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Research Article

Browder-Krasnoselskii-Type Fixed Point Theorems in Banach Spaces

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We present some fixed point theorems for the sum $A+B$ of a weakly-strongly continuous map and a nonexpansive map on a Banach space X. Our results cover several earlier works by Edmunds, Reinermann, Singh, and others.

1. Introduction

Let M be a nonempty subset of a Banach space X and $T : M \to X$ a mapping. We say that T is *weakly-strongly continuous* if for each sequence $\{x_n\}$ in M which converges weakly to x in M, the sequence $\{Tx_n\}$ converges strongly to Tx. The mapping T is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in M$.

In [1], Edmunds proved the following fixed point theorem

Theorem 1.1. *Let M be a nonempty bounded closed convex subset of a Hilbert space H and A, B two maps from M into X such that*

(i) *A is weakly-strongly continuous,*

(ii) *B is a nonexpansive mapping,*

(iii) *$Ax + By \in M$ for all $x, y \in M$.*

Then $A + B$ has a fixed point in M.

It is apparent that Theorem 1.1 is an important supplement to both Krasnoselskii's fixed point [2, Theorem 4.4.1] and Browder's fixed point theorems [2, Theorem 5.1.3]. The proof of Theorem 1.1 depends heavily upon the fact that $F = I - A$ (where I is the identity map) is monotone, that is, $(Fx - Fy, x - y) \geq 0$ for all x, y, and uses the Krasnoselskii fixed point theorem for the sum of a completely continuous and a strict contraction mapping [2, 3]. In [4], Reinermann extended the above result to uniform Banach spaces. The methods used in the Hilbert space setting involving monotone operators do not apply in the more general context of uniform Banach spaces. The author follows another strategy of proof which is based on a demiclosedness principle for nonexpansive mapping defined on a uniformly convex Banach space and uses the fact that every uniformly convex space is reflexive. In [5], Singh extended Theorem 1.1 to reflexive Banach spaces by assuming further that $I - B$ is demiclosed. Notice that all the aforementioned extensions of Theorem 1.1 depend strongly upon the geometry of the ambient Banach space. In this paper we propose an extension of Theorem 1.1 to an arbitrary Banach space. Also, we discuss the existence of a fixed point for the sum of a compact mapping and a nonexpansive mapping for both the weak and the strong topology of a Banach space and under Krasnosel'skii-, Leray Schauder-, and Furi-Pera-type conditions. First we recall the following well-known result.

Theorem 1.2 (see [2, Theorem 5.1.2]). *Let M be a bounded closed convex subset of a Banach space X and T a nonexpansive mapping of M into M. Then for each $\varepsilon > 0$, there is a $x_\varepsilon \in M$ such that $\|Tx_\varepsilon - x_\varepsilon\| < \varepsilon$.*

Now, let us recall some definitions and results which will be needed in our further considerations. Let X be a Banach space, $\Omega(X)$ the collection of all nonempty bounded subsets of X, and $\mathcal{W}(X)$ the subset of $\Omega(X)$ consisting of all weakly compact subsets of X. Let B_r denote the closed ball in X centered at 0 with radius $r > 0$. In [6] De Blasi introduced the following map $w : \Omega(X) \to [0, \infty)$ defined by

$$w(M) = \inf\{r > 0 : \text{there exists a set } N \in \mathcal{W}(X) \text{ such that } M \subseteq N + B_r\}, \tag{1.1}$$

for all $M \in \Omega(X)$. For completeness we recall some properties of $w(\cdot)$ needed below (for the proofs we refer the reader to [6]).

Lemma 1.3. *Let $M_1, M_2 \in \Omega(X)$, then one has the following:*

(i) *if $M_1 \subseteq M_2$, then $w(M_1) \leq w(M_2)$,*

(ii) $w(M_1) = 0$ *if and only if M_1 is relatively weakly compact,*

(iii) $w(\overline{M_1^w}) = w(M_1)$, *where $\overline{M_1^w}$ is the weak closure of M_1,*

(iv) $w(\lambda M_1) = |\lambda| w(M_1)$ *for all $\lambda \in \mathbb{R}$,*

(v) $w(\text{co}(M_1)) = w(M_1)$,

(vi) $w(M_1 + M_2) \leq w(M_1) + w(M_2)$,

(vii) *if $(M_n)_{n\geq 1}$ is a decreasing sequence of nonempty, bounded, and weakly closed subsets of X with $\lim_{n\to\infty} w(M_n) = 0$, then $\bigcap_{n=1}^{\infty} M_n \neq \emptyset$ and $w(\bigcap_{n=1}^{\infty} M_n) = 0$, that is, $w(\bigcap_{n=1}^{\infty} M_n)$ is relatively weakly compact.*

Throughout this paper, a measure of weak noncompactness will be a mapping $\psi : \Omega(X) \to [0, \infty)$ which satisfies assumptions (i)–(vii) cited in Lemma 1.3.

Definition 1.4. Let X be a Banach space, and let ψ be a measure of weak noncompactness on X. A mapping $B : D(B) \subseteq X \to X$ is said to be ψ-contractive if it maps bounded sets into bounded sets and there is $\beta \in [0,1[$ such that $\psi(B(S)) \leq \beta\psi(S)$ for all bounded sets $S \subseteq D(B)$. The mapping $B : D(B) \subseteq X \to X$ is said to be ψ-condensing if it maps bounded sets into bounded sets and $\psi(B(S)) < \psi(S)$ whenever S is a bounded subset of $D(B)$ such that $\psi(S) > 0$.

Let $\mathcal{Q}$ be a nonlinear operator from $D(\mathcal{Q}) \subseteq X$ into X. In what follows, we will use the following two conditions.

($\mathcal{H}1$) If $(x_n)_{n\in\mathbb{N}}$ is a weakly convergent sequence in $D(\mathcal{Q})$, then

$(\mathcal{Q}x_n)_{n\in\mathbb{N}}$ has a strongly convergent subsequence in X.

($\mathcal{H}2$) If $(x_n)_{n\in\mathbb{N}}$ is a weakly convergent sequence in $D(\mathcal{Q})$, then

$(\mathcal{Q}x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence in X.

Remark 1.5. (1) Operators satisfying ($\mathcal{H}1$) or ($\mathcal{H}2$) are not necessarily weakly continuous (see [7–9]).

(2) Every w-contractive map satisfies ($\mathcal{H}2$).

(3) A mapping $\mathcal{Q}$ satisfies ($\mathcal{H}2$) if and only if it maps relatively weakly compact sets into relatively weakly compact ones (use the Eberlein-Šmulian theorem [10], page 430).

(4) A mapping $\mathcal{Q}$ satisfies ($\mathcal{H}1$) if and only if it maps relatively weakly compact sets into relatively compact ones.

(5) Condition ($\mathcal{H}2$) holds true for every bounded linear operator.

(6) Condition ($\mathcal{H}1$) holds true for the class of weakly compact operators acting on Banach spaces with the Dunford-Pettis property.

(7) Continuous mappings satisfying ($\mathcal{H}1$) are sometimes called (ws)-compact operators (see [11, Definition 2]).

The following fixed point theorems are crucial for our purposes.

Theorem 1.6 (see [7, Theorem 2.3]). *Let M be a nonempty closed bounded convex subset of a Banach space X. Suppose that $A : M \to X$ and $B : X \to X$ such that*

(i) *A is continuous, AM is relatively weakly compact, and A satisfies ($\mathcal{H}1$),*

(ii) *B is a strict contraction satisfying ($\mathcal{H}2$),*

(iii) *$Ax + By \in M$ for all $x, y \in M$.*

Then there is a $x \in M$ such that $Ax + Bx = x$.

Theorem 1.7 (see [12, Theorem 2.1]). *Let M be a nonempty closed bounded convex subset of a Banach space X. Suppose that $A : M \to X$ and $B : X \to X$ are sequentially weakly continuous such that*

(i) *AM is relatively weakly compact,*

(ii) *B is a strict contraction,*

(iii) *$Ax + By \in M$ for all $x, y \in M$.*

Then there is a $x \in M$ such that $Ax + Bx = x$.

Theorem 1.8 (see [13, 14]). *Let X be a Banach space with $C \subseteq X$ closed and convex. Assume that U is a relatively open subset of C with $0 \in U, F(\overline{U})$ bounded, and $F : \overline{U} \to C$ a condensing map. Then either F has a fixed point in $\overline{U}$ or there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$, here $\overline{U}$ and ∂U denote the closure of U in C and the boundary of U in C, respectively.*

Theorem 1.9 (see [13, 14]). *Let X be a Banach space and Q a closed convex bounded subset of X with $0 \in Q$. In addition, assume that $F : Q \to X$ is a condensing map with*

$$\begin{aligned} &\text{if}\{(x_j, \lambda_j)\}_{j=1}^{\infty} \text{is a sequence in} \partial Q \times [0,1] \text{converging to} (x,\lambda) \text{with} \\ &x = \lambda F(x) \text{ and } 0 < \lambda < 1, \text{ then } \lambda_j F(x_j) \in Q \text{ for } j \text{ sufficiently large,} \end{aligned} \tag{$\mathcal{FP}$}$$

holding. Then F has a fixed point.

2. Fixed Point Theorems

Now we are ready to state and prove the following result.

Theorem 2.1. *Let M be a nonempty bounded closed convex subset of a Banach space X. Let $A : M \to X$ and $B : X \to X$ satisfy the following:*

(i) *A is weakly-strongly continuous and AM is relatively weakly compact,*

(ii) *B is a nonexpansive mapping satisfying ($\mathcal{H}2$),*

(iii) *if (x_n) is a sequence of M such that $((I - B)x_n)$ is weakly convergent, then the sequence (x_n) has a weakly convergent subsequence,*

(iv) *$I - B$ is demiclosed,*

(v) *$Ax + By \in M$, for all $x, y \in M$.*

Then there is an $x \in M$ such that $Ax + Bx = x$.

Proof. Suppose first that $0 \in M$. By hypothesis (v) we have for each $\lambda \in (0,1)$ and $x, y \in M$

$$\lambda Ax + \lambda By \in M. \tag{2.1}$$

Thus the mappings λA and λB satisfy the conditions of Theorem 1.6. Thus, for all $\lambda \in (0,1)$ there is an $x_\lambda \in M$ such that $\lambda Ax_\lambda + \lambda Bx_\lambda = x_\lambda$. Now, choose a sequence $\{\lambda_n\}$ in $(0,1)$ such that $\lambda_n \to 1$ and consider the corresponding sequence $\{x_n\}$ of elements of M satisfying

$$\lambda_n Ax_n + \lambda_n Bx_n = x_n. \tag{2.2}$$

Using the fact that AM is weakly compact and passing eventually to a subsequence, we may assume that $\{Ax_n\}$ converges weakly to some $y \in M$. Hence

$$(I - \lambda_n B)x_n \rightharpoonup y. \tag{2.3}$$

Since $\{x_n\}$ is a sequence in M, then it is norm bounded and so is $\{Bx_n\}$. Consequently

$$\|(x_n - Bx_n) - (x_n - \lambda_n Bx_n)\| = (1 - \lambda_n)\|Bx_n\| \longrightarrow 0. \tag{2.4}$$

As a result

$$x_n - Bx_n \rightharpoonup y. \tag{2.5}$$

By hypothesis (iii) the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to some $x \in M$. Since A is weakly-strongly continuous, then $\{Ax_{n_k}\}$ converges strongly to Ax. As a result

$$(I - \lambda_{n_k} B)x_{n_k} \longrightarrow Ax. \tag{2.6}$$

Arguing as above we get

$$x_n - Bx_n \longrightarrow Ax. \tag{2.7}$$

The demiclosedness of $I - B$ yields $Ax + Bx = x$.

To complete the proof it remains to consider the case $0 \notin M$. In such a case let us fix any element $x_0 \in M$, and let $M_0 = \{x - x_0,\ x \in M\}$. Define the maps $A_0 : M_0 \to X$ and $B_0 : M_0 \to X$ by $A_0(x-x_0) = Ax-(1/2)x_0$ and $B_0(x-x_0) = Bx-(1/2)x_0$, for $x \in M$. Applying the result of the first case to A_0 and B_0 we get an $x \in M$ such that $A_0(x-x_0)+B_0(x-x_0) = x-x_0$, that is, $Ax + Bx = x$. □

Remark 2.2. (1) The new feature about the result of Theorem 2.1 is that no additional assumption on the Banach space X is required.

(2) If X is reflexive, then the strong continuity plainly implies compactness. Moreover, assumption (iii) of Theorem 2.1 is always verified. Also, every continuous mapping on X satisfies condition ($\mathscr{H}$2). If in addition we suppose that X is a uniformly convex Banach space, then B is nonexpansive implying that $I - B$ is demiclosed (see [4, 15]).

In the light of the aforementioned remarks we obtain the following consequences of Theorem 2.1. The first is proved in [4] while the second in stated in [5].

Corollary 2.3. *Let M be a nonempty bounded closed convex subset of a uniformly convex Banach space X. Let $A : M \to X$ and $B : M \to X$ satisfy the following:*

(i) *A is weakly-strongly continuous,*

(ii) *B is nonexpansive,*

(iii) *$Ax + By \in M$, for all $x, y \in M$.*

Then there is an $x \in M$ such that $Ax + Bx = x$.

Corollary 2.4. *Let M be a nonempty bounded closed convex subset of a reflexive Banach space X. Let $A : M \to X$ and $B : M \to X$ satisfy the following:*

(i) *A is weakly-strongly continuous,*

(ii) *B is nonexpansive and $I - B$ is demiclosed,*

(iii) *$Ax + By \in M$, for all $x, y \in M$.*

Then there is an $x \in M$ such that $Ax + Bx = x$.

Our next result is the following.

Theorem 2.5. *Let M be a nonempty bounded closed convex subset of a Banach space X. Let $A : M \to X$ and $B : M \to X$ satisfy the following:*

(i) *A is sequentially weakly continuous, and AM is relatively weakly compact,*

(ii) *B is sequentially weakly continuous nonexpansive mapping,*

(iii) *if (x_n) is a sequence of M such that $((I - B)x_n)$ is weakly convergent, then the sequence (x_n) has a convergent subsequence,*

(iv) *$Ax + By \in M$, for all $x, y \in M$.*

Then there is an $x \in M$ such that $Ax + Bx = x$.

Proof. Without loss of generality, we may assume that $0 \in M$. By hypothesis (v) we have for each $\lambda \in (0,1)$ and $x, y \in M$

$$\lambda Ax + \lambda By \in M. \tag{2.8}$$

Thus the mappings λA and λB satisfy the conditions of Theorem 1.7. Thus, for all $\lambda \in (0,1)$ there is an $x_\lambda \in M$ such that $\lambda Ax_\lambda + \lambda Bx_\lambda = x_\lambda$. Now choose a sequence $\{\lambda_n\}$ in $(0,1)$ such that $\lambda_n \to 1$ and consider the corresponding sequence $\{x_n\}$ of elements of M satisfying

$$\lambda_n Ax_n + \lambda_n Bx_n = x_n. \tag{2.9}$$

Using the fact that AM is weakly compact and passing eventually to a subsequence, we may assume that $\{Ax_n\}$ converges weakly to some $y \in M$. As a result

$$(I - \lambda_n B)x_n \rightharpoonup y. \tag{2.10}$$

Since $\{x_n\}$ is a sequence in M, then it is norm bounded and so is $\{Bx_n\}$. Consequently

$$\|(x_n - Bx_n) - (x_n - \lambda_n Bx_n)\| = (1 - \lambda_n)\|Bx_n\| \longrightarrow 0. \tag{2.11}$$

This amounts to

$$x_n - Bx_n \rightharpoonup y. \tag{2.12}$$

By hypothesis (iii) the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to some $x \in M$. Since A and B are weakly sequentially continuous, then $\{Ax_{n_k}\}$ converges weakly to Ax and $\{Bx_{n_k}\}$ converges weakly to Bx. Hence, $x = Ax + Bx$. □

We next establish the following result which is a sharpening of [16, Theorem 2.3]. This result is of fundamental importance for our subsequent analysis.

Theorem 2.6. *Let X be a Banach space, and let ψ be a measure of weak noncompactness on X. Let Q and C be closed, bounded, convex subsets of X with $Q \subseteq C$. In addition, let U be a weakly open subset of Q with $0 \in U$, and $F : \overline{U^w} \to C$ a weakly sequentially continuous and ψ-condensing map. Then either*

$$F \text{ has a fixed point,} \tag{2.13}$$

or

$$\text{there is a point } u \in \partial_Q U \text{ and } \lambda \in (0,1) \quad \text{with } u = \lambda F u, \tag{2.14}$$

here $\partial_Q U$ is the weak boundary of U in Q.

Proof. Suppose that (2.14) does not occur and F does not have a fixed point on $\partial_Q U$ (otherwise we are finished since (2.13) occurs). Let

$$M = \left\{ x \in \overline{U^w} : x = \lambda F x \text{ for some } \lambda \in [0,1] \right\}. \tag{2.15}$$

The set M is nonempty since $0 \in U$. Also M is weakly sequentially closed. Indeed let (x_n) be sequence of M which converges weakly to some $x \in \overline{U^w}$, and let (λ_n) be a sequence of $[0,1]$ satisfying $x_n = \lambda_n F x_n$. By passing to a subsequence if necessary, we may assume that (λ_n) converges to some $\lambda \in [0,1]$. Since F is weakly sequentially continuous, then $Fx_n \rightharpoonup Fx$. Consequently $\lambda_n F x_n \rightharpoonup \lambda F x$. Hence $x = \lambda F x$ and therefore $x \in M$. Thus M is weakly sequentially closed. We now claim that M is relatively weakly compact. Suppose that $\psi(M) > 0$. Since $M \subseteq \text{co}(F(M) \cup \{0\})$, then

$$\psi(M) \le \psi(\text{co}(F(M) \cup \{0\})) = \psi(F(M)) < \psi(M), \tag{2.16}$$

which is a contradiction. Hence $\psi(M) = 0$ and therefore $\overline{M^w}$ is compact. This proves our claim. Now let $x \in \overline{M^w}$. Since $\overline{M^w}$ is weakly compact, then there is a sequence (x_n) in M which converges weakly to x. Since M is weakly sequentially closed we have $x \in M$. Thus $\overline{M^w} = M$. Hence M is weakly closed and therefore weakly compact. From our assumptions we have $M \cap \partial_Q U = \emptyset$. Since X endowed with the weak topology is a locally convex space, then there exists a continuous mapping $\rho : \overline{U^w} \to [0,1]$ with $\rho(M) = 1$ and $\rho(\partial_Q U) = 0$ (see [17]). Let

$$T(x) = \begin{cases} \rho(x) F(x), & x \in \overline{U^w}, \\ 0, & x \in C \setminus \overline{U^w}. \end{cases} \tag{2.17}$$

Clearly $T : C \to C$ is weakly sequentially continuous since F is weakly sequentially continuous. Moreover, for any $S \subseteq C$ we have

$$T(S) \subseteq \text{co}(F(S \cap U) \cup \{0\}). \tag{2.18}$$

This implies that

$$\psi(T(S)) \leq \psi(\text{co}(F(S \cap U) \cup \{0\})) = \psi(F(S \cap U)) \leq \psi(F(S)) < \psi(S) \tag{2.19}$$

if $\psi(S) > 0$. Thus $T : C \to C$ is weakly sequentially continuous and ψ-condensing. By [18, Theorem 12] there exists $x \in C$ such that $Tx = x$. Now $x \in U$ since $0 \in U$. Consequently $x = \rho(x)F(x)$ and so $x \in M$. This implies that $\rho(x) = 1$ and so $x = F(x)$. □

Remark 2.7. In [16, Theorem 2.3], $\overline{U^w}$ is assumed to be weakly compact.

Lemma 2.8. *Let X be a Banach space and $B : X \to X$ a k-Lipschitzian map, that is,*

$$\forall x, y \in X, \quad \|Bx - By\| \leq k\|x - y\|. \tag{2.20}$$

In addition, suppose that B verifies $(\mathscr{H}2)$. Then for each bounded subset S of X one has

$$w(BS) \leq kw(S), \tag{2.21}$$

here, w is the De Blasi measure of weak noncompactness.

Proof. Let S be a bounded subset of X and $r > w(S)$. There exist $0 \leq r_0 < r$ and a weakly compact subset K of X such that $S \subseteq K + B_{r_0}$. Now we show that

$$BS \subseteq BK + B_{kr_0} \subseteq \overline{BK^w} + B_{kr_0}. \tag{2.22}$$

To see this let $x \in S$. Then there is a $y \in K$ such that $\|x - y\| \leq r_0$. Since B is k-Lipschizian, then $\|Bx - By\| \leq k\|x - y\| \leq kr_0$. This proves (2.22). Further, since B satisfies $(\mathscr{H}2)$, then the Eberlein-Šmulian theorem [10, page 430] implies that $\overline{BK^w}$ is weakly compact. Consequently

$$w(BS) \leq kr_0 \leq kr. \tag{2.23}$$

Letting $r \to w(S)$ we get

$$w(BS) \leq kw(S). \tag{2.24}$$

□

Now we are in a position to prove our next result.

Theorem 2.9. *Let Q and C be closed, bounded, convex subsets of a Banach space X with $Q \subseteq C$. In addition, let U be a weakly open subset of Q with $0 \in U$. Suppose that $A : \overline{U^w} \to X$ and $B : X \to X$ are two weakly sequentially continuous mappings satisfying the following:*

(i) *$A(\overline{U^w})$ is relatively weakly compact,*

(ii) *B is a nonexpansive map,*

(iii) *if (x_n) is a sequence of M such that $((I-B)x_n)$ is weakly convergent, then the sequence (x_n) has a convergent subsequence,*

(iv) *$Ax + Bx \in C$ for all $x \in \overline{U^w}$.*

Then either

$$A + B \text{ has a fixed point,} \tag{2.25}$$

or

$$\text{there is a point } u \in \partial_Q U \text{ and } \lambda \in (0,1) \quad \text{with } u = \lambda(A+B)u, \tag{2.26}$$

here $\partial_Q U$ is the weak boundary of U in Q.

Proof. Let $\mu \in (0,1)$. We first show that the mapping $F_\mu := \mu A + \mu B$ is w-contractive with constant μ. To see this let S be a bounded subset of $\overline{U^w}$. Using the homogeneity and the subadditivity of the De Blasi measure of weak noncompactness we obtain

$$w(F_\mu(S)) \le w(\mu AS + \mu BS) \le \mu w(AS) + \mu w(BS). \tag{2.27}$$

Keeping in mind that A is weakly compact and using Lemma 2.8 we deduce that

$$w(F_\mu(S)) \le \mu w(S). \tag{2.28}$$

This proves that F_μ is w-contractive with constant μ. Moreover, taking into account that $0 \in U$ and using assumption (iv) we infer that F_μ maps $\overline{U^w}$ into C. Next suppose that (2.26) does not occur and F_μ does not have a fixed point on $\partial_Q U$ (otherwise we are finished since (2.25) occurs). If there exists a $u \in \partial_Q U$ and $\lambda \in (0,1)$ with $u = \lambda F_\mu u$, then $u = \lambda\mu Au + \lambda\mu Bu$ which is impossible since $\lambda\mu \in (0,1)$. By Theorem 2.6 there exists $x_\mu \in \overline{U^w}$ such that $x_\mu = F_\mu(x_\mu)$. Now choose a sequence $\{\mu_n\}$ in $(0,1)$ such that $\mu_n \to 1$ and consider the corresponding sequence $\{x_n\}$ of elements of $\overline{U^w}$ satisfying

$$F_{\mu_n}(x_n) = \mu_n Ax_n + \mu_n Bx_n = x_n. \tag{2.29}$$

Using the fact that $A(\overline{U^w})$ is weakly compact and passing eventually to a subsequence, we may assume that $\{Ax_n\}$ converges weakly to some $y \in \overline{U^w}$. Hence

$$(I - \mu_n B)x_n \rightharpoonup y. \tag{2.30}$$

Since $\{x_n\}$ is a sequence in $\overline{U^w}$, then it is norm bounded and so is $\{Bx_n\}$. Consequently

$$\left\|(x_n - Bx_n) - (x_n - \mu_n Bx_n)\right\| = (1-\mu_n)\|Bx_n\| \longrightarrow 0. \tag{2.31}$$

As a result

$$x_n - Bx_n \rightharpoonup y. \tag{2.32}$$

By hypothesis (iii) the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to some $x \in \overline{U^w}$. The weak sequential continuity of A and B implies that $x = Bx + Ax$. □

The following result is a sharpening of [16, Theorem 2.4].

Theorem 2.10. *Let X be a separable Banach space, C a closed bounded convex subset of X, and Q a closed convex subset of C with $0 \in Q$. Also, assume that $F : Q \to C$ is a weakly sequentially continuous and a weakly compact map. In addition, assume that the following conditions are satisfied:*

(i) *there exists a weakly continuous retraction $r : X \to Q$,*

(ii) *there exists a $\delta > 0$ and a weakly compact set Q_δ with $\Omega_\delta = \{x \in X : d(x,Q) \leq \delta\} \subseteq Q_\delta$, here $d(x,y) = \|x - y\|$,*

(iii) *for any $\Omega_\epsilon = \{x \in X : d(x,Q) \leq \epsilon,\ 0 < \epsilon \leq \delta\}$, if $\{(x_j, \lambda_j)\}_{j=1}^{\infty}$ is a sequence in $Q \times [0,1]$ with $x_j \rightharpoonup x \in \partial_{\Omega_\epsilon} Q$, $\lambda_j \to \lambda$, and $x = \lambda F(x), 0 \leq \lambda < 1$, then $\lambda_j F(x_j) \in Q$ for j sufficiently large, here $\partial_{\Omega_\epsilon} Q$ is the weak boundary of Q relative to Ω_ϵ.*

Then F has a fixed point in Q.

Proof. Consider

$$B = \{x \in X : x = Fr(x)\}. \tag{2.33}$$

We first show that $B \neq \emptyset$. To see this, consider $rF : Q \to Q$. Clearly rF is weakly sequentially continuous, since F is weakly sequentially continuous and r is weakly continuous. Also $rF(Q)$ is relatively weakly compact since $F(Q)$ is relatively weakly compact and r is weakly continuous. Applying the Arino-Gautier Penot fixed point theorem [19] we infer that there exists $y \in Q$ with $rF(y) = y$. Let $z = F(y)$, so $Fr(z) = Fr(F(y)) = F(y) = z$. Thus $z \in B$ and $B \neq \emptyset$. In addition B is weakly sequentially closed, since Fr is weakly sequentially continuous. Moreover, since $B \subseteq Fr(B) \subseteq F(Q)$, then B is relatively weakly compact. Now let $x \in \overline{B^w}$. Since $\overline{B^w}$ is weakly compact, then there is a sequence (x_n) of elements of B which converges weakly to some x. Since B is weakly sequentially closed, then $x \in B$. Thus, $\overline{B^w} = B$. This implies that B is weakly compact. We now show that $B \cap Q \neq \emptyset$. Suppose that $B \cap Q = \emptyset$. Then, since B is weakly compact and Q is weakly closed, we have from [20] that $d(B,Q) > 0$. Thus there exists ϵ, $0 < \epsilon < \delta$, with $\Omega_\epsilon \cap B = \emptyset$, here $\Omega_\epsilon = \{x \in X : d(x,Q) \leq \epsilon\}$. Now Ω_ϵ is closed convex and $\Omega_\epsilon \subseteq Q_\delta$. From our assumptions it follows that Ω_ϵ is weakly compact. Also since X is separable, then the weak topology on Ω_ϵ is metrizable [3, 10]; let d^* denote the metric. For $i \in \{0,1,\ldots\}$, let

$$U_i = \left\{x \in \Omega_\epsilon : d^*(x,Q) < \frac{\epsilon}{i}\right\}. \tag{2.34}$$

For each $i \in \{0,1,\ldots\}$ fixed, U_i is open with respect to d and so U_i is weakly open in Ω_ϵ. Also

$$\overline{U_i^w} = \overline{U_i^d} = \left\{x \in \Omega_\epsilon : d^*(x,Q) \leq \frac{\epsilon}{i}\right\}, \qquad \partial_{\Omega_\epsilon} U_i = \left\{x \in \Omega_\epsilon : d^*(x,Q) = \frac{\epsilon}{i}\right\}. \tag{2.35}$$

Keeping in mind that $\Omega_\epsilon \cap B = \emptyset$, Theorem 2.6 guarantees that there exist $y_i \in \partial_{\Omega_\epsilon} U_i$ and $\lambda_i \in (0,1)$ with $y_i = \lambda_i Fr(y_i)$. We now consider

$$D = \{x \in X : x = \lambda Fr(x), \quad \text{for some } \lambda \in [0,1]\}. \tag{2.36}$$

The same reasoning as above implies that D is weakly compact. Then, up to a subsequence, we may assume that $\lambda_i \to \lambda^* \in [0,1]$ and $y_i \rightharpoonup y^* \in \partial_{\Omega_\epsilon} U_i$. Hence $\lambda_i Fr(y_i) \rightharpoonup \lambda^* Fr(y^*)$ and therefore $y^* = \lambda^* Fr(y^*)$. Notice that $\lambda^* Fr(y^*) \notin Q$ since $y^* \in \partial_{\Omega_\epsilon} U_i$. Thus $\lambda^* \neq 1$ since $B \cap Q = \emptyset$. From assumption (iii) it follows that $\lambda_i Fr(y_i) \in Q$ for j sufficiently large, which is a contradiction. Thus $B \cap Q \neq \emptyset$, so there exists $x \in Q$ with $x = Fr(x)$, that is, $x = Fx$. □

Remark 2.11. In [16, Theorem 2.4], Q is assumed to be weakly compact.

Theorem 2.12. *Let X be a separable Banach space, C a closed bounded convex subset of X, and Q a closed convex subset of C with $0 \in Q$. Suppose that $A : Q \to X$ and $B : X \to X$ are weakly sequentially continuous mappings satisfying the following:*

(i) *$A(Q)$ is relatively weakly compact,*

(ii) *B is a nonexpansive map and $I - B$ is injective,*

(iii) *$A(Q) \subseteq (I - B)(C)$,*

(iv) *if (x_n) is a sequence of M such that $((I - B)x_n)$ is weakly convergent, then the sequence (x_n) has a weakly convergent subsequence,*

(v) *there exists a weakly continuous retraction $r : X \to Q$,*

(vi) *there exists a $\delta > 0$ and a weakly compact set Q_δ with $\Omega_\delta = \{x \in X : d(x,Q) \le \delta\} \subseteq Q_\delta$, here $d(x,y) = \|x - y\|$,*

(vii) *for any $\Omega_\epsilon = \{x \in X : d(x,Q) \le \epsilon,\ 0 < \epsilon \le \delta\}$, if $\{(x_j, \lambda_j)\}_{j=1}^{\infty}$ is a sequence in $Q \times [0,1]$ with $x_j \rightharpoonup x \in \partial_{\Omega_\epsilon} Q$, $\lambda_j \to \lambda$ and $x \in \lambda (I - B)^{-1}(Ax), 0 \le \lambda < 1$ ($(I - B)^{-1}(Ax)$ is the inverse image of Ax under $I - B$), then $\{\lambda_j (I - B)^{-1}(Ax_j)\} \subseteq Q$ for j sufficiently large, here $\partial_{\Omega_\epsilon} Q$ is the weak boundary of Q relative to Ω_ϵ.*

Then $A + B$ has a fixed point in Q.

Proof. Let us denote by F the map which assigns to each $y \in Q$ the point $F(y) \in C$ such that $(I - B)F(y) = Ay$. Since $I - B$ is injective, then $F : Q \to C$ is well defined. Now we show that F fulfills the conditions of Theorem 2.10. We first claim that $F(Q)$ is relatively weakly compact. Indeed let (x_n) be a sequence of elements of Q. Since $A(Q)$ is weakly compact, then, by extracting a subsequence if necessary, we may assume that (Ax_n) converges weakly to some $x \in X$. Hence $(I - B)F(x_n)$ converges weakly to x. By assumption (iv) we deduce that $F(x_n)$ has a weakly convergent subsequence. This proves our claim. Now we show that $F : Q \to C$ is weakly sequentially continuous. To see this let $(x_n)_n$ be a sequence in Q which converges weakly to x. Since $F(Q)$ is relatively weakly compact, there is a subsequence (x_{n_k}) of (x_n) such that

$$F(x_{n_k}) \rightharpoonup z. \tag{2.37}$$

Since $(I - B)F(x_{n_k}) = A(x_{n_k})$, then the weak sequential continuity of A and B implies $(I - B)(z) = Ax$. By the definition of F we have $(I - B)F(x) = Ax$. This gives $z = F(x)$ since $I - B$ is injective. Thus,

$$F(x_{n_k}) \rightharpoonup F(x). \tag{2.38}$$

Now we show that

$$F(x_n) \rightharpoonup F(x). \tag{2.39}$$

Suppose the contrary, then there exists a weak neighborhood N^w of $F(x)$ and a subsequence (x_{n_j}) of (x_n) such that $F(x_{n_j}) \notin N^w$ for all $j \geq 1$. Since (x_{n_j}) converges weakly to x, then arguing as before we may extract a subsequence $(x_{n_{j_k}})$ of (x_{n_j}) such that $F(x_{n_{j_k}}) \rightharpoonup F(x)$. This is not possible since $F(x_{n_{j_k}}) \notin N^w$ for all $k \geq 1$. As a result F is weakly sequentially continuous. Now let $\Omega_\epsilon = \{x \in X : d(x, Q) \leq \epsilon,\ 0 < \epsilon \leq \delta\}$, and let $\{(x_j, \lambda_j)\}_{j=1}^{\infty}$ be a sequence in $Q \times [0,1]$ with $x_j \rightharpoonup x \in \partial_{\Omega_\epsilon} Q$, $\lambda_j \to \lambda$, and $x = \lambda F(x)$, $0 \leq \lambda < 1$. Then $(I - B)(x/\lambda) = (I - B)F(x) = Ax$. Hence $x \in \lambda(I - B)^{-1}(Ax)$. By assumption (vii) we infer that $\{\lambda_j (I - B)^{-1}(Ax_j)\} \subseteq Q$ for j sufficiently large. This implies that $\lambda_j F(x_j) \in Q$ for j sufficiently large. The result follows from Theorem 2.10. □

Theorem 2.13. *Let M be a nonempty bounded closed convex subset of a Banach space X. Suppose that $A, B : M \to X$ are two continuous mappings satisfying the following:*

(i) *the set*

$$\mathcal{F} := \{x \in E : x = Bx + Ay \text{ for some } y \in M\} \tag{2.40}$$

is relatively compact,

(ii) *B is nonexpansive,*

(iii) *if (x_n) is a sequence of M such that $((I - B)x_n)$ is weakly convergent, then the sequence (x_n) has a weakly convergent subsequence,*

(iv) *$I - B$ is injective and demiclosed,*

(v) *$Ax + By \in M$, for all $x, y \in M$.*

Then $A + B$ has at least one fixed point in M.

Proof. Let $z \in A(M)$. The map which assigns to each $x \in M$ the value $Bx + z$ defines a nonexpansive mapping from M into M. In view of Theorem 1.2, there exists a sequence (x_n) in M such that

$$(I - B)x_n - z \longrightarrow 0. \tag{2.41}$$

By assumption (iii) we have that (x_n) has a subsequence, say (x_{n_k}), which converges to some $x \in M$. Since $(I - B)$ is demiclosed, then $z = (I - B)x$. Hence $z \in (I - B)M$. Consequently $A(M) \subseteq (I - B)(M)$. Let us denote by τ the map which assigns to each $y \in M$ the point

$\tau(y) \in M$ such that $(I - B)\tau(y) = Ay$. Since $I - B$ is injective, then $\tau : M \to M$ is well defined. Notice that $\tau(M) \subseteq \mathcal{F}$, then from assumption (i) it follows that $\tau(M)$ is relatively compact. Now we show that $\tau : M \to M$ is continuous. To see this let (x_n) be a sequence of M which converges to some $x \in M$. Since $\tau(M)$ is relatively compact, there is a subsequence (x_{n_k}) of (x_n) such that

$$\tau(x_{n_k}) \longrightarrow u. \tag{2.42}$$

By definition of τ we have

$$\tau(x_{n_k}) = A(x_{n_k}) + B\tau(x_{n_k}). \tag{2.43}$$

The continuity of A and B yields $u = Bu + Ax$. Since $(I - B)\tau(x) = Ax$ and $I - B$ is injective, then we have $u = \tau(x)$. As a result

$$\tau(x_{n_k}) \longrightarrow \tau(x). \tag{2.44}$$

Now we show that

$$\tau(x_n) \longrightarrow \tau(x). \tag{2.45}$$

Suppose the contrary, then there exists a $\epsilon > 0$ and a subsequence (x_{n_j}) of (x_n) such that $\|\tau(x_{n_j}) - \tau(x)\| > \epsilon$ for all $j \geq 1$. Since (x_{n_j}) converges to x, then arguing as before we may extract a subsequence $(x_{n_{j_k}})$ of (x_{n_j}) such that $\tau(x_{n_{j_k}}) \to \tau(x)$. This is not possible since $\|\tau(x_{n_{j_k}}) - \tau(x)\| > \epsilon$ for all $k \geq 1$. Consequently, $\tau : M \to M$ is continuous. Applying the Schauder fixed point theorem we infer that there exists $x \in M$ such that

$$x = \tau(x) = B(\tau(x)) + Ax = Bx + Ax. \tag{2.46}$$

□

An easy consequence of Theorem 2.13 is the following.

Corollary 2.14. *Let M be a nonempty bounded closed convex subset of a reflexive Banach space X. Suppose that $A, B : M \to X$ are two continuous mappings satisfying the following:*

(i) *the set*

$$\mathcal{F} := \{x \in E\ : x = Bx + Ay \text{ for some } y \in M\} \tag{2.47}$$

is relatively compact,

(ii) *B is nonexpansive,*

(iii) *$I - B$ is injective and demi-closed,*

(iv) *$Ax + By \in M$, for all $x, y \in M$.*

Then $A + B$ has at least one fixed point in M.

Proof. Keeping in mind that every bounded subset in a reflexive Banach space is relatively weakly compact, the result follows from Theorem 2.13. □

Corollary 2.15. *Let M be a nonempty bounded closed convex subset of a uniformly convex Banach space X. Suppose that $A, B : M \to X$ are two continuous mappings satisfying the following:*

(i) *the set*

$$\mathcal{F} := \{x \in E \ : x = Bx + Ay \text{ for some } y \in M\} \tag{2.48}$$

is relatively compact,

(ii) *B is nonexpansive and $I - B$ is injective,*

(iii) *$Ax + By \in M$, for all $x, y \in M$.*

Then $A + B$ has at least one fixed point in M.

Proof. Note that in a uniformly convex space we have that B is nonexpansive implying that $I - B$ is demiclosed (see [4, 15]). Moreover, every uniformly convex Banach space is reflexive. The result follows from Corollary 2.14. □

Recall also the following definition.

Definition 2.16 (see [2]). Let X be a Banach space, M a nonempty subset of X and $T : M \to X$ be a mapping. We will call T a shrinking mapping if for all $x, y \in M$ such that $x \neq y$ we have

$$\|Tx - Ty\| < \|x - y\|. \tag{2.49}$$

Thus a shrinking mapping is nonexpansive but need not be a contraction mapping. If T is a shrinking mapping, then $(I - B)^{-1}$ exists but need not be continuous. The following result is also an immediate consequence of Theorem 2.13.

Corollary 2.17. *Let M be a nonempty bounded closed convex subset of a Banach space X. Suppose that $A, B : M \to X$ are two continuous mappings satisfying:*

(i) *The set*

$$\mathcal{F} := \{x \in E : x = Bx + Ay \text{ for some } y \in M\} \tag{2.50}$$

is relatively compact,

(ii) *B is a shrinking map,*

(iii) *if (x_n) is a sequence of M such that $((I - B)x_n)$ is weakly convergent, then the sequence (x_n) has a weakly convergent subsequence,*

(iv) *$I - B$ is demiclosed,*

(v) *$Ax + By \in M$, for all $x, y \in M$.*

Then $A + B$ has at least one fixed point in M.

The following example, which is taken from [21], shows that condition (i) in Theorem 2.13 and Corollary 2.15 cannot be replaced by the compactness of A.

Example 2.18. Let H be a separable Hilbert space and $(e_n)_{n\in\mathbb{Z}}$ an orthonormal basis for H. Let M be the closed unit ball of H. Define A and B as follows. For $x = \sum_{n=-\infty}^{+\infty} x_n e_n$, $Bx = \sum_{n=-\infty}^{+\infty} x_n e_{n+1}$ and $Ax = (1 - \|x\|)e_0$. We have that A is nonexpansive and maps bounded sets into relatively compact sets. The mapping B is weakly continuous and nonexpansive. Moreover, $I - B$ is injective and $(A+B)M \subseteq M$. However, $A+B$ has no fixed point in M.

In the case where A is compact and B is nonexpansive, we add an additional assumption on B to guarantee the existence of a fixed point for the sum $A+B$ as follows.

Theorem 2.19. *Let M be a nonempty bounded closed convex subset of a Banach space X. Suppose that $A, B : M \to X$ are two continuous mappings satisfying the following:*

(i) *A is compact,*

(ii) *B is nonexpansive,*

(iii) *if (x_n) is a sequence of M such that $((I-B)x_n)$ is strongly convergent, then the sequence (x_n) has a strongly convergent subsequence,*

(iv) *$Ax + By \in M$, for all $x, y \in M$.*

Then $A+B$ has at least one fixed point in M.

Proof. Arguing exactly in the same way as in the proof of Theorem 2.5 and using [22, Theorem 2] instead of Theorem 1.7 we get the desired result. □

Now, we state the following fixed point theorem of Furi-Pera type.

Theorem 2.20. *Let Q be a closed convex subset of a Banach space X and $0 \in Q$. Suppose that $A : Q \to X$ and $B : X \to X$ are continuous mapping satisfyingthe following:*

(i) *the set*

$$\mathcal{F} := \{x \in E : x = Bx + Ay \text{ for some } y \in Q\} \tag{2.51}$$

is relatively compact,

(ii) *$I - B$ is injective,*

(iii) *$A(Q) \subseteq (I-B)(X)$,*

(iv) *if $\{(x_j, \lambda_j)\}_{j=1}^{+\infty}$ is a sequence of $\partial Q \times [0,1]$ converging to (x, λ) with $x = \lambda(I-B)^{-1}(Ax)$ and $0 \le \lambda < 1$, then $\lambda_j (I-B)^{-1}(Ax_j) \in Q$ for j sufficiently large.*

Then $A+B$ has a fixed point in Q.

Proof. Let $y \in Q$ be fixed. From assumptions (ii) and (iii) it follows that there is a unique $z_y \in X$ such that $Ay = (I-B)z_y$. Let us denote by $H : Q \to X$ the map which assigns to y the unique point $H(y) = z_y$. Notice that $H(Q) \subseteq \mathcal{F}$, then from assumption (i) it follows that $H(Q)$ is relatively compact. Now we show that $H : Q \to X$ is continuous. To see this let (x_n)

be a sequence of Q which converges to some $x \in Q$. Since $H(Q)$ is relatively compact, then there is a subsequence (x_{n_k}) of (x_n) such that

$$H(x_{n_k}) \longrightarrow u. \tag{2.52}$$

By definition of H we have

$$H(x_{n_k}) = A(x_{n_k}) + BH(x_{n_k}). \tag{2.53}$$

The continuity of A and B yields $u = Bu + Ax$. Since $(I - B)H(x) = Ax$ and $I - B$ is injective, then we have $u = H(x)$. As a result

$$H(x_{n_k}) \longrightarrow H(x). \tag{2.54}$$

The reasoning in Theorem 2.13 shows that

$$H(x_n) \longrightarrow H(x). \tag{2.55}$$

Now let $\{(x_j, \lambda_j)\}_{j=1}^{+\infty}$ be a sequence of $\partial Q \times [0,1]$ converging to (x, λ) with $x = \lambda H(x)$ and $0 \leq \lambda < 1$, then $x = \lambda(I - B)^{-1}(Ax)$. From our assumptions it follows that $\lambda_j (I - B)^{-1}(Ax_j) \in Q$ for j sufficiently large. Thus $\lambda_j H(x_j) \in Q$ for j sufficiently large. Now Theorem 1.9 implies that there is an $x \in Q$ with $x = Hx$ and so $x = Ax + BHx = Ax + Bx$. □

Also, we give the following fixed point theorem of Leray-Schauder type.

Theorem 2.21. *Let X be a Banach space with $C \subseteq X$ closed and convex. Assume that U is a relatively open subset of C with $0 \in U$. Suppose that $A : \overline{U} \to X$ and $B : X \to X$ are continuous mapping satisfying the following:*

(i) *the set*

$$\mathcal{F} := \left\{ x \in E : x = Bx + Ay \text{ for some } y \in \overline{U} \right\} \tag{2.56}$$

is relatively compact,

(ii) *$I - B$ is injective,*

(iii) *$A(\overline{U}) \subseteq (I - B)(C)$.*

Then either

$$A + B \text{ has a fixed point in } \overline{U}, \tag{2.57}$$

or

$$\text{there is a point } u \in \partial U \text{ and } \lambda \in (0,1) \quad \text{with } u = \lambda B\left(\frac{u}{\lambda}\right) + \lambda A u. \tag{2.58}$$

Proof. Let $y \in \overline{U}$ be fixed. From assumptions (ii) and (iii) it follows that there is a unique $z_y \in C$ such that $Ay = (I - B)z_y$. Let us denote by $H : \overline{U} \to C$ the map which assigns to y the unique point $H(y) = z_y$. Notice that $H(Q) \subseteq \mathcal{F}$, then from assumption (i) it follows that $H(Q)$ is relatively compact. The reasoning in Theorem 2.20 shows that H is continuous. Now Theorem 1.8 implies that either there is a $u \in \overline{U}$ such that $u = Hu$, that is, $u = B\mathrm{u} + Au$, or there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda Hu$. Thus $u = \lambda(I - B)^{-1}Au$ which is equivalent to $u = \lambda B\left(\frac{u}{\lambda}\right) + \lambda Au$. □

Theorem 2.22. *Let U be a bounded open convex set in a Banach space X with $0 \in U$. Suppose that $A : \overline{U} \to X$ and $B : X \to X$ are continuous mapping satisfying the following:*

(i) *$A(\overline{U})$ is compact and A is weakly-strongly continuous,*

(ii) *B is nonexpansive and $I - B$ is demiclosed,*

(iii) *if (x_n) is a sequence of $\overline{U}$ such that $((I - B)x_n)$ is weakly convergent, then the sequence (x_n) has a weakly convergent subsequence.*

Then either

$$A + B \text{ has a fixed point in } \overline{U}, \tag{2.59}$$

or

$$\text{there is a point } u \in \partial U \text{ and } \lambda \in (0,1) \text{ with } u = \lambda Bu + \lambda Au. \tag{2.60}$$

Proof. Suppose that (2.60) does not occur, and let $\mu \in (0,1)$. The mapping $F_\mu := \mu A + \mu B$ is the sum of a compact map and a strict contraction. This implies that F_μ is a condensing map (see [23]). By Theorem 1.8 we deduce that there is an $x_\mu \in \overline{U}$ such that $F_\mu x_\mu = \mu A x_\mu + \mu B x_\mu = x_\mu$. Now, choose a sequence $\{\mu_n\}$ in $(0,1)$ such that $\mu_n \to 1$ and consider the corresponding sequence $\{x_n\}$ of elements of $\overline{U}$ satisfying

$$\mu_n A x_n + \mu_n B x_n = x_n. \tag{2.61}$$

Keeping in mind that $A(\overline{U})$ is weakly compact and passing eventually to a subsequence, we may assume that $\{Ax_n\}$ converges weakly to some $y \in \overline{U}$. Hence

$$(I - \mu_n B)x_n \rightharpoonup y. \tag{2.62}$$

Since $\{x_n\}$ is a sequence in $\overline{U}$, then it is norm bounded and so is $\{Bx_n\}$. Consequently

$$\left\|(x_n - Bx_n) - (x_n - \mu_n B x_n)\right\| = (1 - \mu_n)\|Bx_n\| \longrightarrow 0. \tag{2.63}$$

As a result

$$x_n - Bx_n \rightharpoonup y. \tag{2.64}$$

By hypothesis (iii) the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to some $x \in \overline{U}$. Since A is weakly-strongly continuous, then $\{Ax_{n_k}\}$ converges strongly to Ax. Consequently

$$(I - \lambda_{n_k} B)x_{n_k} \longrightarrow Ax. \tag{2.65}$$

Standard arguments yields

$$x_{n_k} - Bx_{n_k} \longrightarrow Ax. \tag{2.66}$$

The demiclosedness of $I - B$ implies $Ax + Bx = x$. □

Theorem 2.23. *Let Q be a closed convex bounded set in a Banach space X with $0 \in Q$. Suppose that $A : Q \to X$ and $B : X \to X$ are continuous mapping satisfying the following:*

(i) *$A(Q)$ is compact and A is weakly-strongly continuous,*

(ii) *B is nonexpansive and $I - B$ is demiclosed,*

(iii) *if (x_n) is a sequence of Q such that $((I - B)x_n)$ is weakly convergent, then the sequence (x_n) has a weakly convergent subsequence,*

(iv) *if $\{(x_j, \lambda_j)\}_{j=1}^{+\infty}$ is a sequence of $\partial Q \times [0,1]$ converging to (x, λ) with $x = \lambda Ax + \lambda Bx$ and $0 \le \lambda < 1$, then $\lambda_j Ax_j + \lambda_j Bx_j \in Q$ for j sufficiently large.*

Then $A + B$ has a fixed point in Q.

Proof. Let $\mu \in (0,1)$ be fixed. Since $F_\mu := \mu A + \mu B$ is the sum of a compact map and a strict contraction, then F_μ is a condensing map (see [23]). Now let $\{(x_j, \lambda_j)\}_{j=1}^{+\infty}$ be a sequence of $\partial Q \times [0,1]$ converging to (x, λ) with $x = \lambda F_\mu(x)$ and $0 \le \lambda < 1$. Then $x = \mu\lambda Ax + \mu\lambda Bx$. From assumption (iv) it follows that $\mu\lambda_j Ax_j + \mu\lambda_j Bx_j \in Q$ for j sufficiently large. Consequently $\lambda_j F_\mu(x_j) \in Q$ for j sufficiently large. Applying Theorem 1.9 to F_μ we deduce that there is an $x_\mu \in Q$ such that $F_\mu x_\mu = \mu Ax_\mu + \mu Bx_\mu = x_\mu$. Now, choose a sequence $\{\mu_n\}$ in $(0,1)$ such that $\mu_n \to 1$ and consider the corresponding sequence $\{x_n\}$ of elements of Q satisfying

$$\mu_n Ax_n + \mu_n Bx_n = x_n. \tag{2.67}$$

Keeping in mind that $A(Q)$ is weakly compact and passing eventually to a subsequence, we may assume that $\{Ax_n\}$ converges weakly to some $y \in Q$. Hence

$$(I - \mu_n B)x_n \rightharpoonup y. \tag{2.68}$$

As in Theorem 2.22 this implies that

$$x_n - Bx_n \rightharpoonup y. \tag{2.69}$$

By hypothesis (iii) the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to some $x \in Q$. Since A is weakly-strongly continuous, then $\{Ax_{n_k}\}$ converges strongly to Ax. Consequently

$$(I - \lambda_{n_k} B)x_{n_k} \longrightarrow Ax. \tag{2.70}$$

Standard arguments yield

$$x_{n_k} - Bx_{n_k} \longrightarrow Ax. \tag{2.71}$$

The demiclosedness of $I - B$ implies that $Ax + Bx = x$. □

Definition 2.24 (see [18, Definition 14]). A mapping $B : D(B) \subseteq X \to X$ is said to be ϕ-expansive if there exists a function $\phi : [0, +\infty[\to [0, +\infty[$ satisfying the following:

(i) $\phi(0) = 0$,

(ii) $\phi(r) > 0$ for $r > 0$,

(iii) either it is continuous or it is nondecreasing, such that, for every $x, y \in D(B)$, the inequality $\|Bx - By\| \geq \phi(\|x - y\|)$ holds.

It was proved in [18] that if M is a nonempty bounded closed and convex subset of a Banach space X and $B : M \to X$ is a nonexpansive mapping such that $I - B : M \to X$ is ϕ-expansive, then $(I - B)^{-1} : (I - B)(M) \to M$ exists and is continuous. In the light of this fact we obtain the following result which is an immediate consequence of Theorem 2.19.

Corollary 2.25. *Let M be a nonempty bounded closed convex subset of a Banach space X. Suppose that $A, B : M \to X$ are two continuous mappings satisfying the following:*

(i) *A is compact,*

(ii) *B is nonexpansive and $I - B$ is ϕ-expansive,*

(iii) *$Ax + By \in M$, for all $x, y \in M$.*

Then $A + B$ has at least one fixed point in M.

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