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Research Article

On Two Iterative Methods for Mixed Monotone Variational Inequalities

Xiwen Lu,[1] Hong-Kun Xu,[2] and Ximing Yin[1]

[1] *Department of Mathematics, East China University of Science and Technology, Shanghai 200237, China*
[2] *Department of Applied Mathematics, National Sun Yat-Sen University, Kaohsiung 80424, Taiwan*

Correspondence should be addressed to Hong-Kun Xu, xuhk@math.nsysu.edu.tw

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A mixed monotone variational inequality (MMVI) problem in a Hilbert space H is formulated to find a point $u^* \in H$ such that $\langle Tu^*, v - u^* \rangle + \varphi(v) - \varphi(u^*) \geq 0$ for all $v \in H$, where T is a monotone operator and φ is a proper, convex, and lower semicontinuous function on H. Iterative algorithms are usually applied to find a solution of an MMVI problem. We show that the iterative algorithm introduced in the work of Wang et al., (2001) has in general weak convergence in an infinite-dimensional space, and the algorithm introduced in the paper of Noor (2001) fails in general to converge to a solution.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let T be an operator with domain $D(T)$ and range $R(T)$ in H. Recall that T is *monotone* if its graph $G(T) := \{(x, y) \in H \times H : x \in D(T), y \in Tx\}$ is a monotone set in $H \times H$. This means that T is monotone if and only if

$$(x, y), (x', y') \in G(T) \Longrightarrow \langle x - x', y - y' \rangle \geq 0. \tag{1.1}$$

A monotone operator T is *maximal* monotone if its graph $G(T)$ is not properly contained in the graph of any other monotone operator on H.

Let $\varphi : H \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}, \not\equiv +\infty$, be a proper, convex, and lower semicontinuous functional. The *subdifferential* of $\varphi, \partial\varphi$ is defined by

$$\partial\varphi(x) := \{z \in H : \varphi(y) \geq \varphi(x) + \langle y - x, z \rangle,\ \forall y \in H\}. \tag{1.2}$$

It is well known (cf. [1]) that $\partial\varphi$ is a maximal monotone operator.

The *mixed monotone variational inequality (MMVI)* problem is to find a point $u^* \in H$ with the property

$$\langle Tu^*, v - u^* \rangle + \varphi(v) - \varphi(u^*) \geq 0, \quad \forall v \in H, \tag{1.3}$$

where T is a monotone operator and φ is a proper, convex, and lower semicontinuous function on H.

If one takes φ to be the indicator of a closed convex subset K of H,

$$\varphi(x) = \begin{cases} 0, & x \in K, \\ +\infty, & x \notin K, \end{cases} \tag{1.4}$$

then the MMVI (1.3) is reduced to the classical *variational inequality* (VI):

$$u^* \in K, \quad \langle Tu^*, v - u^* \rangle \geq 0, \quad v \in K. \tag{1.5}$$

Recall that the *resolvent* of a monotone operator T is defined as

$$J_\rho^T := (I + \rho T)^{-1}, \quad \rho > 0. \tag{1.6}$$

If $T = \partial\varphi$, we write J_ρ^φ for $J_\rho^{\partial\varphi}$. It is known that T is monotone if and only of for each $\rho > 0$, the resolvent J_ρ^T is nonexpansive, and T is maximal monotone if and only of for each $\rho > 0$, the resolvent J_ρ^T is nonexpansive and defined on the entire space H. Recall that a self-mapping of a closed convex subset K of H is said to be

(i) *nonexpansive* if $\|f(x) - f(y)\| \leq \|x - y\|$ for all $x, y \in K$;

(ii) *firmly nonexpansive* if $\|f(x) - f(y)\|^2 \leq \langle x-y, f(x)-f(y) \rangle$ for $x, y \in K$. Equivalently, f is firmly nonexpansive if and only of $2f - I$ is nonexpansive. It is known that each resolvent of a monotone operator is firmly nonexpansive.

We use $\mathrm{Fix}(f)$ to denote the set of fixed points of f; that is, $\mathrm{Fix}(f) = \{x \in K : f(x) = x\}$.

Variational inequalities have extensively been studied; see the monographs by Baiocchi and Capelo [2], Cottle et al. [3], Glowinski et al. [4], Giannessi and Maugeri [5], and Kinderlehrer and Stampacciha [6].

Iterative methods play an important role in solving variational inequalities. For example, if T is a single-valued, strongly monotone (i.e., $\langle Tx - Ty, x - y \rangle \geq \tau\|x - y\|^2$ for all $x, y \in K$ and some $\tau > 0$), and Lipschitzian (i.e., $\|Tx - Ty\| \leq L\|x - y\|$ for some $L > 0$ and all $x, y \in D(T)$) operator on K, then the sequence $\{x^k\}$ generated by the iterative algorithm

$$x^{k+1} = P_K(I - \rho T)x^k, \quad k \geq 0, \tag{1.7}$$

where I is the identity operator and P_K is the metric projection of H onto K, and the initial guess $x^0 \in H$ is chosen arbitrarily, converges strongly to the unique solution of VI (1.5) provided, $\rho > 0$ is small enough.

2. An Inexact Implicit Method

In this section we study the convergence of an inexact implicit method for solving the MMVI (1.3) introduced by Wang et al. [7] (see also [8, 9] for related work).

Let $\{\tau_k\}$ and $\{\pi_k\}$ be two sequences of nonnegative numbers such that

$$\pi_k \in [0,1) \quad \forall k \geq 0, \quad \sum_{k=0}^{\infty} \pi_k < \infty, \quad \sum_{k=0}^{\infty} \tau_k < \infty. \tag{2.1}$$

Let $\gamma \in (0,2)$ and $u^0 \in H$. The inexact implicit method introduced in [7] generates a sequence $\{u^k\}$ defined in the following way. Once u^k has been constructed, the next iterate u^{k+1} is implicitly constructed satisfying the equation

$$(I + \rho_k T)u^{k+1} - (I + \rho_k T)u^k + \gamma e\left(u^k, \rho_k\right) = \theta_k, \tag{2.2}$$

where $\{\rho_k\}$ is a sequence of nonnegative numbers such that

$$\rho_{k+1} \in \left[\rho_k/(1+\tau_k), \rho_k(1+\tau_k)\right] \tag{2.3}$$

for $k \geq 0$, and for $u \in H$ and $\rho > 0$,

$$e(u,\rho) := u - J_{\varphi}^{\rho}(u - \rho T u), \tag{2.4}$$

and where $\theta_k = \theta_k(u^{k+1})$ is such that

$$\|\theta_k\| \leq \delta_k \tag{2.5}$$

with δ_k given as follows:

$$\delta_k = \begin{cases} \pi_k, & \text{if } \gamma(2-\gamma)\left\|e(u^k,\rho_k)\right\|^2 \geq \dfrac{1}{2}, \\ \min\left\{\pi_k, \dfrac{1}{2}\left[1 - \sqrt{1 - 2\gamma(2-\gamma)\left\|e(u^k,\rho_k)\right\|^2}\right]\right\}, & \text{otherwise.} \end{cases} \tag{2.6}$$

We note that u^* is a solution of the MMVI (1.3) if and only if, for each $\rho > 0$, u^* satisfies the fixed point equation

$$u^* = J_{\varphi}^{\rho}(u^* - \rho T u^*). \tag{2.7}$$

Before discussing the convergence of the implicit algorithm (2.2), we look at a special case of (2.2), where $T = 0$. In this case, the MMVI (1.3) reduces to the problem of finding a $u^* \in H$ such that

$$\varphi(v) \geq \varphi(u^*), \quad \forall v \in H, \tag{2.8}$$

in another word, finding an absolute minimizer u^* of φ over H. This is equivalent to solving the inclusion

$$0 \in \partial\varphi(u^*), \tag{2.9}$$

and the algorithm (2.2) is thus reduced to a special case of the Eckastein-Bertsekas algorithm [10]

$$u^{k+1} = (1-\gamma)u^k + \gamma J_{\varphi}^{\rho_k}\left(u^k\right) + e^k, \tag{2.10}$$

where $e^k = \theta_k(u^{k+1})$. If $\gamma = 1$, then algorithm (2.2) is reduced to a special case of Rockafellar's proximal point algorithm [11]

$$u^{k+1} = J_{\varphi}^{\rho_k}\left(u^k\right) + e^k. \tag{2.11}$$

Rockafellar's proximal point algorithm for finding a zero of a maximal monotone operator has received tremendous investigations; see [12–14] and the references therein.

Remark 2.1. Theorem 5.1 of Wang et al. [7] holds true only in the finite-dimensional setting. This is because in the infinite-dimensional setting, a bounded sequence fails, in general, to have a norm-convergent subsequence. As a matter of fact, in the infinite-dimensional case, the special case of (2.2) where $T = 0$ and $\gamma = 1$ corresponds to Rockafellar's proximal point algorithm (2.11) which fails to converge in the norm topology, in general, in the infinite-dimensional setting; see Güler's counterexample [15]. This infinite-dimensionality problem occurred in several papers by Noor (see, e.g., [16–26]).

In the infinite-dimensional setting, whether or not Wang et al.'s implicit algorithm (2.2) converges even in the weak topology remains an open question. We will provide a partial answer by showing that if the operator T is weak-to-strong continuous (i.e., T takes weakly convergent sequences to strongly convergent sequences), then the implicit algorithm (2.2) does converge weakly.

We next collect the (correct) results proved in [7].

Proposition 2.2. *Assume that $\{u^k\}$ is generated by the implicit algorithm* (2.2)*.*

(a) *For $u \in H$, $\|e(u,\rho)\|$ is a nondecreasing function of $\rho \geq 0$.*

(b) *If u^* is a solution to the MMVI* (1.3)*, $u \in H$ and $\rho > 0$, then*

$$\left\langle u - u^* + \rho(Tu - Tu^*), e(u,\rho)\right\rangle \geq \left\|e(u,\rho)\right\|^2 + \rho\langle u - u^*, Tu - Tu^*\rangle. \tag{2.12}$$

(c) *For any solution u^* to the MMVI* (1.3)*,*

$$\left\|u^{k+1} - u^* + \rho_{k+1}(Tu^{k+1} - Tu^*)\right\|^2 \leq \left\|u^k - u^* + \rho_k(Tu^k - Tu^*)\right\|^2 + \sigma_k, \tag{2.13}$$

where $\sigma_k \geq 0$ satisfies $\sum_{k=1}^{\infty} \sigma_k < \infty$.

(d) $\{u^k\}$ *is bounded.*

(e) *There is a* $\overline{\rho} > 0$ *such that* $\lim_{k\to\infty}\|e(u^k, \overline{\rho})\| = 0$.

Since algorithm (2.2) is, in general, not strongly convergent, we turn to investigate its weak convergence. It is however unclear if the algorithm is weakly convergent (if the space is infinite dimensional). We present a partial answer below. But first recall that an operator T is said to be *weak-to-strong* continuous if the weak convergence of a sequence $\{x^k\}$ to a point x implies the strong convergence of the sequence $\{Tx^k\}$ to the point Tx.

Theorem 2.3. *Assume that* $\{u^k\}$ *is generated by algorithm* (2.2)*. If* T *is weak-to-strong continuous, then* $\{u^k\}$ *converges weakly to a solution of the MMVI* (1.3)*.*

Proof. Putting

$$\eta^k = J_{\varphi}^{\rho_k}\left(u^k - \rho_k T u^k\right) - u^k, \tag{2.14}$$

we have

$$J_{\varphi}^{\rho_k}\left(u^k - \rho_k T u^k\right) = u^k + \eta^k, \quad \left\|\eta^k\right\| \longrightarrow 0. \tag{2.15}$$

It follows that

$$u^k - \rho_k T u^k \in \left(I + \rho_k \partial\varphi\right)\left(u^k + \eta^k\right). \tag{2.16}$$

This implies that

$$-\frac{1}{\rho_k}\eta^k - T u^k \in \partial\varphi\left(u^k + \eta^k\right). \tag{2.17}$$

So, if $u^{k_i} \to \overline{u}$ weakly (hence $Tu^{k_i} \to T\overline{u}$ strongly since T is weak-to-strong continuous), it follows that

$$-T\overline{u} \in \partial\varphi(\overline{u}). \tag{2.18}$$

Thus, $\overline{u}$ is a solution.

To prove that the entire sequence of $\{u^k\}$ is weakly convergent, assume that $u^{m_i} \to \widetilde{u}$ weakly. All we have to prove is that $\widetilde{u} = \overline{u}$. Passing through further subsequences if necessary, we may assume that $\lim_{i\to\infty}\|u^{k_i} - \overline{u}\|$ and $\lim_{i\to\infty}\|u^{m_i} - \widetilde{u}\|$ both exist.

For $\varepsilon > 0$, since $Tu^{k_i} \to T\overline{u}$ strongly and since $\{u^k\}$ and $\{\rho_k\}$ are bounded, there exists an integer $i_0 \geq 1$ such that, for $i \geq i_0$,

$$2\rho_{k_i}\langle u^{k_i} - \overline{u}, Tu^{k_i} - T\overline{u}\rangle + \rho_{k_i}^2\left\|Tu^{k_i} - T\overline{u}\right\|^2 + \sum_{j=k_i-1}^{\infty}\sigma_j < \varepsilon. \tag{2.19}$$

It follows that for $k > k_i > k_{i_0}$,

$$\begin{aligned}
\left\|u^k - \overline{u}\right\|^2 &\le \left\|u^k - \overline{u} + \rho_k\left(Tu^k - T\,\overline{u}\right)\right\|^2 \\
&\le \left\|u^{k-1} - \overline{u} + \rho_{k-1}\left(Tu^{k-1} - T\,\overline{u}\right)\right\|^2 + \sigma_{k-1} \\
&\le \cdots \\
&\le \left\|u^{k_i} - \overline{u} + \rho_{k_i}\left(Tu^{k_i} - T\,\overline{u}\right)\right\|^2 + \sum_{j=k_i-1}^{k-1} \sigma_j \\
&= \left\|u^{k_i} - \overline{u}\right\|^2 + 2\rho_{k_i}\left\langle u^{k_i} - \overline{u}, Tu^{k_i} - T\,\overline{u}\right\rangle + \rho_{k_i}^2\left\|Tu^{k_i} - T\,\overline{u}\right\|^2 + \sum_{j=k_i-1}^{\infty} \sigma_j \\
&< \left\|u^{k_i} - \overline{u}\right\|^2 + \varepsilon.
\end{aligned} \tag{2.20}$$

This implies

$$\limsup_{k\to\infty}\left\|u^k - \overline{u}\right\|^2 \le \limsup_{i\to\infty}\left\|u^{k_i} - \overline{u}\right\|^2. \tag{2.21}$$

However,

$$\limsup_{k\to\infty}\left\|u^k - \overline{u}\right\|^2 \ge \limsup_{i\to\infty}\|u^{m_i} - \overline{u}\|^2 = \limsup_{i\to\infty}\|u^{m_i} - \widetilde{u}\|^2 + \|\widetilde{u} - \overline{u}\|^2. \tag{2.22}$$

It follows that

$$\limsup_{i\to\infty}\|u^{m_i} - \widetilde{u}\|^2 + \|\widetilde{u} - \overline{u}\|^2 \le \limsup_{i\to\infty}\left\|u^{k_i} - \overline{u}\right\|^2. \tag{2.23}$$

Similarly, by repeating the argument above we obtain

$$\limsup_{i\to\infty}\left\|u^{k_i} - \overline{u}\right\|^2 + \|\overline{u} - \widetilde{u}\|^2 \le \limsup_{i\to\infty}\|u^{m_i} - \widetilde{u}\|^2. \tag{2.24}$$

Adding these inequalities, we get $\widetilde{u} = \overline{u}$. □

3. A Counterexample

It is not hard to see that $u^* \in H$ solves MMVI (1.3) if and only of $u^* \in H$ solves the inclusion

$$0 \in (T + \partial\varphi)(u^*) \tag{3.1}$$

which is in turn equivalent to the fixed point equation

$$u^* = J_\varphi^\rho [u^* - \rho T u^*], \tag{3.2}$$

where J_φ^ρ is the resolvent of $\partial\varphi$ defined by

$$J_\varphi^\rho(x) = (I + \rho\partial\varphi)^{-1}(x), \quad x \in H. \tag{3.3}$$

Recall that if φ is the indicator of a closed convex subset K of H,

$$\varphi(x) = \begin{cases} 0, & x \in K, \\ +\infty, & x \notin K, \end{cases} \tag{3.4}$$

then MMVI (1.3) is reduced to the classical variational inequality (VI)

$$\langle T u^*, v - u^* \rangle \geq 0, \quad v \in K. \tag{3.5}$$

In [27], Noor introduced a new iterative algorithm [27, Algorithm 3.3, page 36] as follows. Given $u^0 \in H$, compute u^{k+1} by the iterative scheme

$$u^{k+1} = u^k + \rho T u^k - \rho T u^{k+1} - \gamma R\left(u^k\right), \quad k \geq 0, \tag{3.6}$$

where $\rho > 0$ and $\gamma \in (0, 2)$ are constant, and $R(u)$ is given by

$$R(u) = u - J_\varphi^\rho\left[u - \rho T J_\varphi^\rho [u - \rho T u]\right]. \tag{3.7}$$

Noor [27] proved a convergence result for his algorithm (3.6) as follows.

Theorem 3.1 (see [27, page 38]). *Let H be a finite-dimensional Hilbert space. Then the sequence $\{u^k\}$ generated by algorithm* (3.6) *converges to a solution of MMVI* (1.3).

We however found that the conclusion stated in the above theorem is incorrect. It is true that u^* solves MMVI (1.3) if and only if u^* solves the fixed point equation (3.2). The reason that led Noor to his mistake is his claim that u^* solves MMVI (1.3) if and only if u^* solves the following iterated fixed point equation:

$$u^* = J_\varphi^\rho\left[u^* - \rho T J_\varphi^\rho [u^* - \rho T u^*]\right]. \tag{3.8}$$

As a matter of fact, the two fixed point equations (3.2) and (3.8) are not equivalent, as shown in the following counterexample which also shows that the convergence result of Noor [27] is incorrect.

Example 3.2. Take $H = \mathbb{R}$. Define T and φ by

$$Tx = x, \quad \varphi(x) = |x|, \quad x \in \mathbb{R}. \tag{3.9}$$

Notice that (Clarke [28])

$$\partial\varphi(x) = \begin{cases} 1, & \text{if } x > 0, \\ [-1,1], & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases} \tag{3.10}$$

It is easily seen that $u^* = 0$ is the unique solution to the MMVI

$$\langle u^*, v - u^* \rangle + |v| - |u^*| \geq 0, \quad v \in \mathbb{R}. \tag{3.11}$$

Observe that equation $R(u) = 0$ is equivalent to the fixed point equation

$$u = J_\varphi^\rho \Big[u - \rho T J_\varphi^\rho [u - \rho T u] \Big]. \tag{3.12}$$

Now since $Tx = x$ for all $x \in \mathbb{R}$, we get that $u^* \in \mathbf{R}$ solves (3.12) if and only if

$$u^* = J_\varphi^\rho [u^* - \rho v^*], \tag{3.13}$$

where

$$v^* = J_\varphi^\rho [u^* - \rho u^*]. \tag{3.14}$$

It follows from (3.13) that $u^* - \rho v^* \in (I + \rho\partial\varphi)(u^*)$. Hence

$$-v^* \in \partial\varphi(u^*). \tag{3.15}$$

But, since

$$\partial\varphi(u^*) = \begin{cases} 1, & \text{if } u^* > 0, \\ [-1,1], & \text{if } u^* = 0, \\ -1, & \text{if } u^* < 0, \end{cases} \tag{3.16}$$

we deduce that the solution set S of the fixed point equation (3.12) is given by

$$S = \begin{cases} \left\{ \dfrac{\rho+1}{\rho-1}, \dfrac{\rho+1}{1-\rho} \right\}, & \text{if } \rho > 1, \\ 0, & \text{if } \rho = 1, \\ \emptyset, & \text{if } \rho < 1. \end{cases} \tag{3.17}$$

(We therefore conclude that equation $R(u) = 0$ is not equivalent to MMVI (1.3), as claimed by Noor [27].)

Now take the initial guess $u^0 = (\rho + 1)/(\rho - 1)$ for $\rho > 1$. Then $R(u^0) = 0$ and we have that algorithm (3.6) generates a constant sequence $u^k \equiv u^0$ for all $k \geq 1$. However, $u^0 > 0$ is not a solution of MMVI (3.11). This shows that algorithm (3.6) may generate a sequence that fails to converge to a solution of MMVI (1.3) and Noor's result in [27] is therefore false.

Remark 3.3. Noor has repeated his above mistake in a number of his recent articles. A partial search found that articles [20, 21, 26, 29–32] contain the same error.

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