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Research Article

Topological Vector Space-Valued Cone Metric Spaces and Fixed Point Theorems

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We develop the theory of topological vector space valued cone metric spaces with nonnormal cones. We prove three general fixed point results in these spaces and deduce as corollaries several extensions of theorems about fixed points and common fixed points, known from the theory of (normed-valued) cone metric spaces. Examples are given to distinguish our results from the known ones.

1. Introduction

Ordered normed spaces and cones have applications in applied mathematics, for instance, in using Newton's approximation method [1–4] and in optimization theory [5]. K-metric and K-normed spaces were introduced in the mid-20th century ([2], see also [3, 4, 6]) by using an ordered Banach space instead of the set of real numbers, as the codomain for a metric. Huang and Zhang [7] reintroduced such spaces under the name of cone metric spaces but went further, defining convergent and Cauchy sequences in the terms of interior points of the underlying cone. These and other authors (see, e.g., [8–22]) proved some fixed point and common fixed point theorems for contractive-type mappings in cone metric spaces and cone uniform spaces.

In some of the mentioned papers, results were obtained under additional assumptions about the underlying cone, such as normality or even regularity. In the papers [23, 24], the authors tried to generalize this approach by using cones in topological vector spaces (*tvs*) instead of Banach spaces. However, it should be noted that an old result (see, e.g., [3]) shows that if the underlying cone of an ordered *tvs* is solid and normal, then such *tvs* must be an ordered normed space. So, proper generalizations when passing from norm-valued cone metric spaces of [7] to tvs-valued cone metric spaces can be obtained only in the case of nonnormal cones.

In the present paper we develop further the theory of topological vector space valued cone metric spaces (with nonnormal cones). We prove three general fixed point results in these spaces and deduce as corollaries several extensions of theorems about fixed points and common fixed points, known from the theory of (normed-valued) cone metric spaces.

Examples are given to distinguish our results from the known ones.

2. Tvs-Valued Cone Metric Spaces

Let E be a real Hausdorff topological vector space (*tvs* for short) with the zero vector θ. A proper nonempty and closed subset P of E is called a (convex) *cone* if $P + P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \theta$. We will always assume that the cone P has a nonempty interior int P (such cones are called *solid*).

Each cone P induces a partial order $\preceq$ on E by $x \preceq y \Leftrightarrow y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$. The pair (E, P) is an *ordered topological vector space*.

For a pair of elements x, y in E such that $x \preceq y$, put

$$[x, y] = \{z \in E : x \preceq z \preceq y\}. \tag{2.1}$$

The sets of the form $[x, y]$ are called *order intervals*. It is easily verified that order-intervals are convex. A subset A of E is said to be *order-convex* if $[x, y] \subset A$, whenever $x, y \in A$ and $x \preceq y$.

Ordered topological vector space (E, P) is *order-convex* if it has a base of neighborhoods of θ consisting of order-convex subsets. In this case the cone P is said to be *normal*. In the case of a normed space, this condition means that the unit ball is order-convex, which is equivalent to the condition that there is a number k such that $x, y \in E$ and $0 \preceq x \preceq y$ implies that $\|x\| \leq k\|y\|$. Another equivalent condition is that

$$\inf\{\|x + y\| : x, y \in P \text{ and } \|x\| = \|y\| = 1\} > 0. \tag{2.2}$$

It is not hard to conclude from (2.2) that P is a nonnormal cone in a normed space E if and only if there exist sequences $u_n, v_n \in P$ such that

$$0 \preceq u_n \preceq u_n + v_n, \quad u_n + v_n \longrightarrow 0 \quad \text{but } u_n \nrightarrow 0. \tag{2.3}$$

Hence, in this case, the Sandwich theorem does not hold.

Note the following properties of bounded sets.

If the cone P is solid, then each topologically bounded subset of (E, P) is also order-bounded, that is, it is contained in a set of the form $[-c, c]$ for some $c \in \operatorname{int} P$.

If the cone P is normal, then each order-bounded subset of (E, P) is topologically bounded. Hence, if the cone is both solid and normal, these two properties of subsets of E coincide. Moreover, a proof of the following assertion can be found, for example, in [3].

Theorem 2.1. *If the underlying cone of an ordered tvs is solid and normal, then such tvs must be an ordered normed space.*

Example 2.2. (see [5]) Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$, and let $P = \{x \in E : x(t) \geq 0 \text{ on } [0,1]\}$. This cone is solid (it has the nonempty interior) but is not normal. Consider, for example, $x_n(t) = (1 - \sin nt)/(n+2)$ and $y_n(t) = (1 + \sin nt)/(n+2)$. Since $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| = 2/(n+2) \to 0$, it follows that P is a nonnormal cone.

Now consider the space $E = C^1_{\mathbb{R}}[0,1]$ endowed with the strongest locally convex topology t^*. Then P is also t^*-solid (it has the nonempty t^*-interior), but not t^*-normal. Indeed, if it were normal then, according to Theorem 2.1, the space (E, t^*) would be normed, which is impossible since an infinite-dimensional space with the strongest locally convex topology cannot be metrizable (see, e.g., [25]).

Following [7, 23, 24] we give the following.

Definition 2.3. Let X be a nonempty set and $(E \cdot P)$ an ordered *tvs*. A function $d : X \times X \to E$ is called a *tvs-cone metric* and (X, d) is called a *tvs-cone metric, space* if the following conditions hold:

(C1) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;
(C2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(C3) $d(x,z) \preceq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Let $x \in X$ and $\{x_n\}$ be a sequence in X. Then it is said the following.

(i) $\{x_n\}$ tvs-cone converges to x if for every $c \in E$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_n, x) \ll c$ for all $n > n_0$; we denote it by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is a tvs-cone Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exists a natural number n_0 such that $d(x_m, x_n) \ll c$ for all $m, n > n_0$.

(iii) (X, d) is tvs-cone complete if every tvs-Cauchy sequence is tvs-convergent in X.

Taking into account Theorem 2.1, proper generalizations when passing from norm-valued cone metric spaces of [7] to tvs-cone metric spaces can be obtained only in the case of nonnormal cones.

We will prove now some properties of a real *tvs* E with a solid cone P and a tvs-cone metric space (X, d) over it.

Lemma 2.4. *(a) Let $\theta \preceq x_n \to \theta$ in (E, P), and let $\theta \ll c$. Then there exists n_0 such that $x_n \ll c$ for each $n > n_0$.*

(b) It can happen that $\theta \preceq x_n \ll c$ for each $n > n_0$, but $x_n \nrightarrow \theta$ in (E, P).

(c) It can happen that $x_n \to x$, $y_n \to y$ in the tvs-cone metric d, but that $d(x_n, y_n) \nrightarrow d(x,y)$ in (E, P). In particular, it can happen that $x_n \to x$ in d but that $d(x_n, x) \nrightarrow \theta$ (which is impossible if the cone is normal).

(d) $\theta \preceq u \ll c$ for each $c \in \operatorname{int} P$ implies that $u = \theta$.

(e) $x_n \to x \wedge x_n \to y$ (in the tvs-cone metric) implies that $x = y$.

(f) Each tvs-cone metric space is Hausdorff in the sense that for arbitrary distinct points x and y there exist disjoint neighbourhoods in the topology t_c having the local base formed by the sets of the form $K_c(x) = \{z \in X : d(x,z) \ll c\}$, $c \in \operatorname{int} P$.

Proof. (a) It follows from $x_n \to \theta$ that $x_n \in \text{int}[-c,c] = (\text{int } P - c) \cap (c - \text{int } P)$ for $n > n_0$. From $x_n \in c - \text{int } P$, it follows that $c - x_n \in \text{int } P$, that is, $x_n \ll c$.

(b) Consider the sequences $x_n(t) = (1 - \sin nt)/(n+2)$ and $y_n(t) = (1 + \sin nt)/(n+2)$ from Example 2.2. We know that in the ordered Banach space $C^1_{\mathbb{R}}[0,1]$

$$\theta \preceq x_n \preceq x_n + y_n \tag{2.4}$$

and that $x_n + y_n \to \theta$ (in the norm of E) but that $x_n \nrightarrow \theta$ in this norm. On the other hand, since $x_n \preceq x_n + y_n \to \theta$ and $x_n \preceq x_n + y_n \ll c$, it follows that $x_n \ll c$. Then also $x_n \nrightarrow \theta$ in the *tvs* (E, t^*) (the strongest locally convex topology) but $x_n \ll c$ (also considering the interior with respect to t^*).

We can also consider the tvs-cone metric $d : P \times P \to E$ defined by $d(x,y) = x + y$, $x \neq y$, and $d(x,x) = \theta$. Then for the sequence $\{x_n\}$ we have that $d(x_n, \theta) = x_n + \theta = x_n \to \theta$ in the tvs-cone metric, since $x_n \ll c$, but $x_n \nrightarrow \theta$ in the *tvs* (E, t^*) for otherwise it would tend to θ in the norm of the space E.

(c) Take the sequence $\{x_n\}$ from (b) and $y_n = \theta$. Then $x_n \to \theta$, and $y_n \to \theta$ in the cone metric d since $d(x_n, \theta) = x_n + \theta = x_n \ll c$ and $d(y_n, \theta) = y_n + \theta = \theta + \theta = \theta \ll c$, but $d(x_n, y_n) = x_n + y_n = x_n \nrightarrow \theta = d(\theta, \theta)$ in (E, t^*). This means that a tvs-cone metric may be a discontinuous function.

(d) The proof is the same as in the Banach case. For an arbitrary $c \in \text{int } P$, it is $\theta \preceq u \ll (1/n)c$ for each $n \in \mathbb{N}$, and passing to the limit in $\theta \preceq -u + (1/n)c$ it follows that $\theta \preceq -u$, that is, $u \in -P$. Since P is a cone it follows that $u = \theta$.

(e) From $d(x,y) \preceq d(x, x_n) + d(x_n, y) \ll c/2 + c/2 = c$ for each $n > n_0$ it follows that $d(x,y) \ll c$ (for arbitrary $c \in \text{int } P$), which, by (d), means that $x = y$.

(f) Suppose, to the contrary, that for the given distinct points x and y there exists a point $z \in K_c(x) \cap K_c(y)$. Then $d(x,y) \preceq d(x,z) + d(z,y) \ll c/2 + c/2 = c$ for arbitrary $c \in \text{int } P$, implying that $x = y$, a contradiction. □

The following properties, which can be proved in the same way as in the normed case, will also be needed.

Lemma 2.5. *(a) If $u \preceq v$ and $v \ll w$, then $u \ll w$.*

(b) If $u \ll v$ and $v \preceq w$, then $u \ll w$.

(c) If $u \ll v$ and $v \ll w$, then $u \ll w$.

(d) Let $x \in X$, $\{x_n\}$ and $\{b_n\}$ be two sequences in X and E, respectively, $\theta \ll c$, and $0 \preceq d(x_n, x) \preceq b_n$ for all $n \in \mathbb{N}$. If $b_n \to 0$, then there exists a natural number n_0 such that $d(x_n, x) \ll c$ for all $n \geq n_0$.

3. Fixed Point and Common Fixed Point Results

Theorem 3.1. *Let (X, d) be a tvs-cone metric space and the mappings $f, g, h : X \to X$ satisfy*

$$d(fx, gy) \preceq p d(hx, hy) + q d(hx, fx) + r d(hy, gy) + s d(hx, gy) + t d(hy, fx), \tag{3.1}$$

for all $x, y \in X$, where $p, q, r, s, t \geq 0$, $p + q + r + s + t < 1$, and $q = r$ or $s = t$. If $f(X) \cup g(X) \subset h(X)$ and $h(X)$ is a complete subspace of X, then f, g, and h have a unique point of coincidence. Moreover, if (f, h) and (g, h) are weakly compatible, then f, g, and h have a unique common fixed point.

Recall that a point $u \in X$ is called a coincidence point of the pair (f, g) and v is its point of coincidence if $fu = gu = v$. The pair (f, g) is said to be weakly compatible if for each $x \in X$, $fx = gx$ implies that $fgx = gfx$.

Proof. Let $x_0 \in X$ be arbitrary. Using the condition $f(X) \cup g(X) \subset h(X)$ choose a sequence $\{x_n\}$ such that $hx_{2n+1} = fx_{2n}$ and $hx_{2n+2} = gx_{2n+1}$ for all $n \in \mathbb{N}_0$. Applying contractive condition (3.1) we obtain that

$$\begin{aligned} d(hx_{2n+1}, hx_{2n+2}) &= d(fx_{2n}, gx_{2n+1}) \\ &\preceq pd(hx_{2n}, hx_{2n+1}) + qd(hx_{2n}, hx_{2n+1}) + rd(hx_{2n+1}, hx_{2n+2}) \\ &\quad + sd(hx_{2n}, hx_{2n+2}) + td(hx_{2n+1}, hx_{2n+1}) \\ &\preceq pd(hx_{2n}, hx_{2n+1}) + qd(hx_{2n}, hx_{2n+1}) + rd(hx_{2n+1}, hx_{2n+2}) \\ &\quad + s[d(hx_{2n}, hx_{2n+1}) + d(hx_{2n+1}, hx_{2n+2})]. \end{aligned} \tag{3.2}$$

It follows that

$$(1 - r - s)d(hx_{2n+1}, hx_{2n+2}) \preceq (p + q + s)d(hx_{2n}, hx_{2n+1}), \tag{3.3}$$

that is,

$$d(hx_{2n+1}, hx_{2n+2}) \preceq \frac{p + q + s}{1 - (r + s)} d(hx_{2n}, hx_{2n+1}). \tag{3.4}$$

In a similar way one obtains that

$$d(hx_{2n+2}, hx_{2n+3}) \preceq \frac{p + q + t}{1 - (q + t)} \cdot \frac{p + q + s}{1 - (r + s)} d(hx_{2n}, hx_{2n+1}). \tag{3.5}$$

Now, from (3.4) and (3.5), by induction, we obtain that

$$\begin{aligned} d(hx_{2n+1}, hx_{2n+2}) &\preceq \frac{p + q + s}{1 - (r + s)} d(hx_{2n}, hx_{2n+1}) \\ &\preceq \frac{p + q + s}{1 - (r + s)} \cdot \frac{p + r + s}{1 - (q + t)} d(hx_{2n-1}, hx_{2n}) \\ &\preceq \frac{p + q + s}{1 - (r + s)} \cdot \frac{p + r + s}{1 - (q + t)} \cdot \frac{p + q + s}{1 - (r + s)} d(hx_{2n-2}, hx_{2n-1}) \\ &\preceq \cdots \preceq \frac{p + q + s}{1 - (r + s)} \left(\frac{p + r + t}{1 - (q + t)} \cdot \frac{p + q + s}{1 - (r + s)} \right)^n d(hx_0, hx_1), \\ d(hx_{2n+2}, hx_{2n+3}) &\preceq \frac{p + r + t}{1 - (q + t)} d(hx_{2n+1}, hx_{2n+2}) \\ &\preceq \cdots \preceq \left(\frac{p + r + t}{1 - (q + t)} \cdot \frac{p + q + s}{1 - (r + s)} \right)^{n+1} d(hx_0, hx_1). \end{aligned} \tag{3.6}$$

Let

$$A = \frac{p+q+s}{1-(r+s)}, \qquad B = \frac{p+r+t}{1-(q+t)}. \tag{3.7}$$

In the case $q = r$,

$$AB = \frac{p+q+s}{1-(q+s)} \cdot \frac{p+r+t}{1-(q+t)} = \frac{p+q+s}{1-(q+t)} \cdot \frac{p+r+t}{1-(r+s)} < 1 \cdot 1 = 1, \tag{3.8}$$

and if $s = t$,

$$AB = \frac{p+q+s}{1-(r+s)} \cdot \frac{p+r+s}{1-(q+t)} < 1 \cdot 1 = 1. \tag{3.9}$$

Now, for $n < m$, we have

$$\begin{aligned} d(hx_{2n+1}, hx_{2m+1}) &\preceq d(hx_{2n+1}, hx_{2n+2}) + \cdots + d(hx_{2n}, hx_{2m+1}) \\ &\preceq \left(A\sum_{i=n}^{m-1} (AB)^i + \sum_{i=n+1}^{m} (AB)^i \right) d(hx_0, hx_1) \\ &\preceq \left(\frac{A(AB)^n}{1-AB} + \frac{(AB)^{n+1}}{1-AB} \right) d(hx_0, hx_1) \\ &= (1+B)\frac{A(AB)^n}{1-AB} d(hx_0, hx_1). \end{aligned} \tag{3.10}$$

Similarly, we obtain

$$\begin{aligned} d(hx_{2n}, hx_{2m+1}) &\preceq (1+A)\frac{(AB)^n}{1-AB} d(hx_0, hx_1), \\ d(hx_{2n}, hx_{2m}) &\preceq (1+A)\frac{(AB)^n}{1-AB} d(hx_0, hx_1), \\ d(hx_{2n+1}, hx_{2m}) &\preceq (1+B)\frac{A(AB)^n}{1-AB} d(hx_0, hx_1). \end{aligned} \tag{3.11}$$

Hence, for $n < m$

$$d(hx_n, hx_m) \preceq \max\left\{ (1+B)\ \frac{A(AB)^n}{1-AB}, (1+A)\frac{(AB)^n}{1-AB} \right\} d(hx_0, hx_1) = \lambda_n d(hx_0, hx_1), \tag{3.12}$$

where $\lambda_n \to 0$, as $n \to \infty$.

Now, using properties (a) and (d) from Lemma 2.5 and only the assumption that the underlying cone is solid, we conclude that $\{hx_n\}$ is a Cauchy sequence. Since the subspace $h(X)$ is complete, there exist $u, v \in X$ such that $hx_n \to v = hu$ $(n \to \infty)$.

We will prove that $hu = fu = gu$. Firstly, let us estimate that $d(hu, fu) = d(v, fu)$. We have that

$$d(hu, fu) \preceq d(hu, hx_{2n+1}) + d(hx_{2n+1}, fu) = d(v, hx_{2n+1}) + d(fu, gx_{2n+1}). \tag{3.13}$$

By the contractive condition (3.1), it holds that

$$\begin{aligned}
d(fu, gx_{2n+1}) &\preceq pd(hu, hx_{2n+1}) + qd(hu, fu) + rd(hx_{2n+1}, gx_{2n+1}) \\
&\quad + sd(hu, gx_{2n+1}) + td(hx_{2n+1}, fu) \\
&= pd(v, fx_{2n}) + qd(v, fu) + rd(fx_{2n}, gx_{2n+1}) \\
&\quad + sd(v, gx_{2n+1}) + td(fx_{2n}, fu) \\
&\preceq pd(v, fx_{2n}) + qd(v, fu) + rd(fx_{2n}, gx_{2n+1}) \\
&\quad + sd(v, gx_{2n+1}) + td(fx_{2n}, v) + td(v, fu).
\end{aligned} \tag{3.14}$$

Now it follows from (3.13) that

$$\begin{aligned}
(1 - q - t)d(v, fu) &\preceq d(v, hx_{2n+1}) + pd(v, fx_{2n}) \\
&\quad + rd(fx_{2n}, gx_{2n+1}) + sd(v, gx_{2n+1}) + td(fx_{2n}, v).
\end{aligned} \tag{3.15}$$

that is,

$$\begin{aligned}
&(1 - q - t)d(v, fu) \preceq (1 + s)d(v, gx_{2n+1}) + (p + t)d(v, fx_{2n}) + rd(fx_{2n}, gx_{2n+1}), \\
&d(v, fu) \preceq \frac{1 + s}{1 - q - t}d(v, gx_{2n+1}) + \frac{p + t}{1 - q - t}d(v, fx_{2n}) + \frac{r}{1 - q - t}d(fx_{2n}, gx_{2n+1}).
\end{aligned} \tag{3.16}$$

Let $c \in \operatorname{int} P$. Then there exists n_0 such that for $n > n_0$ it holds that

$$d(v, gx_{2n+1}) \ll \frac{1 - q - t}{3(1 + s)}c, d(v, fx_{2n}) \ll \frac{1 - q - t}{3(p + t)}c \tag{3.17}$$

and $d(fx_{2n}, gx_{2n+1}) \ll (1 - q - t)/(3r)c$, that is, $d(v, fu) \ll c$ for $n > n_0$. Since $c \in \operatorname{int} P$ was arbitrary, it follows that $d(v, fu) = 0$, that is, $fu = hu = v$.

Similarly using that

$$\begin{aligned}
d(hu, gu) &\preceq d(hu, hx_{2n+1}) + d(hx_{2n+1}, gu) \\
&= d(hu, hx_{2n+1}) + d(fx_{2n}, gu),
\end{aligned} \tag{3.18}$$

it can be deduced that $hu = gu = v$. It follows that v is a common point of coincidence for f, g, and h, that is,

$$v = fu = gu = hu. \tag{3.19}$$

Now we prove that the point of coincidence of f, g, h is unique. Suppose that there is another point $v_1 \in X$ such that

$$v_1 = fu_1 = gu_1 = hu_1 \tag{3.20}$$

for some $u_1 \in X$. Using the contractive condition we obtain that

$$\begin{aligned} d(v, v_1) &= d(fu, gu_1) \\ &\preceq pd(hu, hu_1) + qd(hu, fu) + rd(hu_1, gu_1) + sd(hu, gu_1) + td(hu_1 fu) \\ &= pd(v, v_1) + q \cdot 0 + r \cdot 0 + sd(v, v_1) + td(v, v_1) = (p + s + t)d(v, v_1). \end{aligned} \tag{3.21}$$

Since $p + s + t < 1$, it follows that $d(v, v_1) = 0$, that is, $v = v_1$.

Using weak compatibility of the pairs (f, h) and (g, h) and proposition 1.12 from [16], it follows that the mappings f, g, h have a unique common fixed point, that is, $fv = gv = hv = v$. □

Corollary 3.2. *Let (X, d) be a tvs-cone metric space and the mappings $f, g, h : X \to X$ satisfy*

$$d(fx, gy) \preceq \alpha d(hx, hy) + \beta[d(hx, fx) + d(hy, gy)] + \gamma[d(hx, gy) + d(hy, fx)] \tag{3.22}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. If $f(X) \cup g(X) \subset h(X)$ and $h(X)$ is a complete subspace of X, then f, g, and h have a unique point of coincidence. Moreover, if (f, h) and (g, h) are weakly compatible, then f, g, and h have a unique common fixed point.

Putting in this corollary $h = i_X$ and taking into account that each self-map is weakly compatible with the identity mapping, we obtain the following.

Corollary 3.3. *Let (X, d) be a complete tvs-cone metric space, and let the mappings $f, g : X \to X$ satisfy*

$$d(fx, gy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, gy)] + \gamma[d(x, gy) + d(y, fx)] \tag{3.23}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then f and g have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g, and conversely.

In the case of a cone metric space with a normal cone, this result was proved in [14].

Now put first $g = f$ in Theorem 3.1 and then $h = g$. Choosing appropriate values for coefficients, we obtain the following.

Corollary 3.4. *Let (X, d) be a tvs-cone metric space. Suppose that the mappings $f, g : X \to X$ satisfy the contractive condition*

$$d(fx, fy) \preceq \lambda \cdot d(gx, gy), \tag{3.24}$$

$$d(fx, fy) \preceq \lambda \cdot (d(fx, gx) + d(fy, gy)), \tag{3.25}$$

or

$$d(fx, fy) \preceq \lambda \cdot (d(fx, gy) + d(fy, gx)), \tag{3.26}$$

for all $x, y \in X$, where λ is a constant ($\lambda \in [0,1)$ in (3.24) and $\lambda \in [0,1/2)$ in (3.25) and (3.26)). If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

In the case when the space E is normed and the cone P is normal, these results were proved in [9].

Similarly one obtains the following.

Corollary 3.5. *Let (X, d) be a tvs-cone metric space, and let $f, g : X \to X$ be such that $f(X) \subset g(X)$. Suppose that*

$$d(fx, fy) \preceq \alpha d(fx, gx) + \beta d(fy, gy) + \gamma d(gx, gy), \tag{3.27}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \in [0,1)$ and $\alpha + \beta + \gamma < 1$, and let $fx = gx$ imply that $fgx = ggx$ for each $x \in X$. If $f(X)$ or $g(X)$ is a complete subspace of X, then the mappings f and g have a unique common fixed point in X. Moreover, for any $x_0 \in X$, the f-g-sequence $\{fx_n\}$ with the initial point x_0 converges to the fixed point.

Here, an f-g-sequence (also called a Jungck sequence) $\{fx_n\}$ is formed in the following way. Let $x_0 \in X$ be arbitrary. Since $f(X) \subset g(X)$, there exists $x_1 \in X$ such that $fx_0 = gx_1$. Having chosen $x_n \in X$, $x_{n+1} \in X$ is chosen such that $gx_{n+1} = fx_n$.

In the case when the space E is normed and under the additional assumption that the cone P is normal, these results were firstly proved in [10].

Corollary 3.6. *Let (X, d) be a complete tvs-cone metric space. Suppose that the mapping $f : X \to X$ satisfies the contractive condition*

$$d(fx, fy) \preceq \lambda \cdot d(x, y), \tag{3.28}$$

$$d(fx, fy) \preceq \lambda \cdot (d(fx, x) + d(fy, y)), \tag{3.29}$$

or

$$d(fx, fy) \preceq \lambda \cdot (d(fx, y) + d(fy, x)) \tag{3.30}$$

for all $x, y \in X$, where λ is a constant ($\lambda \in [0,1)$ in (3.28) and $\lambda \in [0,1/2)$ in (3.29) and (3.30)). Then f has a unique fixed point in X, and for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.

In the case when the space E is normed and under the additional assumption that the cone P is normal, these results were firstly proved in [7]. The normality condition was removed in [8].

Finally, we give an example of a situation where Theorem 3.1 can be applied, while the results known so far cannot.

Example 3.7 (see [26, Example 3.3]). Let $X = \{1, 2, 3\}$, $E = C^1_{\mathbb{R}}[0, 1]$ with the cone P as in Example 2.2 and endowed with the strongest locally convex topology t^*. Let the metric $d : X \times X \to E$ be defined by $d(x, y)(t) = 0$ if $x = y$ and $d(1, 2)(t) = d(2, 1)(t) = 6e^t$, $d(1, 3)(t) = d(3, 1)(t) = (30/7)e^t$, and $d(2, 3)(t) = d(3, 2)(t) = (24/7)e^t$. Further, let $f, g : X \to X$ be given by, $fx = 1$, $x \in X$ and $g1 = g3 = 1$, $g2 = 3$. Finally, let $h = I_X$.

Taking $p = q = r = s = 0$, $t = 5/7$, all the conditions of Theorem 3.1 are fulfilled. Indeed, since $f1 = g1 = f3 = g3 = 1$, we have only to check that

$$d(f3, g2) \preceq 0 \cdot d(3, 2) + 0 \cdot d(3, f3) + 0 \cdot d(2, g2) + 0 \cdot d(3, g2) + \frac{5}{7} d(2, f3), \tag{3.31}$$

which is equivalent to

$$\frac{30}{7} e^t \leq \frac{5}{7} d(2, f3)(t) = \frac{5}{7} d(2, 1)(t) = \frac{5}{7} \cdot 6e^t = \frac{30}{7} e^t. \tag{3.32}$$

Hence, we can apply Theorem 3.1 and conclude that the mappings f, g, h have a unique common fixed point ($u = 1$).

On the other hand, since the space (E, P, t^*) is not an ordered Banach space and its cone is not normal, neither of the mentioned results from [7–10, 14] can be used to obtain such conclusion. Thus, Theorem 3.1 and its corollaries are proper extensions of these results.

Note that an example of similar kind is also given in [24].

The following example shows that the condition "$p = q$ or $s = t$" in Theorem 3.1 cannot be omitted.

Example 3.8 (see [26, Example 3.4]). Let $X = \{x, y, u, v\}$, where $x = (0, 0, 0)$, $y = (4, 0, 0)$, $u = (2, 2, 0)$, and $v = (2, -2, 1)$. Let d be the Euclidean metric in $\mathbb{R}^3$, and let the tvs-cone metric $d_1 : X \times X \to E$ (E, P, and t^* are as in the previous example) be defined in the following way: $d_1(a, b)(t) = d(a, b) \cdot \varphi(t)$, where $\varphi \in P$ is a fixed function, for example, $\varphi(t) = e^t$. Consider the mappings

$$f = \begin{pmatrix} x & y & u & v \\ u & v & v & u \end{pmatrix}, \qquad g = \begin{pmatrix} x & y & u & v \\ y & x & y & x \end{pmatrix}, \tag{3.33}$$

and let $h = i_X$. By a careful computation it is easy to obtain that

$$d(fa, gb) \leq \frac{3}{4} \max\{d(a, b), d(a, fa), d(b, gb), d(a, gb), d(b, fa)\}, \tag{3.34}$$

for all $a, b \in X$. We will show that f and g satisfy the following contractive condition: there exist $p, q, r, s, t \geq 0$ with $p + q + r + s + t < 1$ and $q \neq r$, $s \neq t$ such that

$$d_1(fa, gb) \preceq pd_1(a, b) + qd_1(a, fa) + rd_1(b, gb) + sd_1(a, gb) + td_1(b, fa) \tag{3.35}$$

holds true for all $a, b \in X$. Obviously, f and g do not have a common fixed point.

Taking (3.34) into account, we have to consider the following cases.

(1) In case $d_1(fa, gb) \preceq (3/4)d_1(a, b)$, then (3.35) holds for $p = 3/4$, $r = t = 0$ and $q = s = 1/9$.

(2) In case $d_1(fa, gb) \preceq (3/4)d_1(a, fa)$, then (3.35) holds for $q = 3/4$, $p = r = t = 0$ and $s = 1/5$.

(3) In case $d_1(fa, gb) \preceq (3/4)d_1(b, gb)$, then (3.35) holds for $r = 3/4$, $p = q = t = 0$ and $s = 1/5$.

(4) In case $d_1(fa, gb) \preceq (3/4)d_1(a, gb)$, then (3.35) holds for $s = 3/4$, $p = r = t = 0$ and $q = 1/5$.

(5) In case $d_1(fa, gb) \preceq (3/4)d_1(b, fa)$, then (3.35) holds for $t = 3/4$, $p = r = s = 0$ and $q = 1/5$.

4. Quasicontractions in Tvs-Cone Metric Spaces

Definition 4.1. Let (X, d) be a tvs-cone metric space, and let $f, g : X \to X$. Then, f is called a *quasi-contraction* (resp., a *g-quasi-contraction*) if for some constant $\lambda \in [0, 1)$ and for all $x, y \in X$, there exists

$$\begin{gathered} u \in C(x,y) = \{d(x,y), d(x,fx), d(x,fy), d(y,fy), d(y,fx)\}, \\ (\text{resp.}, u \in C(g;x,y) = \{d(gx,gy), d(gx,fx), d(gx,fy), d(gy,fy), d(gy,fx)\}), \end{gathered} \tag{4.1}$$

such that

$$d(fx, fy) \preceq \lambda \cdot u. \tag{4.2}$$

Theorem 4.2. *Let (X, d) be a complete tvs-cone metric space, and let $f, g : X \to X$ be such that $fX \subset gX$ and gX is closed. If f is a g-quasi-contraction with $\lambda \in [0, 1/2)$, then f and g have a unique point of coincidence. Moreover, if the pair (f, g) is weakly compatible or, at least, occasionally weakly compatible, then f and g have a unique common fixed point.*

Recall that the pair (f, g) of self-maps on X is called occasionally weakly compatible (see [27] or [28]) if there exists $x \in X$ such that $fx = gx$ and $fgx = gfx$.

Proof. Let us remark that the condition $fX \subset gX$ implies that starting with an arbitrary $x_0 \in X$, we can construct a sequence $\{y_n\}$ of points in X such that $y_n = fx_n = gx_{n+1}$ for all $n \geq 0$. We will prove that $\{y_n\}$ is a Cauchy sequence. First, we show that

$$d(y_n, y_{n+1}) \preceq \frac{\lambda}{1-\lambda} d(y_{n-1}, y_n) \tag{4.3}$$

for all $n \geq 1$. Indeed,

$$d(y_n, y_{n+1}) = d(fx_n, fx_{n+1}) \preceq \lambda u_n, \tag{4.4}$$

where

$$\begin{aligned} u_n \in\ & \{d(gx_n, gx_{n+1}), d(gx_n, fx_n), d(gx_{n+1}, fx_{n+1}), d(gx_n, fx_{n+1}), d(gx_{n+1}, fx_n)\} \\ &= \{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), d(y_{n-1}, y_{n+1}), d(y_n, y_n)\} \\ &= \{d(y_{n-1}, y_n), d(y_n, y_{n+1}), d(y_{n-1}, y_{n+1}), \theta\}. \end{aligned} \tag{4.5}$$

The following four cases may occur:

(1) First, $d(y_n, y_{n+1}) \preceq \lambda d(y_{n-1}, y_n) \preceq \lambda/(1-\lambda) d(y_{n-1}, y_n)$.

(2) Second, $d(y_n, y_{n+1}) \preceq \lambda d(y_n, y_{n+1})$ and so $d(y_n, y_{n+1}) = \theta$. In this case, (4.3) follows immediately, because $\lambda < \lambda/(1-\lambda)$.

(3) Third, $d(y_n, y_{n+1}) \preceq \lambda d(y_{n-1}, y_{n+1}) \preceq \lambda d(y_{n-1}, y_n) + \lambda d(y_n, y_{n+1})$. It follows that (4.3) holds.

(4) Fourth, $d(y_n, y_{n+1}) \preceq \lambda \cdot \theta = \theta$ and so $d(y_n, y_{n+1}) = \theta$. Hence, (4.3) holds.

Thus, by putting $h = \lambda/(1-\lambda) < 1$, we have that $d(y_n, y_{n+1}) \preceq h d(y_{n-1}, y_n)$. Now, using (4.3), we have

$$d(y_n, y_{n+1}) \preceq h d(y_{n-1}, y_n) \preceq \cdots \preceq h^n d(y_0, y_1), \tag{4.6}$$

for all $n \geq 1$. It follows that

$$\begin{aligned} d(y_n, y_m) &\preceq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \cdots + d(y_{m+1}, y_m) \\ &\preceq \left(h^{n-1} + h^{n-2} + \ldots + h^m\right) d(y_0, y_1) \\ &\preceq \frac{h^m}{1-h} d(y_0, y_1) \longrightarrow \theta, \quad \text{as } m \longrightarrow \infty. \end{aligned} \tag{4.7}$$

Using properties (a) and (d) from Lemma 2.5, we obtain that $\{y_n\}$ is a Cauchy sequence. Therefore, since X is complete and gX is closed, there exists $z \in X$ such that

$$y_n = fx_n = gx_{n+1} \longrightarrow gz, \quad \text{as } n \longrightarrow \infty. \tag{4.8}$$

Now we will show that $fz = gz$.

By the definition of g-quasicontraction, we have that

$$d(fx_n, fz) \preceq \lambda \cdot u_n, \tag{4.9}$$

where $u_n \in \{d(gx_n, gz), d(gx_n, fx_n), d(gz, fz), d(gz, fx_n), d(gx_n, fz)\}$. Observe that $d(gz, fz) \preceq d(gz, fx_n) + d(fx_n, fz)$ and $d(gx_n, fz) \preceq d(gx_n, fx_n) + d(fx_n, fz)$. Now let $0 \ll c$ be given. In all of the possible five cases there exists $n_0 \in \mathbb{N}$ such that (using (4.9)) one obtains that $d(fx_n, fz) \ll c$:

(1) $d(fx_n, fz) \preceq \lambda \cdot d(gx_n, gz) \ll \lambda(c/\lambda) = c$;

(2) $d(fx_n, fz) \preceq \lambda \cdot d(gx_n, fx_n) \ll \lambda(c/\lambda) = c$;

(3) $d(fx_n, fz) \preceq \lambda \cdot d(gz, fz) \preceq \lambda d(gz, fx_n) + \lambda d(fx_n, fz)$; it follows that $d(fx_n, fz) \preceq (\lambda/(1-\lambda))d(gz, fx_n) \ll (\lambda/(1-\lambda))((1-\lambda)c/\lambda) = c$;

(4) $d(fx_n, fz) \preceq \lambda \cdot d(gz, fx_n) \ll \lambda(c/\lambda) = c$;

(5) $d(fx_n, fz) \preceq \lambda \cdot d(gx_n, fz) \preceq \lambda d(gx_n, fx_n) + \lambda d(fx_n, fz)$; it follows that $d(fx_n, fz) \preceq (\lambda/(1-\lambda))d(gx_n, fx_n) \ll (\lambda/(1-\lambda))((1-\lambda)c/\lambda) = c$.

It follows that $fx_n \to fz$ $(n \to \infty)$. The uniqueness of limit in a cone metric space implies that $fz = gz = t$. Thus, z is a coincidence point of the pair (f, g), and t is its point of coincidence. It can be showed in a standard way that this point of coincidence is unique. Using lemma 1.6 of [27] one readily obtains that, in the case when the pair (f, g) is occasionally weakly compatible, the point t is the unique common fixed point of f and g. □

In the normed case and assuming that the cone is normal (but letting $\lambda \in [0, 1)$), this theorem was proved in [11].

Puting $g = i_X$ in Theorem 4.2 we obtain the following.

Corollary 4.3. *Let (X, d) be a complete tvs-cone metric space, and let the mapping $f : X \to X$ be a quasi-contraction with $\lambda \in [0, 1/2)$. Then f has a unique fixed point in X, and for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.*

In the case of normed-valued cone metric spaces and under the assumption that the underlying cone P is normal (and with $\lambda \in [0, 1)$), this result was obtained in [12]. Normality condition was removed in [13].

From Theorem 4.2, as corollaries, among other things, we again recover and extend the results of Huang and Zhang [7] and Rezapour and Hamlbarani [8]. The following three corollaries follow in a similar way.

In the next corollary, we extend the well-known result [29, (9′)].

Corollary 4.4. *Let (X, d) be a complete tvs-cone metric space, and let $f, g : X \to X$ be such that $fX \subset gX$ and gX is closed. Further, let for some constant $\lambda \in [0, 1)$ and every $x, y \in X$ there exists*

$$u = u(x, y) \in \{d(gx, gy), d(gx, fx), d(gy, fy)\} \tag{4.10}$$

such that

$$d(fx, fy) \preceq \lambda \cdot u. \tag{4.11}$$

Then f and g have a unique point of coincidence. Moreover, if the pair (f, g) is occasionally weakly compatible, then they have a unique common fixed point.

We can also extend the well-known Bianchini's result [29, (5)]

Corollary 4.5. *Let (X, d) be a complete tvs-cone metric space, and let $f, g : X \to X$ be such that $fX \subset gX$ and gX is closed. Further, let for some constant $\lambda \in [0, 1)$ and every $x, y \in X$, there exists*

$$u = u(x, y) \in \{d(gx, fx), d(gy, fy)\} \tag{4.12}$$

such that

$$d(fx, fy) \leq \lambda \cdot u. \tag{4.13}$$

Then f and g have a unique point of coincidence. Moreover, if the pair (f, g) is occasionally weakly compatible, then they have a unique common fixed point.

In the next corollary, we extend the well-known result of Jungck [30, Theorem 1.1].

Corollary 4.6. *Let (X, d) be a complete tvs-cone metric space, and let $f, g : X \to X$ be such that $fX \subset gX$ and gX is closed. Further, let for some constant $\lambda \in [0, 1)$ and every $x, y \in X$,*

$$d(fx, fy) \preceq \lambda \cdot d(gx, gy). \tag{4.14}$$

Then f and g have a unique point of coincidence. Moreover, if the pair (f, g) is occasionally weakly compatible, then they have a unique common fixed point.

Remark 4.7. Note that in the previous three corollaries it is possible that the parameter λ takes values from $[0, 1)$ (and not only in $[0, 1/2)$ as in Theorem 4.2). Namely, it is possible to show that the sequence $\{y_n\}$ used in the proof, is a Cauchy sequence because the condition on u is stronger.

Now, we prove the main result of Das and Naik [31] in the frame of tvs-cone metric spaces in which the cone need not be normal.

Theorem 4.8. *Let (X, d) be a complete tvs-cone metric space. Let g be a self-map on X such that g^2 is continuous, and let f be any self-map on X that commutes with g. Further let f and g satisfy*

$$fgX \subset g^2X, \tag{4.15}$$

and let f be a g-quasi-contraction. Then f and g have a unique common fixed point.

Proof. By (4.15), starting with an arbitrary $x_0 \in gX$, we can construct a sequence $\{x_n\}$ of points in fX such that $y_n = fx_n = gx_{n+1}$, $n \geq 0$ (as in Theorem 4.2). Now $gy_{n+1} = gfx_{n+1} = fgx_{n+1} = fy_n = z_n$, $n \geq 1$. It can be proved as in Theorem 4.2 that $\{z_n\}$ is a Cauchy sequence and hence convergent to some $z \in X$. Further, we will show that $g^2z = fgz$. Since

$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} gfx_n = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} fy_n = \lim_{n\to\infty} z_n = z, \tag{4.16}$$

it follows that

$$\lim_{n\to\infty} g^4x_n = \lim_{n\to\infty} g^3fx_n = \lim_{n\to\infty} fg^3x_n = g^2z, \tag{4.17}$$

because g^2 is continuous. Now, we obtain

$$d\left(g^2z, fgz\right) \preceq d\left(g^2z, g^3fx_n\right) + d\left(g^3fx_n, fgz\right) \preceq d\left(g^2z, g^3fx_n\right) + \lambda \cdot u_n, \tag{4.18}$$

where

$$u_n \in \left\{d\left(g^4x_n, f^2z\right), d\left(g^4x_n, fg^3x_n\right), d\left(g^2z, fgz\right), d\left(g^4x_n, fgz\right), d\left(g^2z, fg^3x_n\right)\right\}. \tag{4.19}$$

Let $\theta \ll c$ be given. Since $g^3fx_n \to g^2z$ and $g^4x_n \to g^2z$, choose a natural number n_0 such that for all $n \geq n_0$ we have $d(g^2z, g^3fx_n) \ll c(1-\lambda)/2$ and $d(g^4x_n, fg^3x_n) \ll (1-\lambda)c/2\lambda$. Again, we have the following cases:

(a)

$$d\left(g^2z, fgz\right) \preceq d\left(g^2z, g^3fx_n\right) + \lambda d\left(g^4x_n, g^2z\right) \ll \frac{c}{2} + \lambda\frac{c}{2\lambda} = c. \tag{4.20}$$

(b)

$$\begin{aligned} d\left(g^2z, fgz\right) &\preceq d\left(g^2z, g^3fx_n\right) + \lambda d\left(g^4x_n, fg^3z\right) \\ &\preceq d\left(g^2z, g^3fx_n\right) + \lambda d\left(g^4x_n, g^2z\right) + \lambda d\left(g^2z, fg^3x_n\right) \\ &= (1+\lambda)d\left(g^2z, g^3fx_n\right) + \lambda d\left(g^4x_n, g^2z\right) \\ &\ll (1+\lambda)\frac{c(1-\lambda)}{2} + \lambda\frac{(1-\lambda)c}{2\lambda} \ll c. \end{aligned} \tag{4.21}$$

(c)

$$\begin{aligned} d\left(g^2z, fgz\right) &\preceq d\left(g^2z, g^3fx_n\right) + \lambda d\left(g^2z, fgz\right). \text{ Hence,} \\ d\left(g^2z, fgz\right) &\preceq \frac{1}{1-\lambda}d\left(g^2z, g^3fx_n\right) \ll \frac{1}{1-\lambda}\frac{c(1-\lambda)}{2} = c. \end{aligned} \tag{4.22}$$

(d)

$$\begin{aligned} d\left(g^2z, fgz\right) &\preceq d\left(g^2z, g^3fx_n\right) + \lambda d\left(g^4x_n, fgz\right) \\ &\preceq d\left(g^2z, g^3fx_n\right) + \lambda d\left(g^4x_n, g^2z\right) + d\left(g^2z, fgz\right). \text{ Hence,} \\ d\left(g^2z, fgz\right) &\preceq \frac{1}{1-\lambda}d\left(g^2z, g^3fx_n\right) + \frac{\lambda}{1-\lambda}d\left(g^4x_n, g^2z\right) \\ &\ll \frac{1}{1-\lambda}\frac{c(1-\lambda)}{2} + \frac{\lambda}{1-\lambda}\frac{(1-\lambda)c}{2\lambda} = c. \end{aligned} \tag{4.23}$$

(e)

$$d\left(g^2z, fgz\right) \preceq d\left(g^2z, g^3fx_n\right) + \lambda d\left(g^2z, fg^3x_n\right) \ll \frac{c}{2} + \lambda\frac{c}{2\lambda} = c. \tag{4.24}$$

Therefore, $d(g^2z, fgz) \ll c$ for all $\theta \ll c$. By property (d) of Lemma 2.4, $g^2z = fgz$, and so fgz is a common fixed point for f and g. Indeed, putting in the contractivity condition $x = fgz, y = gz$, we get $f(fgz) = fgz$. Since $g^2z = fgz$, that is, $g(gz) = f(gz)$, we have that $g(fgz) = fg^2z = f(fgz) = fgz$. □

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