1.4}$$

Then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \mathrm{Fix}(S) \cap \mathrm{EP}(\Phi)$, *where*

$$z = P_{\mathrm{Fix}(S) \cap \mathrm{EP}(\Phi)} f(z) \tag{1.5}$$

(P is the metric projection of H onto C and $P_{\mathrm{Fix}(S) \cap \mathrm{EP}(\Phi)} f(z)$ denotes nearest point in $\mathrm{Fix}(S) \cap \mathrm{EP}(\Phi)$ *from $f(z)$).*

Recently, many results on equilibrium problems and fixed points problems in the context of the Hilbert space and Banach space are introduced (see, e.g., [4–8]).

Let $F : H \to H$ be a nonlinear mapping. The variational inequality problem corresponding to the mapping F is to find a point $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.6}$$

The variational inequality problem is denoted by $\mathrm{VI}(F, C)$ [9].

The mapping F is called κ-Lipschitzian and η-strongly monotone if there exist constants $\kappa, \eta > 0$ such that

$$\|Fx - Fy\| \leq \kappa \|x - y\|, \quad \forall x, y \in H, \tag{1.7}$$

$$\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in H, \tag{1.8}$$

respectively. It is well known that if F is strongly monotone and Lipschitzian on C, then $\mathrm{VI}(F, C)$ has a unique solution. An important problem is how to find a solution of $\mathrm{VI}(F, C)$. Recently, there are many results to solve the $\mathrm{VI}(F, C)$ (see, e.g., [10–14]).

Let C be a nonempty closed and convex subset of a Hilbert space H, $\{T_n\}_{n=1}^{\infty} : H \to H$ be a countable family of nonexpansive mappings, and $\{\Phi_i\}_{i=1}^{m} : C \times C \to \mathbb{R}$ be m bifunctions satisfying conditions (A1)–(A4) such that $\Omega = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{EP}(\Phi_1) \cap \cdots \cap \mathrm{EP}(\Phi_m) \neq \emptyset$. Let $r_1, \ldots, r_m \in (0, \infty)$. For each $i = 1, \ldots, m$, define the mapping $T_{r_i} : H \to C$ by

$$T_{r_i}(x) = \left\{ z \in C : \Phi_i(z, y) + \frac{1}{r_i} \langle y - z, z - x \rangle \geq 0, \ \forall y \in C \right\}, \quad \forall x \in H. \tag{1.9}$$

Lemma 2.5 (see below) shows that, for each $1 \leq i \leq m$, T_{r_i} is firmly nonexpansive and hence nonexpansive and $\mathrm{Fix}(T_{r_i}) = \mathrm{EP}(\Phi_i)$. Suppose that $F : H \to H$ is a κ-Lipschitzian and η-strong monotone operator and let $\mu \in (0, 2\eta/\kappa^2)$. Assume that $\mathrm{VI}(\Phi_i(F, \Omega) \neq \emptyset$.

In this paper, motivated and inspired by the above research results, we introduce the following iterative process for finding an element in Ω: for an arbitrary initial point $x_1 \in H$,

$$\begin{gathered} z_n = \gamma_1 T_{r_1} x_n + \gamma_2 T_{r_2} x_n + \cdots + \gamma_m T_{r_m} x_n, \\ x_{n+1} = \alpha_n x_n + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \sigma_n T_i x_n + (1 - \alpha_n)(1 - \sigma_n) T^{\lambda_n} z_n, \quad \forall n \geq 1, \end{gathered} \tag{1.10}$$

where $T^{\lambda_n} z_n = z_n - \lambda_n \mu F(z_n)$, $\alpha_0 = 1$, $\{\alpha_n\}_{n=1}^{\infty}$ is a strictly decreasing sequence in $(0, \alpha)$ with $0 < \alpha < 1$, $\{\lambda_n\}_{n=1}^{\infty} \subset (0, 1)$, $\{\gamma_i\}_{i=1}^{m} \subset (0, 1)$ with $\sum_{i=1}^{m} \gamma_i = 1$, and $\{\sigma_n\}_{n=1}^{\infty} \subset (a, b)$ with $0 < a, b < 1$. Then we prove that the iterative process $\{x_n\}$ defined by (1.10) strongly converge to an element $x^* \in \Omega$, which is the unique solution of the variational inequality

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega. \tag{1.11}$$

As an application of our main result, we solve a multiobjective optimization problem.

2. Preliminaries

Let H be a Hilbert space and T a nonexpansive mapping of H into itself such that $\mathrm{Fix}(T) \neq \emptyset$. For all $\hat{x} \in \mathrm{Fix}(T)$ and $x \in H$, we have

$$\begin{aligned} \|x-\hat{x}\|^2 &\geq \|Tx-T\hat{x}\|^2 = \|Tx-\hat{x}\|^2 = \|Tx-x+(x-\hat{x})\|^2 \\ &= \|Tx-x\|^2 + \|x-\hat{x}\|^2 + 2\langle Tx-x, x-\hat{x}\rangle \end{aligned} \tag{2.1}$$

and hence

$$\|Tx-x\|^2 \leq 2\langle x-Tx, x-\hat{x}\rangle, \quad \forall \hat{x} \in \mathrm{Fix}(T), \quad x \in H. \tag{2.2}$$

It is well known that, for all $x, y \in H$ and $t \in [0,1]$,

$$\|tx+(1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2, \tag{2.3}$$

which implies that

$$\left\|\sum_{i=1}^{n} t_i x_i\right\|^2 \leq \sum_{i=1}^{n} t_i \|x_i\|^2 \tag{2.4}$$

for all $\{x_i\}_{i=1}^n \subset H$ and $\{t_i\}_{i=1}^n \subset [0,1]$ with $\sum_{i=1}^n t_i = 1$.

Let C be a nonempty closed and convex subset of H and, for any $x \in H$, there exists unique nearest point in C, denoted by $P_C x$, such that

$$\|P_C x - x\| \leq \|y-x\|, \quad \forall y \in C. \tag{2.5}$$

Moreover, we have the following:

$$z = P_C x \Longleftrightarrow \langle x-z, z-y\rangle \geq 0, \quad \forall y \in C. \tag{2.6}$$

Let I denote the identity operator of H and let $\{x_n\}$ be a sequence in a Hilbert space H and $x \in H$. Throughout this paper, $x_n \to x$ denotes that $\{x_n\}$ strongly converges to x and $x_n \rightharpoonup x$ denotes that $\{x_n\}$ weakly converges to x.

We need the following lemmas for our main results.

Lemma 2.1 (see [15]). *Let C be a nonempty closed and convex subset of a Hilbert space H and T a nonexpansive mapping from C into itself. Then $I-T$ is demiclosed at zero, that is,*

$$x_n \rightharpoonup x, \quad x_n - Tx_n \longrightarrow 0 \quad \text{implies } x = Tx. \tag{2.7}$$

Lemma 2.2 (see [10, Lemma 3.1(b)]). *Let H be a Hilbert space and $T : H \to H$ be a nonexpansive mapping. Let $F : H \to H$ be a mapping which is κ-Lipschitzian and η-strong monotone on $T(H)$. Assume that $\lambda \in (0,1)$ and $\mu \in (0, 2\eta/\kappa^2)$. Define a mapping $T^{\lambda} : H \to H$ by*

$$T^{\lambda}x = Tx - \lambda\mu F(Tx), \quad \forall x \in H. \tag{2.8}$$

Then $\|T^{\lambda}x - T^{\lambda}y\| \le (1-\lambda\tau)\|x-y\|$ for all $x, y \in H$, where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)} \in (0,1)$.

If $T = I$, Lemma 2.2 still holds.

Lemma 2.3 (see [16]). *Let $\{s_n\}$, $\{c_n\}$ be the sequences of nonnegative real numbers and $\{a_n\} \subset (0,1)$. Suppose that $\{b_n\}$ is a real number sequence such that*

$$s_{n+1} \le (1-a_n)s_n + b_n + c_n, \quad \forall n \ge 0. \tag{2.9}$$

Assume that $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold.

(1) If $b_n \le \beta a_n$ for all $n \ge 0$, where $\beta \ge 0$, then $\{s_n\}$ is a bounded sequence.

(2) If

$$\sum_{n=0}^{\infty} a_n = \infty, \quad \limsup_{n\to\infty} \frac{b_n}{a_n} \le 0, \tag{2.10}$$

then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 (see [17]). *Let C be a nonempty closed and convex subset of a Hilbert space H and $\Phi : C \times C \to \mathbb{R}$ be a bifunction which satisfies the conditions (A1)–(A4). Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$\Phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C. \tag{2.11}$$

Lemma 2.5 (see [2]). *Let H be a Hilbert space and C be a nonempty closed and convex subset of H. Assume that $\Phi : C \times C \to \mathbb{R}$ satisfies the conditions (A1)–(A4). For all $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : \Phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,\ \forall y \in C \right\}, \quad \forall x \in H. \tag{2.12}$$

Then the following holds:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle; \tag{2.13}$$

(3) $\mathrm{Fix}(T_r) = \mathrm{EP}(\Phi)$;

(4) $\mathrm{EP}(\Phi)$ *is closed and convex.*

The following lemma is an immediate consequence of an inner product.

Lemma 2.6. *Let H be a real Hilbert space. Then the following identity holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H. \tag{2.14}$$

3. Main Results

First, we prove some lemmas as follows.

Lemma 3.1. *The sequence $\{x_n\}$ generated by* (1.10) *is bounded.*

Proof. Let $u_{in} = T_{r_i}x_n$ for each $i = 1, 2, \ldots, m$. Lemma 2.5 shows that each T_{r_i} is firmly-nonexpansive and hence nonexpansive. Hence, for each $1 \leq i \leq m$ and $p \in \Omega$, we have

$$\|u_{in} - p\| = \|T_{r_i}x_n - T_{r_i}p\| \leq \|x_n - p\|, \quad \forall n \geq 1, \tag{3.1}$$

$$\|z_n - p\| \leq \sum_{i=1}^{m} \gamma_i \|u_{in} - p\| \leq \|x_n - p\|, \quad \forall n \geq 1. \tag{3.2}$$

By Lemma 2.2, we have

$$\left\|T^{\lambda_n}x - T^{\lambda_n}y\right\| \leq (1-\lambda_n\tau)\|x-y\|, \quad \forall x, y \in H, \tag{3.3}$$

where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)} \in (0,1)$. Therefore, by (3.2) and (3.3), we obtain (note that $\{\alpha_n\}$ is strictly decreasing and $T^{\lambda_n}p - p = -\lambda_n\mu F(p)$)

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\|\alpha_n(x_n - p) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\sigma_n(T_i x_n - p) + (1-\alpha_n)(1-\sigma_n)\left(T^{\lambda_n}z_n - p\right)\right\| \\
&\leq \alpha_n\|x_n - p\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\sigma_n\|T_i x_n - p\| + (1-\alpha_n)(1-\sigma_n)\left\|T^{\lambda_n}z_n - p\right\| \\
&\leq \alpha_n\|x_n - p\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\sigma_n\|x_n - p\| \\
&\quad + (1-\alpha_n)(1-\sigma_n)\left[\left\|T^{\lambda_n}z_n - T^{\lambda_n}p\right\| + \left\|T^{\lambda_n}p - p\right\|\right] \\
&\leq \alpha_n\|x_n - p\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\sigma_n\|x_n - p\| \\
&\quad + (1-\alpha_n)(1-\sigma_n)\left[(1-\lambda_n\tau)\|z_n - p\| + \lambda_n\mu\|F(p)\|\right]
\end{aligned}$$

$$\begin{aligned}&\leq \alpha_n\|x_n-p\|+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\sigma_n\|x_n-p\|\\&\quad+(1-\alpha_n)(1-\sigma_n)\left[(1-\lambda_n\tau)\|x_n-p\|+\lambda_n\mu\|F(p)\|\right]\\&=(1-(1-\alpha_n)(1-\sigma_n)\lambda_n\tau)\|x_n-p\|+(1-\alpha_n)(1-\sigma_n)\lambda_n\mu\|F(p)\|.\end{aligned} \tag{3.4}$$

By induction, we obtain $\|x_{n+1}\| \leq \max\{\|x_1-p\|,(\mu/\tau)\|F(p)\|\}$. Hence $\{x_n\}$ is bounded and so are $\{z_n\}$ and $\{u_{in}\}$ for each $i=1,2,\ldots,m$. Since F is κ-Lipschitzian, we have

$$\begin{aligned}\|F(z_n)\| &\leq \|F(z_n)-F(p)\|+\|F(p)\|\\&\leq \kappa\|z_n-p\|+\|F(p)\| \leq \kappa\|z_n\|+\kappa\|p\|+\|F(p)\|,\end{aligned} \tag{3.5}$$

which shows that $\{F(z_n)\}$ is bounded. This completes the proof. □

Lemma 3.2. *If the following conditions hold:*

$$\sum_{n=1}^{\infty}\lambda_n=\infty,\qquad \sum_{n=1}^{\infty}|\lambda_n-\lambda_{n+1}|<\infty,\qquad \sum_{n=1}^{\infty}|\sigma_n-\sigma_{n+1}|<\infty, \tag{3.6}$$

then $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Proof. For each $i=1,2,\ldots,m$, since each T_{r_i} is nonexpansive, we have

$$\|u_{in-1}-u_{in}\|=\|T_{r_i}x_{n-1}-T_{r_i}x_n\| \leq \|x_{n-1}-x_n\|,\quad \forall n\geq 1. \tag{3.7}$$

By (3.7), we have

$$\begin{aligned}\|z_n-z_{n-1}\| &= \|\gamma_1(u_{1n}-u_{1n-1})+\gamma_2(u_{2n}-u_{2n-1})+\cdots+\gamma_m(u_{mn}-u_{mn-1})\|\\&\leq \sum_{i=1}^{m}\gamma_i\|u_{in}-u_{in-1}\| \leq \sum_{i=1}^{m}\gamma_i\|x_n-x_{n-1}\|\\&=\|x_{n-1}-x_n\|,\quad \forall n\geq 1.\end{aligned} \tag{3.8}$$

By the definition of the iterative sequence (1.10), we have

$$\begin{aligned}x_{n+1}-x_n &= \alpha_n(x_n-x_{n-1})+\alpha_n x_{n-1}+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\sigma_n(T_i x_n-T_i x_{n-1})\\&\quad+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\sigma_n T_i x_{n-1}+(1-\alpha_n)(1-\sigma_n)\left(T^{\lambda_n}z_n-T^{\lambda_n}z_{n-1}\right)\\&\quad+(1-\alpha_n)(1-\sigma_n)T^{\lambda_n}z_{n-1}-\alpha_{n-1}x_{n-1}-\sum_{i=1}^{n-1}(\alpha_{i-1}-\alpha_i)\sigma_{n-1}T_i x_{n-1}\\&\quad-(1-\alpha_{n-1})(1-\sigma_{n-1})T^{\lambda_{n-1}}z_{n-1}\end{aligned}$$

$$
\begin{aligned}
&= \alpha_n(x_n - x_{n-1}) + (\alpha_n - \alpha_{n-1})x_{n-1} + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\sigma_n(T_i x_n - T_i x_{n-1}) \\
&\quad + (1-\alpha_n)(1-\sigma_n)\left(T^{\lambda_n} z_n - T^{\lambda_n} z_{n-1}\right) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\sigma_n T_i x_{n-1} \\
&\quad - \sum_{i=1}^{n-1}(\alpha_{i-1} - \alpha_i)\sigma_{n-1} T_i x_{n-1} + (1-\alpha_n)(1-\sigma_n)T^{\lambda_n} z_{n-1} \\
&\quad - (1-\alpha_{n-1})(1-\sigma_{n-1})T^{\lambda_{n-1}} z_{n-1} \\
&= \alpha_n(x_n - x_{n-1}) + (\alpha_n - \alpha_{n-1})x_{n-1} + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\sigma_n(T_i x_n - T_i x_{n-1}) \\
&\quad + (1-\alpha_n)(1-\sigma_n)\left(T^{\lambda_n} z_n - T^{\lambda_n} z_{n-1}\right) + \sum_{i=1}^{n-1}(\alpha_{i-1} - \alpha_i)(\sigma_n - \sigma_{n-1})T_i x_{n-1} \\
&\quad + (\alpha_{n-1} - \alpha_n)\sigma_n T_n x_{n-1} + [(\alpha_{n-1} - \alpha_n)(1-\sigma_n) + (\sigma_{n-1} - \sigma_n)(1-\alpha_{n-1})]z_{n-1} \\
&\quad + \{(1-\alpha_{n-1})(1-\sigma_{n-1})(\lambda_{n-1} - \lambda_n) \\
&\qquad -[(\alpha_{n-1} - \alpha_n)(1-\sigma_n) + (\sigma_{n-1} - \sigma_n)(1-\alpha_{n-1})]\lambda_n\}\mu F(z_{n-1}),
\end{aligned}
\tag{3.9}
$$

and hence

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\le \alpha_n\|x_n - x_{n-1}\| + (\alpha_{n-1} - \alpha_n)\|x_{n-1}\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\sigma_n\|x_n - x_{n-1}\| \\
&\quad + (1-\alpha_n)(1-\sigma_n)(1-\lambda_n\tau)\|z_n - z_{n-1}\| + \sum_{i=1}^{n-1}(\alpha_{i-1} - \alpha_i)|\sigma_n - \sigma_{n-1}|\|T_i x_{n-1}\| \\
&\quad + (\alpha_{n-1} - \alpha_n)\|T_n x_{n-1}\| + [(\alpha_{n-1} - \alpha_n) + |\sigma_{n-1} - \sigma_n|]\|z_{n-1}\| \\
&\quad + [|\lambda_{n-1} - \lambda_n| + (\alpha_{n-1} - \alpha_n) + |\sigma_{n-1} - \sigma_n|]\mu\|F(z_{n-1})\| \\
&= \alpha_n\|x_n - x_{n-1}\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\sigma_n\|x_n - x_{n-1}\| \\
&\quad + (1-\alpha_n)(1-\sigma_n)(1-\lambda_n\tau)\|z_n - z_{n-1}\| \\
&\quad + (\alpha_{n-1} - \alpha_n)\left[\|x_{n-1}\| + \|T_n x_{n-1}\| + \|z_{n-1}\| + \mu\|F(z_{n-1})\|\right] \\
&\quad + \sum_{i=1}^{n-1}(\alpha_{i-1} - \alpha_i)|\sigma_n - \sigma_{n-1}|\|T_i x_{n-1}\| + |\sigma_{n-1} - \sigma_n|\left(\|z_{n-1}\| + \mu\|F(z_{n-1})\|\right) \\
&\quad + |\lambda_{n-1} - \lambda_n|\mu\|F(z_{n-1})\|.
\end{aligned}
\tag{3.10}
$$

It follows from (3.8) and (3.10) that

$$
\begin{aligned}
\|x_{n+1} - x_n\| \leq\ & \alpha_n\|x_n - x_{n-1}\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\sigma_n\|x_n - x_{n-1}\| \\
& + (1-\alpha_n)(1-\sigma_n)(1-\lambda_n\tau)\|x_{n-1} - x_n\| \\
& + (\alpha_{n-1} - \alpha_n)\left[\|x_{n-1}\| + \|T_n x_{n-1}\| + \|z_{n-1}\| + \mu\|F(z_{n-1})\|\right] \\
& + \sum_{i=1}^{n-1}(\alpha_{i-1} - \alpha_i)|\sigma_n - \sigma_{n-1}|\|T_i x_{n-1}\| + |\sigma_{n-1} - \sigma_n|\left(\|z_{n-1}\| + \mu\|F(z_{n-1})\|\right) \\
& + |\lambda_{n-1} - \lambda_n|\mu\|F(z_{n-1})\| \\
\leq\ & (1 - (1-\alpha_n)(1-\sigma_n)\lambda_n\tau)\|x_n - x_{n-1}\| + (\alpha_{n-1} - \alpha_n)(3+\mu)M \\
& + |\sigma_n - \sigma_{n-1}|(2+\mu)M + |\lambda_{n-1} - \lambda_n|\mu M \\
\leq\ & (1 - (1-a)(1-b)\lambda_n\tau)\|x_n - x_{n-1}\| + (\alpha_{n-1} - \alpha_n)(3+\mu)M \\
& + |\sigma_n - \sigma_{n-1}|(2+\mu)M + |\lambda_{n-1} - \lambda_n|\mu M,
\end{aligned}
\tag{3.11}
$$

where $M = \max\{\sup_{n\geq 1}\|x_n\|, \sup_{n\geq 1}\|z_n\|, \sup_{i\geq 1, n\geq 1}\|T_i x_n\|, \sup_{n\geq 1}\|F(z_n)\|\}$. Since $\{\alpha_n\}$ is strictly decreasing, we have $\sum_{n=2}^{\infty}(\alpha_{n-1} - \alpha_n) = \alpha_1 < \infty$. Further, from the assumptions, it follows that

$$
\sum_{n=2}^{\infty}\left\{(\alpha_{n-1} - \alpha_n)(3+\mu)M + |\sigma_n - \sigma_{n-1}|(2+\mu)M + |\lambda_{n-1} - \lambda_n|\mu M\right\} < \infty. \tag{3.12}
$$

Therefore, by Lemma 2.3, we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. This completes the proof. □

Lemma 3.3. *If the following conditions hold:*

$$
\lim_{n\to\infty}\lambda_n = 0, \qquad \sum_{n=1}^{\infty}\lambda_n = \infty, \qquad \sum_{n=1}^{\infty}|\lambda_n - \lambda_{n+1}| < \infty, \qquad \sum_{n=1}^{\infty}|\sigma_n - \sigma_{n+1}| < \infty, \tag{3.13}
$$

then $\lim_{n\to\infty}\|x_n - u_{in}\| = 0$ *for each* $i = 1, 2, \ldots, m$.

Proof. For any $p \in \Omega$ and $i = 1, 2, \ldots, m$, it follows from Lemma 2.5(2) that

$$
\begin{aligned}
\|u_{in} - p\|^2 &= \|T_{r_i}x_n - T_{r_i}p\|^2 \leq \langle T_{r_i}x_n - T_{r_i}p, x_n - p\rangle = \langle u_{in} - p, x_n - p\rangle \\
&= \frac{1}{2}\left(\|u_{in} - p\|^2 + \|x_n - p\|^2 - \|u_{in} - x_n\|^2\right),
\end{aligned}
\tag{3.14}
$$

and hence $\|u_{in} - p\|^2 \leq \|x_n - p\|^2 - \|u_{in} - x_n\|^2$. Further, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \left\|\sum_{i=1}^{m}\gamma_i(u_{in} - p)\right\|^2 \leq \sum_{i=1}^{m}\gamma_i\|u_{in} - p\|^2 \\
&\leq \sum_{i=1}^{m}\gamma_i\left(\|x_n - p\|^2 - \|u_{in} - x_n\|^2\right) \\
&= \|x_n - p\|^2 - \sum_{i=1}^{m}\gamma_i\|u_{in} - x_n\|^2, \quad \forall n \geq 1.
\end{aligned} \tag{3.15}$$

Therefore, from (2.4) and (3.3), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\|\alpha_n(x_n - p) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\sigma_n(T_i x_n - p) + (1 - \alpha_n)(1 - \sigma_n)(T^{\lambda_n} z_n - p)\right\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\sigma_n\|T_i x_n - p\|^2 + (1 - \alpha_n)(1 - \sigma_n)\left\|T^{\lambda_n} z_n - p\right\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\sigma_n\|x_n - p\|^2 \\
&\quad + (1 - \alpha_n)(1 - \sigma_n)\left[\left\|T^{\lambda_n} z_n - T^{\lambda_n} p\right\| + \left\|T^{\lambda_n} p - p\right\|\right]^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\sigma_n\|x_n - p\|^2 \\
&\quad + (1 - \alpha_n)(1 - \sigma_n)\left[(1 - \lambda_n\tau)\|z_n - p\| + \lambda_n\mu\|F(p)\|\right]^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\sigma_n\|x_n - p\|^2 + (1 - \alpha_n)(1 - \sigma_n) \\
&\quad \times \left[(1 - \lambda_n\tau)\|z_n - p\|^2 + 2\lambda_n(1 - \lambda_n\tau)\mu\|z_n - p\|\|F(p)\| + \lambda_n\mu^2\|F(p)\|^2\right] \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\sigma_n\|x_n - p\|^2 + (1 - \alpha_n)(1 - \sigma_n) \\
&\quad \times \Bigg[(1 - \lambda_n\tau)\left(\|x_n - p\|^2 - \sum_{i=1}^{m}\gamma_i\|u_{in} - x_n\|^2\right) \\
&\qquad + 2\lambda_n(1 - \lambda_n\tau)\mu\|z_n - p\|\|F(p)\| + \lambda_n\mu^2\|F(p)\|^2\Bigg] \\
&= (1 - (1 - \alpha_n)(1 - \sigma_n)\lambda_n\tau)\|x_n - p\|^2 - (1 - \alpha_n)(1 - \sigma_n)(1 - \lambda_n\tau)\sum_{i=1}^{m}\gamma_i\|u_{in} - x_n\|^2 \\
&\quad + 2\lambda_n\mu(1 - \alpha_n)(1 - \sigma_n)(1 - \lambda_n\tau)\|z_n - p\|\|F(p)\| + (1 - \alpha_n)(1 - \sigma_n)\lambda_n\mu^2\|F(p)\|^2
\end{aligned}$$

$$\begin{aligned} &\le \left\|x_n - p\right\|^2 - (1-\alpha_n)(1-\sigma_n)(1-\lambda_n\tau)\sum_{i=1}^{m}\gamma_i\|u_{in} - x_n\|^2 \\ &\quad + (1-\alpha_n)(1-\sigma_n)\lambda_n\mu^2\left\|F(p)\right\|^2 \\ &\quad + 2\lambda_n\mu(1-\alpha_n)(1-\sigma_n)(1-\lambda_n\tau)\left\|z_n - p\right\|\left\|F(p)\right\|. \end{aligned} \tag{3.16}$$

It follows that

$$\begin{aligned} &\gamma_i(1-\alpha_n)(1-\sigma_n)(1-\lambda_n\tau)\|u_{in} - x_n\|^2 \\ &\quad \le \left(\left\|x_n - p\right\| + \left\|x_{n+1} - p\right\|\right)\|x_n - x_{n+1}\| + \lambda_n\left[\mu^2\left\|F(p)\right\|^2 + 2\mu\left\|z_n - p\right\|\left\|F(p)\right\|\right] \end{aligned} \tag{3.17}$$

for each $i = 1, 2, \ldots, m$. Note that $0 < \gamma_i < 1$ for $i = 1, 2, \ldots, m$. From the assumptions, Lemma 3.2, and the previous inequality, we conclude that $\|u_{in} - x_n\| \to 0$ as $n \to \infty$ for each $i = 1, 2, \ldots, m$. Further, we have

$$\|z_n - x_n\| \le \sum_{i=1}^{m}\gamma_i\|u_{in} - x_n\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.18}$$

This completes the proof. □

Lemma 3.4. *If the following conditions hold:*

$$\lim_{n\to\infty}\lambda_n = 0, \qquad \sum_{n=1}^{\infty}\lambda_n = \infty, \qquad \sum_{n=1}^{\infty}|\lambda_n - \lambda_{n+1}| < \infty, \qquad \sum_{n=1}^{\infty}|\sigma_n - \sigma_{n+1}| < \infty, \tag{3.19}$$

then $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ *for all* $i \ge 1$.

Proof. By the definition of the iterative sequence (1.10), we have

$$x_{n+1} + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\sigma_n(x_n - T_i x_n) - (1-\alpha_n)\sigma_n x_n = \alpha_n x_n + (1-\alpha_n)(1-\sigma_n)T^{\lambda_n}z_n, \tag{3.20}$$

that is,

$$\begin{aligned} \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\sigma_n(x_n - T_i x_n) &= x_n - x_{n+1} - x_n + \alpha_n x_n + (1-\alpha_n)\sigma_n x_n + (1-\alpha_n)(1-\sigma_n)T^{\lambda_n}z_n \\ &= x_n - x_{n+1} + (1-\alpha_n)(\sigma_n - 1)x_n + (1-\alpha_n)(1-\sigma_n)T^{\lambda_n}z_n \\ &= x_n - x_{n+1} + (1-\alpha_n)(1-\sigma_n)\left(T^{\lambda_n}z_n - x_n\right). \end{aligned} \tag{3.21}$$

Hence, for any $p \in \Omega$, we get

$$\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\sigma_n\langle x_n - T_i x_n, x_n - p\rangle = (1-\alpha_n)(1-\sigma_n)\langle T^{\lambda_n} z_n - x_n, x_n - p\rangle + \langle x_n - x_{n+1}, x_n - p\rangle. \tag{3.22}$$

Since each T_i is nonexpansive, by (2.2), we have

$$\|T_i x_n - x_n\|^2 \leq 2\langle x_n - T_i x_n, x_n - p\rangle. \tag{3.23}$$

Hence, combining this inequality with (3.22), we get

$$\frac{1}{2}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\sigma_n\|T_i x_n - x_n\|^2 \leq (1-\alpha_n)(1-\sigma_n)\langle T^{\lambda_n} z_n - x_n, x_n - p\rangle + \langle x_n - x_{n+1}, x_n - p\rangle, \tag{3.24}$$

which implies that (note that $\{\alpha_n\}$ is a strictly decreasing sequence)

$$\begin{aligned}\|T_i x_n - x_n\|^2 &\leq \frac{2(1-\alpha_n)(1-\sigma_n)}{(\alpha_{i-1}-\alpha_i)\sigma_n}\langle T^{\lambda_n} z_n - x_n, x_n - p\rangle + \frac{2}{(\alpha_{i-1}-\alpha_i)\sigma_n}\langle x_n - x_{n+1}, x_n - p\rangle \\ &\leq \frac{2(1-\alpha_n)(1-\sigma_n)}{(\alpha_{i-1}-\alpha_i)\sigma_n}\left\|T^{\lambda_n} z_n - x_n\right\|\left\|x_n - p\right\| + \frac{2}{(\alpha_{i-1}-\alpha_i)\sigma_n}\|x_n - x_{n+1}\|\left\|x_n - p\right\|.\end{aligned} \tag{3.25}$$

From Lemma 3.3, $\lim_{n\to\infty}\lambda_n = 0$, and the inequality

$$\left\|T^{\lambda_n} z_n - x_n\right\| \leq \|z_n - x_n\| + \lambda_n \mu \|F(z_n)\|, \tag{3.26}$$

we obtain

$$\lim_{n\to\infty}\left\|T^{\lambda_n} z_n - x_n\right\| = 0. \tag{3.27}$$

Therefore, from Lemma 3.2, (3.25), and (3.27), it follows that

$$\lim_{n\to\infty}\|T_i x_n - x_n\| = 0, \quad \forall i \geq 1. \tag{3.28}$$

This completes the proof. □

Next we prove the main results of this paper.

Theorem 3.5. *Assume that the following conditions hold:*

$$\lim_{n\to\infty}\lambda_n = 0, \qquad \sum_{n=1}^{\infty}\lambda_n = \infty, \qquad \sum_{n=1}^{\infty}|\lambda_n - \lambda_{n+1}| < \infty,$$
$$\sum_{n=1}^{\infty}|\gamma_n - \gamma_{n+1}| < \infty, \qquad \sum_{n=1}^{\infty}|\sigma_n - \sigma_{n+1}| < \infty. \tag{3.29}$$

Then the sequence $\{x_n\}$ *generated by* (1.10) *converges strongly to an element in* Ω*, which is the unique solution of the variational inequality* $\mathrm{VI}(F,\Omega)$*.*

Proof. Since $\mathrm{VI}(F,\Omega) \neq \emptyset$, we can select an element $x^* \in \mathrm{VI}(F,\Omega)$, which implies that

$$\langle F(x^*), x^* - x\rangle \geq 0, \quad \forall x \in \Omega. \tag{3.30}$$

First, we prove that

$$\limsup_{n\to\infty}\langle -F(x^*), x_{n+1} - x^*\rangle \leq 0. \tag{3.31}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle -F(x^*), x_n - x^*\rangle = \lim_{j\to\infty}\langle -F(x^*), x_{n_j} - x^*\rangle. \tag{3.32}$$

Without loss of generality, we may further assume that $x_{n_j} \rightharpoonup \hat{x}$ for some $\hat{x} \in H$. From Lemmas 3.4 and 2.1, we get $\hat{x} \in \mathrm{Fix}(T_n)$ for all $n \geq 1$. Hence we have $\hat{x} \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$. It follows from Lemma 2.5 that each T_{r_i} is firmly nonexpansive and hence nonexpansive. Lemma 3.3 shows that $\|T_{r_i}x_n - x_n\| \to 0$ as $n \to \infty$. Therefore, from Lemma 2.1, it follows that $\hat{x} \in \mathrm{Fix}(T_{r_i})$ for each $i = 1,\ldots,m$, which shows that $\hat{x} \in \bigcap_{i=1}^{m}\mathrm{Fix}(T_{r_i})$. Lemma 2.5 shows that $\mathrm{Fix}(T_{r_i}) = \mathrm{EP}(\Phi_i)$ for each $i = 1,\ldots,m$. Hence $\hat{x} \in \bigcap_{i=1}^{m}\mathrm{EP}(\Phi_i)$. By using the above argument, we conclude that

$$\hat{x} \in \Omega = \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{EP}(\Phi_1) \cap \cdots \cap \mathrm{EP}(\Phi_m). \tag{3.33}$$

Noting that x^* is a solution of the $\mathrm{VI}(F,\Omega)$, we obtain

$$\limsup_{n\to\infty}\langle -F(x^*), x_n - x^*\rangle = \langle -F(x^*), \hat{x} - x^*\rangle \leq 0. \tag{3.34}$$

It follows from Lemma 2.6 that

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&= \left\| \left[\alpha_n(x_n-x^*) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_n)\sigma_n(T_ix_n-x^*) + (1-\alpha_n)(1-\sigma_n)\left(T^{\lambda_n}z_n - T^{\lambda_n}x^*\right) \right] \right. \\
&\quad \left. +(1-\alpha_n)(1-\sigma_n)\left(T^{\lambda_n}x^*-x^*\right) \right\|^2 \\
&\le \left\| \alpha_n(x_n-x^*) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_n)\sigma_n(T_ix_n-x^*) + (1-\alpha_n)(1-\sigma_n)(T^{\lambda_n}z_n - T^{\lambda_n}x^*) \right\|^2 \\
&\quad + 2(1-\alpha_n)(1-\sigma_n)\langle T^{\lambda_n}x^*-x^*, x_{n+1}-x^*\rangle \\
&\le \alpha_n\|x_n-x^*\|^2 + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_n)\sigma_n\|T_ix_n-x^*\|^2 + (1-\alpha_n)(1-\sigma_n)\left\|T^{\lambda_n}z_n - T^{\lambda_n}x^*\right\|^2 \\
&\quad + 2(1-\alpha_n)(1-\sigma_n)\lambda_n\mu\langle -F(x^*), x_{n+1}-x^*\rangle \\
&\le \alpha_n\|x_n-x^*\|^2 + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_n)\sigma_n\|x_n-x^*\|^2 + (1-\alpha_n)(1-\sigma_n)(1-\lambda_n\tau)\|z_n-x^*\|^2 \\
&\quad + 2(1-\alpha_n)(1-\sigma_n)\lambda_n\mu\langle -F(x^*), x_{n+1}-x^*\rangle \\
&= \alpha_n\|x_n-x^*\|^2 + (1-\alpha_n)\sigma_n\|x_n-x^*\|^2 + (1-\alpha_n)(1-\sigma_n)(1-\lambda_n\tau)\|z_n-x^*\|^2 \\
&\quad + 2(1-\alpha_n)(1-\sigma_n)\lambda_n\mu\langle -F(x^*), x_{n+1}-x^*\rangle \\
&\le \alpha_n\|x_n-x^*\|^2 + (1-\alpha_n)\sigma_n\|x_n-x^*\|^2 + (1-\alpha_n)(1-\sigma_n)(1-\lambda_n\tau)\|x_n-x^*\|^2 \\
&\quad + 2(1-\alpha_n)(1-\sigma_n)\lambda_n\mu\langle -F(x^*), x_{n+1}-x^*\rangle \\
&= (1-(1-\alpha_n)(1-\sigma_n)\lambda_n\tau)\|x_n-x^*\| + 2(1-\alpha_n)(1-\sigma_n)\lambda_n\mu\langle -F(x^*), x_{n+1}-x^*\rangle.
\end{aligned}
\tag{3.35}
$$

Let $a_n = (1-\alpha_n)(1-\sigma_n)\lambda_n\tau$ and $b_n = 2(1-\alpha_n)(1-\sigma_n)\lambda_n\mu\langle -F(x^*), x_{n+1}-x^*\rangle$ for all $n \ge 1$. Then, from the assumptions and (3.31), we have

$$
0 < a_n < 1, \qquad \sum_{n=1}^{\infty} a_n = \infty, \qquad \limsup_{n\to\infty}\frac{b_n}{a_n} = 0. \tag{3.36}
$$

Therefore, by applying Lemma 2.3 to (3.35), we conclude that the sequence $\{x_n\}$ strongly converges to a point x^*.

In order to prove the uniqueness of solution of the $\mathrm{VI}(F,\Omega)$, we assume that u^* is another solution of $\mathrm{VI}(F,\Omega)$. Similarly, we can conclude that $\{x_n\}$ converges strongly to a point u^*. Hence $x^* = u^*$, that is, x^* is the unique solution of $\mathrm{VI}(F,\Omega)$. This completes the proof. □

As direct consequences of Theorem 3.5, we obtain the following corollaries.

Corollary 3.6. *Let C be a nonempty closed and convex subset of a Hilbert space H. For each $i = 1, 2, \ldots, m$ let $\Phi_i : C \times C \to \mathbb{R}$ be m bifunctions which satisfy conditions (A1)–(A4) such that $\bigcap_{i=1}^{m} \mathrm{EP}(\Phi_i) \neq \emptyset$. Let $\mu \in (0, 2)$, and let $\{\alpha_n\}_{n=1}^{\infty} \subset (0, \alpha)$ be a strictly decreasing sequence with $0 < \alpha < 1$, $\{\lambda_n\}_{n=1}^{\infty} \subset (0, 1)$, $\{\gamma_i\}_{i=1}^{m} \subset (0, 1)$ with $\sum_{i=1}^{m} \gamma_i = 1$, $r_1, r_2, \ldots, r_m \in (0, \infty)$, and $\{\sigma_n\}_{n=1}^{n} \subset (a, b)$ with $0 < a, b < 1$. For an arbitrary initial $x_1 \in H$, define the iterative sequence $\{x_n\}$ by*

$$\begin{aligned} z_n &= \gamma_1 T_{r_1} x_n + \gamma_2 T_{r_2} x_n + \cdots + \gamma_m T_{r_m} x_n, \\ x_{n+1} &= (\alpha_n + (1 - \alpha_n)\sigma_n) x_n + (1 - \alpha_n)(1 - \sigma_n)(1 - \lambda_n \mu) z_n, \quad \forall n \geq 1. \end{aligned} \tag{3.37}$$

If the following conditions hold:

$$\lim_{n \to \infty} \lambda_n = 0, \qquad \sum_{n=1}^{\infty} \lambda_n = \infty, \qquad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty, \qquad \sum_{n=1}^{\infty} |\sigma_n - \sigma_{n+1}| < \infty, \tag{3.38}$$

then the sequence $\{x_n\}$ converges strongly to an element $x^* \in \bigcap_{i=1}^{m} \mathrm{EP}(\Phi_i)$.

Proof. Put $F = I$ and $T_i = I$ for each $i \geq 1$ in Theorem 3.5. Then we know that F is 1-Lipschitzian and 1-strongly monotone, $\sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) T_i x_n = (1 - \alpha_n) x_n$ and $T^{\lambda_n} z_n = (1 - \lambda_n \mu) z_n$. Therefore, by Theorem 3.5, we conclude the desired result. □

Corollary 3.7. *Let C be a nonempty closed and convex subset of a Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be a countable family of nonexpansive mappings of H such that $C = \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$ and $F : H \to H$ an operator which is κ-Lipschitzian and η-strong monotone on H. Let $\mu \in (0, 2\eta/\kappa^2)$. Assume that $\mathrm{VI}(F, C) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty} \subset (0, \alpha)$ with $0 < \alpha < 1$ be a strictly decreasing sequence, $\{\lambda_n\}_{n=1}^{\infty} \subset (0, 1)$ and $\{\sigma_n\}_{n=1}^{\infty} \subset (a, b)$ with $0 < a, b < 1$. For an arbitrary initial $x_1 \in H$, define the iterative sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_n x_n + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \sigma_n T_i x_n + (1 - \alpha_n)(1 - \sigma_n)\big(P_C x_n - \lambda_n \mu F(P_C x_n)\big), \quad \forall n \geq 1, \tag{3.39}$$

where $\alpha_0 = 1$. If the following conditions hold:

$$\lim_{n \to \infty} \lambda_n = 0, \qquad \sum_{n=1}^{\infty} \lambda_n = \infty, \qquad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty, \qquad \sum_{n=1}^{\infty} |\sigma_n - \sigma_{n+1}| < \infty, \tag{3.40}$$

then the sequence $\{x_n\}$ strongly converges to an element $x^* \in C$, which is the unique solution of the variational inequality

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{3.41}$$

Proof. Put $\Phi_i(x, y) = 0$ for each $i = 1, 2, \ldots, m$ and $x, y \in C$. Set $r_1 = r_2 = \cdots = r_m = 1$ in Theorem 3.5. Then, by (2.6), we have $T_{r_1} x_n = T_{r_2} x_n = \cdots = T_{r_m} x_n = P_C x_n$. Therefore, by Theorem 3.5, we conclude the desired result. □

Remark 3.8. (1) Recently, many authors have studied the iteration sequences for infinite family of nonexpansive mappings. But our iterative sequence (1.10) is very different from others because we do not use W-mapping generated by the infinite family of nonexpansive mappings and we have no any restriction with the infinite family of nonlinear mappings.

(2) We do not use Suzuki's lemma [18] for obtaining the result that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. However, many authors have used Suzuki's lemma [18] for obtaining the result that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ in the process of studying the similar algorithms. For example, see [5, 19, 20] and so on.

4. Application

In this section, we study a kind of multiobjective optimization problem based on the result of this paper. That is, we give an iterative sequence which solves the following multiobjective optimization problem with nonempty set of solutions:

$$\begin{gathered} \min h_1(x), \\ \min h_2(x), \\ x \in C, \end{gathered} \tag{4.1}$$

where $h_1(x)$ and $h_2(x)$ are both convex and lower semicontinuous functions defined on a nonempty closed and convex subset of C of a Hilbert space H. We denote by A the set of solutions of (4.1) and assume that $A \neq \emptyset$.

We denote the sets of solutions of the following two optimization problems by A_1 and A_2, respectively,

$$\begin{gathered} \min h_1(x) \\ x \in C, \\ \min h_2(x) \\ x \in C. \end{gathered} \tag{4.2}$$

Obviously, if we find a solution $x \in A_1 \cap A_2$, then one must have $x \in A$.

Now, let Φ_1 and Φ_2 be two bifunctions from $C \times C$ to $\mathbb{R}$ defined by $\Phi_1(x,y) = h_1(y) - h_1(x)$ and $\Phi_2(x,y) = h_2(y) - h_2(x)$, respectively. It is easy to see that $\mathrm{EP}(\Phi_1) = A_1$ and $\mathrm{EP}(\Phi_2) = A_2$, where $\mathrm{EP}(\Phi_i)$ denotes the set of solutions of the equilibrium problem:

$$\Phi_i(x,y) \geq 0, \quad \forall y \in C,\ i = 1,2, \tag{4.3}$$

respectively. In addition, it is easy to see that Φ_1 and Φ_2 satisfy the conditions (A1)–(A4). Therefore, by setting $m = 2$ in Corollary 3.6, we know that, for any initial guess $x_1 \in H$,

$$\begin{gathered} h_1(y) - h_1(u_{1n}) + \frac{1}{r_{1n}}\langle y - u_{1n}, u_{1n} - x_n\rangle \geq 0, \quad \forall y \in C, \\ h_2(y) - h_2(u_{2n}) + \frac{1}{r_{2n}}\langle y - u_{2n}, u_{2n} - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = \gamma_1 u_{1n} + (1-\gamma_1)u_{2n}, \\ x_{n+1} = (\alpha_n + (1-\alpha_n)\sigma_n)x_n + (1-\alpha_n)(1-\sigma_n)(1-\lambda_n\mu)z_n, \quad \forall n \geq 1. \end{gathered} \tag{4.4}$$

By Corollary 3.6, we know that the sequence $\{x_n\}$ converges strongly to a solution $x^* \in \mathrm{EP}(\Phi_1) \cap \mathrm{EP}(\Phi_2) = A_1 \cap A_2$, which is a solution of the multiobjective optimization problem (4.1).

Acknowledgment

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00050).

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