Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 303640, 7 pages
doi:10.1155/2010/303640

Research Article

Fixed Simplex Property for Retractable Complexes

Adam Idzik[1, 2] and Anna Zapart[3]

[1] *Institute of Mathematics, Jan Kochanowski University, 15 Świętokrzyska street, 25-406 Kielce, Poland*
[2] *Institute of Computer Science, Polish Academy of Sciences, 21 Ordona street, 01-237 Warsaw, Poland*
[3] *Faculty of Mathematics and Information Science, Warsaw University of Technology, Pl. Politechniki 1, 00-661 Warsaw, Poland*

Correspondence should be addressed to Adam Idzik, adidzik@ipipan.waw.pl

Received 16 December 2009; Revised 10 August 2010; Accepted 9 September 2010

Academic Editor: L. Górniewicz

Retractable complexes are defined in this paper. It is proved that they have the fixed simplex property for simplicial maps. This implies the theorem of Wallace and the theorem of Rival and Nowakowski for finite trees: every simplicial map transforming vertices of a tree into itself has a fixed vertex or a fixed edge. This also implies the Hell and Nešetřil theorem: any endomorphism of a dismantlable graph fixes some clique. Properties of recursively contractible complexes are examined.

1. Preliminaries

We apply some combinatorial methods in the fixed point theory [1]. These methods allow us to extend some known theorems for graphs [2] and to suggest algorithmic procedures finding fixed simplices for simplicial maps defined on some classes of complexes.

By $\mathbf{N}$ we denote the set of natural numbers. Let V be a finite set and $I_n = \{0, \ldots, n\}$, $n \in \mathbf{N}$. By $\mathbb{P}(V)$ we denote the family of all nonempty subsets of V, and $\mathbb{P}_n(V)$ ($\mathbb{P}_{\leqslant n}(V)$) is the family of all subsets of V of the cardinality $n+1$ (at most $n+1$), $n \in \mathbf{N}$. A subset $\mathbb{H}_n \subset \mathbb{P}_{\leqslant n}(V)$ is called a *hypergraph* and its elements are called *edges* (a subset $\mathbb{H}_1 \subset \mathbb{P}_{\leqslant 1}(V)$ is called a *graph* [3]). An element of $\mathbb{P}_n(V)$ is called an *n-simplex* defined on the set V, and a nonempty family $\mathbb{K}_n \subset \mathbb{P}_n(V)$ of n-simplices defined on V is called an *n-complex* defined on the set V.

A *complex generated by an n-simplex S* is the complex $\mathbb{K}_{\leqslant n}(S) = \{V : V \subset S\}$.

Generally, a complex $\mathbb{K}_{\leqslant n}$ (or an *$\leqslant n$-complex* $\mathbb{K}$) defined on the set V is a family consisting of some complexes generated by i-simplices, $i \in I_n$, that is, $\mathbb{K}_{\leqslant n} \subset \mathbb{P}_{\leqslant n}(V)$, and for any simplex $S \in \mathbb{K}_{\leqslant n}$, $\mathbb{K}_{\leqslant n}(S) \subset \mathbb{K}_{\leqslant n}$.

Vertices of a complex are *adjacent* if they are vertices of some of its simplex.

Simplices of a complex are *adjacent* if they have a common vertex.

A *star at a vertex* p in an $\leqslant n$-complex $\mathbb{K}$ is the $\leqslant n$-complex $\mathrm{st}_{\mathbb{K}}(p) = \{S : p \in S \in \mathbb{K}\}$; the vertex p is also called a *center of the star*.

Let $S \in \mathbb{K}_{\leqslant n}$ be an i-simplex of a complex $\mathbb{K}_{\leqslant n}$. Then the i-simplex S is a *single* i-*simplex* if there exists exactly one $(i+1)$-simplex $T \in \mathbb{K}_{\leqslant n}$ such that $S \subset T$, $i \in I_{n-1}$; compare [4, Definition 2.60] of a free face.

Observe that an $\leqslant n$-complex $\mathbb{K}$ is precisely defined by its vertices $V(\mathbb{K}) := \bigcup_{S\in\mathbb{K}} S$ and its maximal simplices $\max \mathbb{K} := \{S : S \in \mathbb{K}$; there is no T such that $S \subset T \in \mathbb{K}$ and $S \neq T\}$.

For complexes $\mathbb{K}_{\leqslant n}$ and $\mathbb{L}_{\leqslant m}$ a map $f : V(\mathbb{K}_{\leqslant n}) \to V(\mathbb{L}_{\leqslant m})$ is called *simplicial* if every simplex of $\mathbb{K}_{\leqslant n}$ is mapped onto some simplex of $\mathbb{L}_{\leqslant m}$.

For a simplex $S = \{p_0, \dots, p_n\} \in \mathbb{K}_{\leqslant n}$ by $\partial S := \{\{p_0, \dots, \widehat{p}_i, \dots, p_n\} : i \in I_n\} \subset \mathbb{K}_{\leqslant n}$ we denote the *boundary of a simplex* S and denotation $\widehat{p}_i$ means that the vertex p_i is omitted.

Notice that for an $(n+1)$-simplex S, ∂S is an n-complex consisting of all n-subsimplices of S.

Let u, v be adjacent vertices of a complex $\mathbb{K}_{\leqslant n}$, and let V be the set of its vertices. A map $r : V \to V \setminus \{u\}$ defined by $r(u) = v$ and $r(x) = x$ for $x \in V \setminus \{u\}$ is called a *retraction* if:

(i) u and v do not belong to the boundary $\partial S \subset \mathbb{K}_{\leqslant n}$ of some simplex $S \notin \mathbb{K}_{\leqslant n}$,

(ii) the complex $\mathbb{K}'_{\leqslant n}$ defined on vertices $V \setminus \{u\}$ with simplices $S \in \mathbb{K}_{\leqslant n}$, such that $u \notin S$ or $S = S' \setminus \{u\} \cup \{v\}$ for some $S' \in \mathbb{K}_{\leqslant n}$ and $S' \ni u$, is the subcomplex of $\mathbb{K}_{\leqslant n}$.

A complex $\mathbb{K}_{\leqslant n}$ is *retractable* if it can be reduced to one vertex by a sequence of retractions.

A *union of complexes* $\mathbb{K}_i$, $i \in I_n$, is the complex $\mathbb{L} = \bigcup_{i\in I_n} \mathbb{K}_i$ with vertices $V(\mathbb{L}) = \bigcup_{i\in I_n} V(\mathbb{K}_i)$.

Analogously, the *intersection of complexes* $\mathbb{K}_i$, $i \in I_n$, is the complex $\mathbb{L} = \bigcap_{i\in I_n} \mathbb{K}_i$ with vertices $V(\mathbb{L}) = \bigcap_{i\in I_n} V(\mathbb{K}_i)$.

2. Fixed Simplex Property

We say that an $\leqslant n$-complex $\mathbb{K}$ has the *fixed simplex property* if for every simplicial map $f : V(\mathbb{K}) \to V(\mathbb{K})$, there exists a simplex $S \in \mathbb{K}$ which is mapped onto itself, that is, $f(S) = S$.

Observe that the following lemma is true.

Lemma 2.1. *For an* n*-simplex* S*, the complex* $\mathbb{K}_{\leqslant n}(S)$ *has the fixed simplex property.*

Proof. Let the complex $\mathbb{K}_{\leqslant n}(S)$ be generated by an n-simplex S, and let $f : S \to S$ be a simplicial map. Notice that $f^{k+1}(S) \subset f^k(S)$, where $k \in \mathbf{N}$ and $f^0(S) := S$.

Because S is a finite set, we have $f(f^i(S)) = f^i(S)$ for some iteration $i \in I_n$, that is, $f^i(S)$ is a fixed simplex. □

Lemma 2.1 can be extended to the following.

Theorem 2.2. *A star has the fixed simplex property.*

Proof. Assume that $\mathrm{st}_{\mathbb{K}}(p)$ is a star at a vertex p in an $\leqslant n$-complex $\mathbb{K}$. It consists of a finite number of simplices. All simplices have the common vertex p: the center of the star. We show that for any simplicial map $f : V(\mathrm{st}_{\mathbb{K}}(p)) \to V(\mathrm{st}_{\mathbb{K}}(p))$ there is a simplex in $\mathrm{st}_{\mathbb{K}}(p)$ which is mapped onto itself. Denote $p_0 := p$.

(1) If $f(p_0) = p_0$, then $\{p_0\}$ is a fixed simplex.

(2) If $f(p_0) \neq p_0$, then denote $p_1 := f(p_0)$. The vertices p_0 and p_1 are adjacent because the center of the star p_0 is adjacent to all vertices.

Observe that all succesive iterations of the vertex p_0 (including p_0) are in one simplex. By Lemma 2.1 there exists a fixed simplex of the map f. More precisely, consider any vertex $p_i = f^i(p_0)$ such that $f(p_i) = p_k$, where $k \in I_i$, $i \in I_n$. Observe that the simplex $\{p_k, \dots, p_i\}$ is the fixed simplex of the map f. □

The method used in the second step of the proof of Theorem 2.2 can be applied to show the following.

Theorem 2.3. *If an $\leqslant n$-complex is retractable, then it has the fixed simplex property.*

Proof. We proceed by induction on the number m of vertices of a retractable complex. The theorem is true for the 0-complex. Let $\mathbb{K}_{\leqslant n}$ be retractable complex with $m + 1$ vertices and let f be a simplicial map defined on $V(\mathbb{K}_{\leqslant n})$. By the definition of a retractable complex $\mathbb{K}_{\leqslant n}$ there exists a retraction r of a vertex u to a vertex v. The complex $\mathbb{K}'_{\leqslant n}$ with m vertices obtained by the retraction r has the fixed simplex property. Of course r is the simplicial map from $V(\mathbb{K}_{\leqslant n})$ to $V(\mathbb{K}'_{\leqslant n})$, indeed all simplices of $\mathbb{K}'_{\leqslant n}$ are mapped onto themselves, simplices containing $\{u\}$ are mapped onto respective simplices containing v, simplices containing u and v are mapped onto simplices of a smaller dimension. Define a simplicial map $f' := r \circ f$ on $V(\mathbb{K}_{\leqslant n})$. Let $S \in \mathbb{K}'_{\leqslant n}$ be a fixed simplex of the map $f'|_{V(\mathbb{K}'_{\leqslant n})}$. If $f(S) \in \mathbb{K}'_{\leqslant n}$, then S is the fixed simplex of f. If not, then there is some vertex $x \in S$ such that $f(x) = u$, $u \notin S$, and $f'(x) = v$, $v \in S$. For all the other vertices $y \in S \setminus \{x\}$ we have $f(y) = f'(y) \in S \setminus \{v\}$. We consider successive iterations of $f(x)$ and show that all $f^i(x)$, $i \in \mathbf{N}$, $f^0(x) := x$, are in some simplex of $\mathbb{K}_{\leqslant n}$. Because f is the simplicial map, the simplex $\{u\} \cup (S \setminus \{v\}) \in \mathbb{K}_{\leqslant n}$. By (i) for any $T \subset S \setminus \{v\}$ the simplex $\{u, v\} \cup T$ belongs to $\mathbb{K}_{\leqslant n}$ because u, v are on some boundary $\partial T' \subset \mathbb{K}_{\leqslant n}$ for some $T' \subset \{u, v\} \cup S$. In particular the simplex $f(x) \cup S$ is in $\mathbb{K}_{\leqslant n}$. Analogously, by induction on k we prove that $\bigcup_{i \in I_k} \{f^i(x)\} \cup S \in \mathbb{K}_{\leqslant n}$, $k \in I_m$. Observe that any vertex adjacent to the vertex u is also adjacent to the vertex v, because of condition (ii) of the retraction r. So all simplices $\{f^i(x), v\}$ belong to $\mathbb{K}_{\leqslant n}$, $i \in \mathbf{N} \setminus \{0, 1\}$. Thus, by Lemma 2.1 applied to the simplex $\bigcup_{i \in \mathbf{N}} \{f^i(x)\} \cup S$, the complex $\mathbb{K}_{\leqslant n}$ has the fixed simplex property. □

3. Recursively Contractible Complexes

A complex is *recursively contractible* [5] if it is generated by an n-simplex or, recursively, it is a union of two recursively contractible complexes whose intersection is also a recursively contractible complex.

A complex is *s-recursively contractible* (or a *tree-like*) if it is generated by n-simplex or, recursively, it is a union of two s-recursively contractible complexes whose intersection is a complex generated by some simplex.

Theorem 3.1. *From an s-recursively retractable complex $\mathbb{K}_{\leqslant n}$, by a sequence of retractions, one can obtain the complex generated by any simplex $S \in \mathbb{K}_{\leqslant n}$.*

Proof. We proceed by induction on the number of recursive steps in the definition of $\mathbb{K}_{\leqslant n}$. Our theorem is obviously true for complexes consisting of two complexes generated by some simplices with a common complex generated by some simplex. Assume that our theorem is true for s-recursively complexes $\mathbb{K}_{\leqslant n}$ and $\mathbb{L}_{\leqslant n}$. Let the complex $\mathbb{K}_{\leqslant n} \cup \mathbb{L}_{\leqslant n}$ be their union and a complex $\mathbb{M}_{\leqslant n}(S)$ (generated by some simplex S) be their intersection. Let $T \in \mathbb{K}_{\leqslant n}$. Then we construct a sequence of retractions from $\mathbb{L}_{\leqslant n}$ to the complex $\mathbb{M}_{\leqslant n}(S)$ and successively in the complex $\mathbb{K}_{\leqslant n}$ to obtain the complex generated by T. □

Corollary 3.2. *Every s-recursively contractible complex $\mathbb{K}_{\leqslant n}$ is retractable.*

Now from Corollary 3.2 and Theorem 2.3 we have the following.

Corollary 3.3. *If an $\leqslant n$-complex $\mathbb{K}$ is s-recursively contractible, then it has the fixed simplex property.*

Notice that the recursive contractibility of complexes is not equivalent to the topological contractibility (see Figure 1).

Theorem 3.4. *Any triangulation of the dunce cap is not recursively contractible.*

Proof. Let an $\leqslant 2$-complex $\mathbb{K}$ be a triangulation of the dunce cap. Assume that $\mathbb{K}$ is recursively contractible. Then it can be represented as a union of two recursively contractible $\leqslant 2$-complexes $\mathbb{A}$ and $\mathbb{B}$ such that their intersection $\mathbb{C}$ is also a recursively contractible complex. Each of complexes $\mathbb{A}$ and $\mathbb{B}$ must contain at least one 2-simplex which does not belong to $\mathbb{C}$. Let us remove all 2-simplices, 1-simplices and 0-simplices of $\mathbb{A}$ and $\mathbb{B}$ which do not belong to $\mathbb{C}$, respectively. The remaining simplices compose a complex $\mathbb{C}$. We successively remove all single 1-simplices and respective 2-complexes of $\mathbb{C}$. Observe that the remaining part of $\mathbb{C}$ contains a 1-dimensional cycle and it cannot be recursively contractible. □

4. Graph Complexes

Now we present some applications to the graph theory.

A graph is represented by an $\leqslant 1$-complex. A vertex of a graph is considered also as a 0-simplex and an edge is considered as a 1-simplex [7].

A *graph* G is a nonempty finite set $V(G)$, whose elements are called vertices, and a finite set $\mathbb{E}(G) \subset \mathbb{P}_{\leqslant 1}(V(G))$ of unordered pairs of the set $V(G)$ called edges. In case $\mathbb{E}(G) = \mathbb{P}_1(V(G))$ it is called a *clique* or a *complete graph*.

An edge of the form $\{v\} \in \mathbb{P}_0(V(G))$ is called a *loop* in $\mathbb{E}(G)$.

Assumption 4.1. In this paragraph we assume that $\mathbb{P}_0(V(G)) \subset \mathbb{E}(G)$ for every graph G.

A vertex u is a *neighbour* of a vertex v if there is an edge $e = \{u, v\} \in \mathbb{E}(G)$.

A *subgraph* of a graph $G = (V, \mathbb{E})$ is a graph $H = (V_1, \mathbb{E}_1)$, where $V_1 \subset V$ and $\mathbb{E}_1 \subset \mathbb{E}$. In this case we denote $H \Subset G$.

A path $P = (W, \mathbb{F})$ in a graph $G = (V, \mathbb{E})$ is a subgraph $P \Subset G$ with pairwise different vertices $W = \{v_0, v_1, \ldots, v_{k+1}\}$, such that $\{v_i, v_{i+1}\} \in \mathbb{F}$ for $i \in I_k$ and some $k \in \mathbf{N}$. The path P is denoted by $v_0 \cdots v_{k+1}$.

Furthermore, a path $v_0 \cdots v_{k+1} = (W, \mathbb{F})$ is a *cycle* if $\{v_0, v_{k+1}\} \in \mathbb{F}$, $k \in \mathbf{N}$.

A graph is *connected* if every two vertices can be joined by a path.

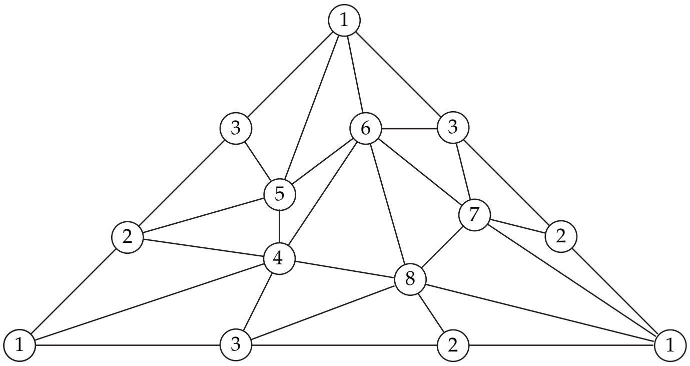

Figure 1: Dunce cap is topologically contractible [6].

A connected graph without cycles is called a *tree*.

Let G_i be a graph, $V(G_i)$ be a set of its vertices and $\mathbb{E}(G_i)$ be a set of its edges. A *union of the graphs* G_i, $i \in I_n$, is a graph $H = \biguplus_{i\in I_n} G_i$, where $V(H) = \bigcup_{i\in I_n} V(G_i)$ and $E(H) = \bigcup_{i\in I_n} \mathbb{E}(G_i)$.

Analogously, the *intersection of the graphs* G_i, $i \in I_n$, is a graph $H = \Cap_{i\in I_n} G_i$, where $V(H) = \bigcap_{i\in I_n} V(G_i)$ and $\mathbb{E}(H) = \bigcap_{i\in I_n} \mathbb{E}(G_i)$.

Let the vertices of a graph G be covered by its maximal cliques (the covering is unique). These cliques generate maximal simplices. The graph G is identified with a graph complex $\mathbb{K}_G$ consisting of these simplices and its subsimplices. There is one to one correspondence between the graph G and the graph complex $\mathbb{K}_G$ defined in that way.

We know that a tree has the fixed edge property [8] or the fixed point property [9]. To formulate this theorem for graph complexes we consider a tree as a union of 1-simplices, where the intersection of some two 1-simplices is a vertex or an empty set.

Fact 1 (see [8, Theorem 3]). A tree with loops has the fixed clique property.

Similarly, we conclude that a union of graphs, having the fixed clique property, with a clique as their intersection also has the fixed clique property. We just consider complexes generated by these graphs with simplices generated by respective cliques.

The fixed clique property is analogous to the fixed simplex property. Simplicial maps on complexes correspond to edge-preserving maps on graphs.

Theorem 4.2. *If each of a finite number of graphs $G_1 = (V_1, \mathbb{E}_1)$, $G_2 = (V_2, \mathbb{E}_2)$, $\ldots, G_k = (V_k, \mathbb{E}_k)$ has the fixed clique property and the intersection of these graphs is a clique, then their union $G_1 \uplus G_2 \uplus \cdots \uplus G_k = (V_1 \cup V_2 \cup \cdots \cup V_k, \mathbb{E}_1 \cup \mathbb{E}_2 \cup \cdots \cup \mathbb{E}_k)$ has also the fixed clique property.*

A graph G which generate the retractable graph complex $\mathbb{K}_G$ is called a *retractable graph*.

A graph G is triangulated [10] if every cycle of the length greater than 3 possesses a chord, that is, an edge joining two nonconsecutive vertices of the cycle.

Let H be a graph and u, v be its vertices such that every neighbour of v (including v) is also a neighbour of u. Then there is a *fold* of the graph H to $H - v$ (a graph obtained from H by removing the vertex v with all edges e such that $v \in e$), mapping v to u and fixing other vertices. A graph is *dismantlable* if it can be reduced, by a sequence of such folds, to one vertex.

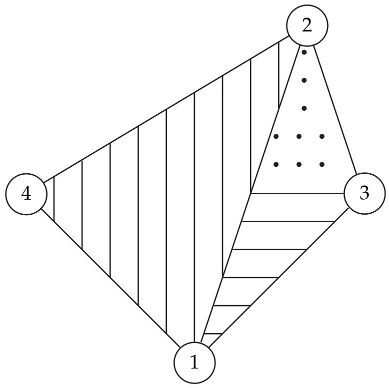

Figure 2: The retractable complex $\mathbb{M}_{\leqslant 2}$ cannot be obtained from the dismantlable graph K_4 (by covering by maximal cliques).

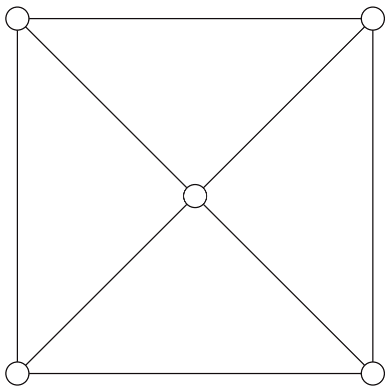

Figure 3: The dismantlable graph which is not triangulated.

Observe that a fold in a dismantlable graph G corresponds to a retraction in the respective graph complex $\mathbb{K}_G$.

Theorem 4.3 (see [2, Theorem 2.65]). *Every endomorphism of a dismantlable graph fixes some clique.*

Fact 2. A dismantlable graph is a retractable graph.

A dismantlable graph always generate a retractable complex. However, there are some retractable complexes which cannot be obtained from the dismantlable graph. Consider a clique K_4 with four vertices. Covering its vertices by simplices we obtain a complex $\mathbb{L}_{\leqslant 3}(K_4)$. This complex contains all edges of the clique K_4 but these edges are also contained in the complex $\mathbb{M}_{\leqslant 2}$ obtained from $\mathbb{L}_{\leqslant 3}(K_4)$ by removing simplices 1234 and 123 (see Figure 2).

Observe that triangulated graphs are dismantlable. One can find some dismantlable graphs which are not triangulated (see Figure 3).

Acknowledgment

The authors are thankful to their referees for their suggestions and comments.

References

[1] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer Monographs in Mathematics, Springer, New York, NY, USA, 2003.
[2] P. Hell and J. Nešetřil, *Graphs and Homomorphisms*, vol. 28 of *Oxford Lecture Series in Mathematics and Its Applications*, Oxford University Press, Oxford, UK, 2004.
[3] C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam, The Netherlands, 1973.
[4] T. Kaczynski, K. Mischaikow, and M. Mrozek, *Computational Homology*, vol. 157 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 2004.
[5] A. Wieczorek, "The Kakutani property and the fixed point property of topological spaces with abstract convexity," *Journal of Mathematical Analysis and Applications*, vol. 168, no. 2, pp. 483–499, 1992.
[6] J. R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, Menlo Park, Calif, USA, 1984.
[7] A. K. Dewdney, *Extensions and generalizations of graph theorems to complexes and hypergraphs*, Ph.D. thesis, Department of Combinatorics and Optimalization, University of Waterloo, August 1974.
[8] R. Nowakowski and I. Rival, "Fixed-edge theorem for graphs with loops," *Journal of Graph Theory*, vol. 3, no. 4, pp. 339–350, 1979.
[9] A. D. Wallace, "A fixed-point theorem for trees," *Bulletin of the American Mathematical Society*, vol. 47, pp. 757–760, 1941.
[10] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York, NY, USA, 1980.