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Research Article

Some Fixed Point Theorems of Integral Type Contraction in Cone Metric Spaces

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We define a new concept of integral with respect to a cone. Moreover, certain fixed point theorems in those spaces are proved. Finally, an extension of Meir-Keeler fixed point in cone metric space is proved.

1. Introduction

In 2007, Huang and Zhang in [1] introduced cone metric space by substituting an ordered Banach space for the real numbers and proved some fixed point theorems in this space. Many authors study this subject and many fixed point theorems are proved; see [2–5]. In this paper, the concept of integral in this space is introduced and a fixed point theorem is proved. In order to do this, we recall some definitions, examples, and lemmas from [1, 4] as follows.

Let E be a real Banach space. A subset P of E is called a cone if and only if the following hold:

(i) P is closed, nonempty, and $P \neq \{0\}$,

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply that $ax + by \in P$,

(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for

$y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$, for all $x, y \in E$. The least positive number satisfying above is called the normal constant [1].

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent [1]. Also every regular cone is normal [4]. In addition, there are some nonnormal cones.

Example 1.1. Suppose $E = C^2_{\mathbb{R}}([0,1])$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and consider the cone $P = \{f \in E\colon f \geq 0\}$. For all $K \geq 1$, set $f(x) = x$ and $g(x) = x^{2K}$. Then $0 \leq g \leq f, \|f\| = 2$ and $\|g\| = 2K + 1$. Since $K\|f\| < \|g\|, K$ is not normal constant of P. Therefore, P is non-normal cone.

From now on, we suppose that E is a real Banach space, P is a cone in E with *int* $P \neq \emptyset$, and $\leq$ is partial ordering with respect to P. Let X be a nonempty set. As it has been defined in [1], a function $d : X \times X \to E$ is called a cone metric on X if it satisfies the following conditions:

(i) $d(x,y) \geq 0$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$, for all $x, y \in X$,

(iii) $d(x,y) \leq d(x,z) + d(y,z)$, for all $x, y, z \in X$.

Then (X,d) is called a cone metric space.

Example 1.2. Suppose $E = l^1, P = \{\{x_n\}_{n\in\mathbb{N}} \in E : x_n \geq 0, \text{for all } n, (X,\rho)$ is a metric space and $d : X\times X \to E$ is defined by $d(x,y) = \{\rho(x,y)/2^n\}_{n\in\mathbb{N}}$. Then (X,d) is a cone metric space and the normal constant of P is equal to 1.

Definition 1.3. Let (X,d) be a cone metric space. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. If for any $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent to x, and x is the limit of $\{x_n\}_{n\in\mathbb{N}}$. We denote this by

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \longrightarrow x \quad (n \longrightarrow \infty). \tag{1.1}$$

Definition 1.4. Let (X,d) be a cone metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. If for any $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that for all $m, n > n_0, d(x_n, x_m) \ll c$, then $\{x_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence in X.

Definition 1.5. Let (X,d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Definition 1.6. Let (X,d) be a cone metric space. Let T be a self-map on X. If for all sequence $\{x_n\}_{n\in\mathbb{N}}$ in X,

$$\lim_{n\to\infty} x_n = x \ \text{ implies } \lim_{n\to\infty} T(x_n) = T(x), \tag{1.2}$$

then T is called continuous on X.

The following lemmas are useful for us to prove the main result.

Lemma 1.7. *Let (X, d) be a cone metric space and P a normal cone with normal constant K. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then $\{x_n\}_{n\in\mathbb{N}}$ converges to x if and only if*

$$\lim_{n\in\mathbb{N}} d(x_n, x) = 0. \tag{1.3}$$

Lemma 1.8. *Let (X, d) be a cone metric space and P a normal cone with normal constant K. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if*

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0. \tag{1.4}$$

Lemma 1.9. *Let (X, d) be a cone metric space and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X. If $\{x_n\}_{n\in\mathbb{N}}$ is convergent, then it is a Cauchy sequence.*

Lemma 1.10. *Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \to x$, $y_n \to y$ $(n \to \infty)$. Then*

$$d(x_n, y_n) \longrightarrow d(x, y) \quad (n \longrightarrow \infty). \tag{1.5}$$

The following example is a cone metric space.

Example 1.11. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in Ex, y \geq 0\}$, and $X = \mathbb{R}$. Suppose that $d : X \times X \to E$ is defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Theorem 1.12. *Let (X, d) be a complete cone metric space and P a normal cone with normal constant K. Suppose the mapping $f : X \to X$ satisfies the contractive condition*

$$d(fx, fy) \leq \beta d(x, y) \tag{1.6}$$

for all $x, y \in X$, where $\beta \in (0, 1)$ is a constant. Then f has a unique fixed point $x_0 \in X$. Also, for all $x \in X$, the sequence $\{f^n(x)\}_{n=1}^{\infty}$ converges to x_0.

2. Certain Integral Type Contraction Mapping in Cone Metric Space

In 2002, Branciari in [6] introduced a general contractive condition of integral type as follows.

Theorem 2.1. *Let (X, d) be a complete metric space, $\alpha \in (0, 1)$, and $f : X \to X$ is a mapping such that for all $x, y \in X$,*

$$\int_0^{d(f(x), f(y))} \phi(t)dt \leq \alpha \int_0^{d(x,y)} \phi(t)dt, \tag{2.1}$$

where $\phi : [0,+\infty) \to [0,+\infty)$ is nonnegative and Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0,+\infty)$ such that for each $\epsilon > 0$, $\int_0^{\epsilon}\phi(t)dt > 0$, then f has a unique fixed point $a \in X$, such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

In this section we define a new concept of integral with respect to a cone and introduce the Branciari's result in cone metric spaces.

Definition 2.2. Suppose that P is a normal cone in E. Let $a, b \in E$ and $a < b$. We define

$$\begin{aligned} [a,b] &:= \{x \in E : x = tb + (1-t)a, \quad \text{for some } t \in [0,1]\}, \\ [a,b) &:= \{x \in E : x = tb + (1-t)a, \quad \text{for some } t \in [0,1)\}. \end{aligned} \tag{2.2}$$

Definition 2.3. The set $\{a = x_0, x_1, \ldots, x_n = b\}$ is called a partition for $[a,b]$ if and only if the sets $\{[x_{i-1}, x_i)\}_{i=1}^{n}$ are pairwise disjoint and $[a,b] = \{\bigcup_{i=1}^{n}[x_{i-1}, x_i)\} \cup \{b\}$.

Definition 2.4. For each partition Q of $[a,b]$ and each increasing function $\phi : [a,b] \to P$, we define cone lower summation and cone upper summation as

$$\begin{aligned} L_n^{\mathrm{Con}}(\phi, Q) &= \sum_{i=0}^{n-1}\phi(x_i)\|x_i - x_{i+1}\|, \\ U_n^{\mathrm{Con}}(\phi, Q) &= \sum_{i=0}^{n-1}\phi(x_{i+1})\|x_i - x_{i+1}\|, \end{aligned} \tag{2.3}$$

respectively.

Definition 2.5. Suppose that P is a normal cone in E. $\phi : [a,b] \to P$ is called an integrable function on $[a,b]$ with respect to cone P or to simplicity, Cone integrable function, if and only if for all partition Q of $[a,b]$

$$\lim_{n\to\infty} L_n^{\mathrm{Con}}(\phi, Q) = S^{\mathrm{Con}} = \lim_{n\to\infty} U_n^{\mathrm{Con}}(\phi, Q), \tag{2.4}$$

where S^{Con} must be unique.

We show the common value S^{Con} by

$$\int_a^b \phi(x)\,d_P(x) \quad \text{or to simplicity} \quad \int_a^b \phi\, d_p. \tag{2.5}$$

We denote the set of all cone integrable function $\phi : [a,b] \to P$ by $\mathcal{L}^1([a,b], P)$.

Lemma 2.6. *(1) If $[a,b] \subseteq [a,c]$, then $\int_a^b f d_p \le \int_a^c f d_p$, for $f \in \mathcal{L}^1(X,P)$.(2) $\int_a^b(\alpha f + \beta g)d_p = \alpha\int_a^b f d_p + \beta\int_a^b g d_p$, for $f, g \in \mathcal{L}^1(X,P)$ and $\alpha, \beta \in \mathbb{R}$.*

Proof. (1) Suppose that P and R are partitions for $[a,b]$ and $[b,c]$, respectively. That is,

$$R = \{x_n, x_{n+1}, \ldots, x_{m-1}, x_m = c\}, \qquad P = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\},. \tag{2.6}$$

Let $Q = P \cup R$. Q is a partition for $[a, c]$. Therefore one can write

$$\begin{aligned} L_n^{\text{Con}}(f, P) &= \sum_{i=0}^{n-1} f(x_i)\|x_i - x_{i+1}\| \le \sum_{i=0}^{n-1} f(x_i)\|x_i - x_{i+1}\| + \sum_{i=n}^{m-1} f(x_i)\|x_i - x_{i+1}\| \\ &= L_n^{\text{Con}}(f, P) + L_n^{\text{Con}}(f, R) = L_n^{\text{Con}}(f, Q). \end{aligned} \tag{2.7}$$

So

$$\int_a^b f\,d_p \le \int_a^c f\,d_p. \tag{2.8}$$

(2) Suppose P is an partition for $[a, b]$, that is

$$P = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}. \tag{2.9}$$

Then

$$\begin{aligned} L_n^{\text{Con}}(f, P) &= \sum_{i=0}^{n-1} (\alpha f(x_i) + \beta g(x_i))\|x_i - x_{i+1}\| \\ &= \alpha \sum_{i=0}^{n-1} f(x_i)\|x_i - x_{i+1}\| + \beta \sum_{i=0}^{n-1} g(x_i)\|x_i - x_{i+1}\| = \alpha L_n^{\text{Con}}(f, P) + \beta L_n^{\text{Con}}(g, P). \end{aligned} \tag{2.10}$$

Thus

$$\int_a^b (\alpha f + \beta g)\,d_p = \alpha \int_a^b f\,d_p + \beta \int_a^b g\,d_p. \tag{2.11}$$

□

Definition 2.7. The function $\phi : P \to E$ is called subadditive cone integrable function if and only if for all $a, b \in P$

$$\int_0^{a+b} \phi\,d_P \le \int_0^a \phi\,d_P + \int_0^b \phi\,d_P. \tag{2.12}$$

Example 2.8. Let $E = X = \mathbb{R}$, $d(x, y) = |x - y|$, $P = [0, +\infty)$, and $\phi(t) = 1/(t + 1)$ for all $t > 0$. Then for all $a, b \in P$,

$$\int_0^{a+b} \frac{dt}{t+1} = \ln(a + b + 1), \qquad \int_0^a \frac{dt}{t+1} = \ln(a + 1), \qquad \int_0^b \frac{dt}{t+1} = \ln(b + 1). \tag{2.13}$$

Since $ab \geq 0$, then $a+b+1 \leq a+b+1+ab = (a+1)(b+1)$. Therefore

$$\ln(a+b+1) \leq \ln((a+1)(b+1)) = \ln(a+1) + \ln(b+1). \tag{2.14}$$

This shows that ϕ is an example of subadditive cone integrable function.

Theorem 2.9. *Let (X,d) be a complete cone metric space and P a normal cone. Suppose that $\phi : P \to P$ is a nonvanishing map and a subadditive cone integrable on each $[a,b] \subset P$ such that for each $\epsilon \gg 0$, $\int_0^{\epsilon} \phi d_p \gg 0$. If $f : X \to X$ is a map such that, for all $x, y \in X$*

$$\int_0^{d(f(x),f(y))} \phi d_p \leq \alpha \int_0^{d(x,y)} \phi d_p, \tag{2.15}$$

for some $\alpha \in (0,1)$, then f has a unique fixed point in X.

Proof. Let $x_1 \in P$. Choose $x_{n+1} = f(x_n)$. We have

$$\begin{aligned}
\int_0^{d(x_{n+1},x_n)} \phi d_p &= \int_0^{d(f(x_n),f(x_{n-1}))} \phi d_p \\
&\leq \alpha \int_0^{d(x_n,x_{n-1})} \phi d_p \\
&\vdots \\
&\leq \alpha^{n-1} \int_0^{d(x_2,x_1)} \phi d_p.
\end{aligned} \tag{2.16}$$

Since $\alpha \in (0,1)$ thus

$$\lim_{n\to\infty} \int_0^{d(x_{n+1},x_n)} \phi d_p = 0. \tag{2.17}$$

If $\lim_{n\to\infty} d(x_{n+1},x_n) \neq 0$ then $\lim_{n\to\infty} \int_0^{d(x_{n+1},x_n)} \phi d_p \neq 0$ and this is a contradiction, so

$$\lim_{n\to\infty} d(x_{n+1},x_n) = 0. \tag{2.18}$$

We now show that (x_n) is a Cauchy sequence. Due to this, we show that

$$\lim_{m,n\to\infty} d\big(f(x_m), f(x_n)\big) = 0. \tag{2.19}$$

By triangle inequality

$$\int_0^{d(f(x_m),f(x_n))} \phi d_p \leq \int_0^{d(f(x_n),f(x_{n+1}))+d(f(x_{n+1}),f(x_{n+2}))+\cdots+d(f(x_{m-1}),f(x_m))} \phi d_p \tag{2.20}$$

and by sub-additivity of ϕ we get

$$\begin{aligned}\int_0^{d(f(x_m),f(x_n))}\phi d_p &\le \int_0^{d(f(x_n),f(x_{n+1}))}\phi \ \ d_p + \cdots + \int_0^{d(f(x_{m-1}),f(x_m))}\phi d_p \\ &\le \left(\alpha^n + \alpha^{n-1} + \cdots + \alpha^{m-1}\right)\int_0^{d(x_2,x_1)}\phi \ \ d_p \le \frac{\alpha^n}{1-\alpha}\int_0^{d(x_2,x_1)}\phi \ \ d_p \longrightarrow 0.\end{aligned} \tag{2.21}$$

Thus

$$\lim_{m,n\to\infty} d\left(f(x_n), f(x_m)\right) = 0. \tag{2.22}$$

This means that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and since X is a complete cone metric space, thus $\{x_n\}_{n\in\mathbb{N}}$ is convergent to $x_0 \in X$. Finally, since

$$\int_0^{d(x_{n+1},f(x_0))}\phi \ \ d_p = \int_0^{d(f(x_n),f(x_0))}\phi d_p \le \alpha\int_0^{d(x_n,x_0)}\phi \ \ d_p, \tag{2.23}$$

thus $\lim_{n\to\infty} d(x_{n+1}, f(x_0)) = 0$. This means that $f(x_0) = x_0$. If x_0, y_0 are two distinct fixed points of f, then

$$\int_0^{d(x_0,y_0)}\phi d_p = \int_0^{d(f(x_0),f(y_0))}\phi d_p \le \alpha\int_0^{d(x_0,y_0)}\phi d_p \tag{2.24}$$

which is a contradiction. Thus f has a unique fixed point $x_0 \in X$. □

Lemma 2.10. *Let $E = \mathbb{R}^2, P = \{(x,y) \in Ex, y \ge 0\}$, and $X = \mathbb{R}$. Suppose that $d : X \times X \to E$ is defined by $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \ge 0$ is a constant. Suppose that $\phi : [(0,0),(a,b)] \to P$ is defined by $\phi(x,y) = (\phi_1(x), \phi_2(y))$, where $\phi_1, \phi_2 : [0,+\infty) \to [0,+\infty)$ are two Riemann-integrable functions. Then*

$$\int_{(0,0)}^{(a,b)}\phi d_P = \sqrt{a^2+b^2}\left(\frac{1}{a}\int_0^a\phi_1(t)dt, \frac{1}{b}\int_0^b\phi_2(t)dt\right). \tag{2.25}$$

Proof. Let $Q = \{(x_i, y_i)\}_{i=0}^{n}$ be a partition of set $[(0,0),(a,b)]$ such that $x_i = (a/n)i$ and $y_i = (b/n)i$, then (by Definitions 2.4 and 2.5)

$$\begin{aligned}
\int_{(0,0)}^{(a,b)} \phi d_P &= \lim_{n\to\infty} L_n^{Con}(\phi, Q) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \phi(x_i, y_i) \left\| (x_{i+1}, y_{i+1}) - (x_i, y_i) \right\| \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(\phi_1\left(\frac{a}{n}i\right), \phi_2\left(\frac{b}{n}i\right) \right) \left\| \left(\frac{a}{n}, \frac{b}{n} \right) \right\| \\
&= \sqrt{a^2+b^2} \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{n} \left(\phi_1\left(\frac{a}{n}i\right), \phi_2\left(\frac{b}{n}i\right) \right) \\
&= \sqrt{a^2+b^2} \left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi_1\left(\frac{a}{n}i\right), \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi_2\left(\frac{b}{n}i\right) \right) \\
&= \sqrt{a^2+b^2} \left(\frac{1}{a} \lim_{n\to\infty} \frac{a}{n} \sum_{i=0}^{n-1} \phi_1\left(\frac{a}{n}i\right), \frac{1}{b} \lim_{n\to\infty} \frac{b}{n} \sum_{i=0}^{n-1} \phi_2\left(\frac{b}{n}i\right) \right) \\
&= \sqrt{a^2+b^2} \left(\frac{1}{a} \int_0^a \phi_1(t)\,dt, \frac{1}{b} \int_0^b \phi_2(t)\,dt \right).
\end{aligned} \tag{2.26}$$

Thus

$$\int_{(0,0)}^{(a,b)} \phi d_P = \sqrt{a^2+b^2} \left(\frac{1}{a} \int_0^a \phi_1(t)\,dt, \frac{1}{b} \int_0^b \phi_2(t)\,dt \right). \tag{2.27}$$

□

Example 2.11. Let $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$, $E = \mathbb{R}^2$ and $P = \{(x,y) \in E : x \geq 0, y \geq 0\}$. Suppose $d(x,y) = (|x-y|, \alpha|x-y|)$, for some constant $\alpha > 0$. Firstly, (X, d) is a complete cone metric space. Secondly, if $f : X \to X$ and $\phi : P \to E$ are defined by

$$\begin{gathered}
f(x) = \begin{cases} \dfrac{1}{n+1} & \text{if } x = \dfrac{1}{n},\ n \in \mathbb{N}, \\ 0 & \text{if } x = 0, \end{cases} \\
\phi(t,s) = \begin{cases} \left(t^{1/(t-2)}(1-\ln(t)), s^{1/(s-2)}(1-\ln(s))\right), & (t,s) \in P \setminus \{(0,0)\}, \\ (0,0), & (t,s) = (0,0), \end{cases}
\end{gathered} \tag{2.28}$$

respectively, then

$$\int_0^{d(fx,fy)} \phi d_P \leq \frac{1}{2} \int_0^{d(x,y)} \phi d_P. \tag{2.29}$$

In order to obtain inequality (2.29), set $x = 1/n$ and $y = 1/m$, where $m > n$. Hence

$$\begin{gathered} d(fx, fy) = \left(\frac{m-n}{(m+1)(n+1)}, \frac{\alpha(m-n)}{(m+1)(n+1)}\right), \\ d(x, y) = \left(\frac{m-n}{mn}, \frac{\alpha(m-n)}{mn}\right). \end{gathered} \tag{2.30}$$

Suppose $\phi_1(t) = \phi_2(t) = t^{1/(t-2)}(1-\ln(t))$ for all $t > 0$ and $\phi_1(0) = \phi_2(0) = 0$. Thus $\phi(t,s) = (\phi_1(t), \phi_2(s))$. By Lemma 2.10

$$\begin{aligned} \int_0^{d(fx,fy)} \phi d_P &= \int_{(0,0)}^{((m-n)/(m+1)(n+1),\alpha(m-n)/(m+1)(n+1))} (\phi_1, \phi_2) d_P \\ &= \left(\frac{m-n}{(m+1)(n+1)}\sqrt{1+\alpha^2}\right) \\ &\left(\frac{(m+1)(n+1)}{m-n}\int_0^{(m-n)/(m+1)(n+1)} \phi_1(t)dt, \frac{(m+1)(n+1)}{\alpha(m-n)}\int_0^{\alpha(m-n)/(m+1)(n+1)} \phi_2(t)dt\right) \\ &= \left(\sqrt{1+\alpha^2}\right)\left(\int_0^{(m-n)/(m+1)(n+1)} \phi_1(t)dt, \frac{1}{\alpha}\int_0^{\alpha(m-n)/(m+1)(n+1)} \phi_2(t)dt\right). \end{aligned} \tag{2.31}$$

Since $\int_0^{\tau} t^{1/(t-2)}(1-\ln(t))dt = \tau^{1/\tau}$, thus

$$\begin{aligned} \int_0^{(m-n)/(m+1)(n+1)} \phi_1(t)dt &= \left[\frac{m-n}{(m+1)(n+1)}\right]^{(m+1)(n+1)/(m-n)}, \\ \int_0^{\alpha(m-n)/(m+1)(n+1)} \phi_2(t)dt &= \left[\frac{\alpha(m-n)}{(m+1)(n+1)}\right]^{(m+1)(n+1)/\alpha(m-n)}. \end{aligned} \tag{2.32}$$

It means that

$$\int_0^{d(fx,fy)} \phi d_P = \sqrt{1+\alpha^2}\left(\left[\frac{m-n}{(m+1)(n+1)}\right]^{(m+1)(n+1)/(m-n)}, \frac{1}{\alpha}\left[\frac{\alpha(m-n)}{(m+1)(n+1)}\right]^{(m+1)(n+1)/\alpha(m-n)}\right). \tag{2.33}$$

On the other side, Branciari in [6] shows that

$$\left[\frac{m-n}{(n+1)(m+1)}\right]^{(n+1)(m+1)/(m-n)} \le \frac{1}{2}\left[\frac{m-n}{nm}\right]^{nm/(m-n)}, \tag{2.34}$$

for all $m, n \in \mathbb{N}$. Therefore

$$\begin{aligned}&\left(\left[\frac{m-n}{(m+1)(n+1)}\right]^{(m+1)(n+1)/(m-n)}, \frac{1}{\alpha}\left[\frac{\alpha(m-n)}{(m+1)(n+1)}\right]^{(m+1)(n+1)/\alpha(m-n)}\right)\\ &\quad\leq \frac{1}{2}\left(\left[\frac{m-n}{mn}\right]^{mn/(m-n)}, \frac{1}{\alpha}\left[\frac{\alpha(m-n)}{mn}\right]^{mn/\alpha(m-n)}\right).\end{aligned} \tag{2.35}$$

Thus inequalities (2.33) and (2.35) imply that

$$\int_0^{d(fx,fy)} \phi d_P \leq \frac{1}{2}\sqrt{1+\alpha^2}\left(\left[\frac{m-n}{mn}\right]^{mn/(m-n)}, \frac{1}{\alpha}\left[\frac{\alpha(m-n)}{mn}\right]^{mn/\alpha(m-n)}\right) = \frac{1}{2}\int_0^{d(x,y)} \phi d_P, \tag{2.36}$$

or in other words

$$\int_0^{d(fx,fy)} \phi d_P \leq \frac{1}{2}\int_0^{d(x,y)} \phi d_P. \tag{2.37}$$

Thus by Theorem 2.9, f has a fixed point. But, on the other hand,

$$d(fx, fy) < d(x, y), \tag{2.38}$$

and this means that f does not satisfy in Theorem 1.12.

3. Extension of Meir-Keeler Contraction in Cone Metric Space

In 2006, Suzuki in [7] proved that the integral type contraction (see [6]) is a special case of Meir-Keeler contraction (see [8]). Haghi and Rezapour in [5] extended Meir-Keeler contraction in cone metric space as follows.

Theorem 3.1 (see[5]). *Let (X, d) be a complete regular cone metric space and f has the property (KMC) on X; that is, for all $0 \neq \epsilon \in P$, there exists $\delta \gg 0$ such that*

$$d(x, y) < \epsilon + \delta \quad \text{implies} \quad d(fx, fy) < \epsilon \tag{3.1}$$

for all $x, y \in X$. Then f has a unique fixed point.

An extension of Theorem 3.1 is as follows.

Theorem 3.2. *Let (X, d) be a complete regular cone metric space and f a mapping on X. Suppose that there exists a function θ from P into itself satisfying the following:*

(B1) $\theta(0) = 0$ *and* $\theta(t) \gg 0$ *for all* $t \gg 0$,

(B2) θ *is nondecreasing and continuous function. Moreover, its inverse is continuous,* (B3) *for all* $0 \neq \epsilon \in P$*, there exists* $\delta \gg 0$ *such that for all* $x, y \in X$

$$\theta(d(x,y)) < \epsilon + \delta \quad \text{implies } \theta(d(fx,fy)) < \epsilon \tag{3.2}$$

(B4) *for all* $x, y \in X$

$$\theta(x+y) \leq \theta(x) + \theta(y). \tag{3.3}$$

Then f *has a unique fixed point.*

Proof. First, note that $\theta(d(f(x), f(y))) < \theta(d(x,y))$ for all $x, y \in X$ with $x \neq y$. Since θ^{-1} exists, thus $d(f(x), f(y)) < d(x,y)$ for all $x, y \in X$ with $x \neq y$. Now Let $x_0 \in X$. Set $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$. If, there is a natural $m \in N$ such that $d(x_{m+1}, x_m) = 0$, then $f(x_m) = x_m$ and so f has a fixed point. If $d(x_{n+1}, x_n) \neq 0$ for all $n \in \mathbb{N}$, then $\theta(d(x_{n+1}, x_n)) < \theta(d(x_n, x_{n-1}))$. Hence, according to regularity of P, there exists $\alpha \in P$ such that $\theta(d(x_{n+1}, x_n)) \downarrow \alpha$. We claim $\alpha = 0$. If $\alpha \neq 0$, then according to $(B3)$, there is $0 \ll d$ such that $\theta(d(f(x), f(y)) < \alpha$ for all $x, y \in X$ with $\theta(d(x,y)) < \alpha + d$. Choose $r > 0$ such that $(d/2) + N_r(0) \subseteq P$ and take the natural number N such that $\|\theta(d(x_{n+1}, x_n)) - \alpha\| < r$, for all $n \geq N$. We obtain

$$\left\| \frac{d}{2} - (\theta(d(x_{n+1} - x_n)) - \alpha) - \frac{d}{2} \right\| < r. \tag{3.4}$$

Thus

$$\frac{d}{2} - (\theta(d(x_{n+1} - x_n)) - \alpha) \in \frac{d}{2} + N_r(0) \subseteq P. \tag{3.5}$$

So, $\theta(d(x_{n+1}, x_n)) - \alpha \ll d$. Since f has the property $(B3)$, $\theta(d(x_{n+2}, x_{n+1})) < \alpha$ for all $n \geq N$. This is a contradiction because $\alpha < \theta(d(x_{i+1}, x_i))$ for all $i \geq 1$. Thus

$$\lim_{n \to \infty} \theta(d(x_{n+1}, x_n)) = 0. \tag{3.6}$$

Now, we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. If this is not, then there is a $0 \ll c$ such that for all natural number k, there are $m_k, n_k > k$ so that the relation $d(x_{m_k}, x_{n_k}) \ll c$ does not hold. Since θ has continuous inverse thus there exists $0 \ll c$ such that for all natural number k, there are $m_k, n_k > k$ so that the relation $\theta(d(x_{m_k}, x_{n_k})) \ll c$ does not hold. For each $0 \ll e \ll c$, there exists $0 \ll d$ such that $\theta(d(f(x), f(y))) < e$, for all $x, y \in X$ with $\theta(d(x,y)) < e + d$. Choose a natural number M such that $\theta(d(x_{i+1}, x_i)) \ll d/2$ for all $i \geq M$.

Also, take $m_M \geq n_M > M$ so that the relation $\theta(d(x_{m_M}, x_{n_M})) \ll c$ does not hold. Then $(B4)$ yields

$$\begin{aligned}\theta(d(x_{n_M-1}, x_{n_M+1})) &\leq \theta(d(x_{n_M-1}, x_{n_M})) + \theta(d(x_{n_M}, x_{n_M+1})) \\ &\ll \frac{d}{2} + \frac{d}{2} \\ &\ll d + e.\end{aligned} \tag{3.7}$$

Hence, $\theta(d(x_{n_M}, x_{n_M+2})) \ll e$. Similarly, $\theta(d(x_{n_M}, x_{n_M+3})) \ll e$. Thus

$$\theta(d(x_{n_M}, x_{m_M})) \ll e \ll c \tag{3.8}$$

which is a contradiction. Therefore $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since (X, d) is a complete cone metric space, there is $u \in X$ such that $x_n \to u$. Since $d(fx, fy) < d(x, y)$, for all $x, y \in X$ with $x \neq y$, thus for each $\epsilon \gg 0$, there is a natural number $N > 0$ such that for all $n > N$, $d(x_n, x) \ll \epsilon$. Since $d(fx_n, fu) < d(x_n, u)$ thus $d(fx_n, fu) \ll \epsilon$ for all $n > N$. It means that $fx_n \to fu$. In the other side, $f(x_n) = x_{n+1} \to u$ and the limit point is unique in cone metric spaces. Thus f has at least one fixed point. Now, if u, v are two distinct fixed points for f, then

$$d(u, v) = d(fu, fv) < d(u, v) \tag{3.9}$$

which is a contradiction. Therefore f has a unique fixed point. □

Remark 3.3. (1) Set $\theta(x) = x$, then Theorem 3.1 is a direct result of Theorem 3.2.

(2) Let $\phi : P \to P$ be a nonvanishing map and a subadditive cone integrable on each $[a, b] \subset P$ such that for each $\epsilon \gg 0$, $\int_0^{\epsilon} \phi d_p \gg 0$. If $\theta(x) = \int_0^x \phi d_P$, then θ satisfies all conditions of Theorem 3.2. In other words, Theorem 2.9 is a direct result of Theorem 3.2.

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