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Research Article

On Uniformly Generalized Lipschitzian Mappings

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We consider another class of generalized Lipschitzian type mappings and utilize the same to prove fixed point theorems for asymptotically regular and uniformly generalized Lipschitzian one-parameter semigroups of self-mappings defined on bounded metric spaces equipped with uniform normal structure which yield corresponding results in respect of semigroup of iterates of a self-mapping as corollaries. Our results also generalize some relevant results due to the work of Lim and Xu (1995), Yao and Zeng (2007), and Soliman (2009) .

1. Introduction

In 1989, Khamsi [1] defined normal structure and uniform normal structure for metric spaces and utilized the same to prove that nonexpansive mappings on a complete bounded metric space (X, d) equipped with uniform normal structure have fixed point property and further satisfy a kind of intersection property which extends results of Maluta [2] to metric spaces. In 1995, Lim and Xu [3] proved a fixed point theorem for uniformly Lipschitzian mappings in metric spaces equipped with both property (P) and uniform normal structure which, in turn, extends a result of Khamsi [1]. This result is indeed the metric space version of a result contained in the work of Casini and Maluta [4]. Recently, Yao and Zeng [5] established a fixed point theorem for an asymptotically regular semigroup of uniformly Lipschitzian mappings in a complete bounded metric space equipped with uniform normal structure and the property $(*)$ which is an improvement over certain relevant results contained in Lim and Xu [3]. Here, it may be pointed out that there exists extensive literature on weak and strong convergence theorems via iterative procedures in respect of semigroups of nonexpansive operators in Banach spaces (e.g., [6–8]).

In this paper, we introduce yet another class of uniformly generalized Lipschitzian one-parameter semigroups of self mappings defined on bounded metric spaces equipped with uniform normal structure and utilize the same to prove our results. Our results are generalizations of certain results due to Yao and Zeng [5] and also Soliman [9].

2. Preliminaries

Throughout this paper, (X, d) (also X) stands for a metric space. In what follows, we recall some relevant definitions and results in respect of uniformly Lipschitzian mappings and uniformly generalized Lipschitzian mappings in metric spaces.

Definition 2.1 (see [4]). A mapping $T : X \to X$ is said to be a Lipschitzian mapping if for each integer $n \geq 1$, there exists a constant $k_n > 0$, such that

$$d(T^n x, T^n y) \leq k_n d(x, y) \quad \forall x, y \in X. \tag{2.1}$$

If $k_n = k$ for all $n \geq 1$, then T is called uniformly Lipschitzian, and if $k_n = 1$ for all $n \geq 1$, then T is called nonexpansive.

In 2001, Jung and Thakur [10] introduced and studied the following class of mappings.

Definition 2.2 (see [10]). A mapping $T : X \to X$ is said to be generalized Lipschitzian mapping (in short G1-Lipschitzian) if

$$\begin{aligned} d(T^n x, T^n y) \leq{}& a_n d(x, y) + b_n (d(x, T^n x) + d(y, T^n y)) \\ &+ c_n (d(x, T^n y) + d(y, T^n x)), \end{aligned} \tag{2.2}$$

for each $x, y \in X$ and $n \geq 1$, where $a_n, b_n,$ and c_n are nonnegative constants such that there exists an integer n_0 such that $b_n + c_n < 1$ for all $n > n_0$. Here it may be pointed out that this class of generalized Lipschitzian mappings is relatively larger than the classes of nonexpansive, asymptotically nonexpansive, Lipschitzian, and uniformly k-Lipschitzian mappings. The earlier mentioned facts can be realized by choosing constants a_n, b_n, and c_n suitably.

Recently, in 2009, Soliman [9] defined another class of generalized Lipschitzian mappings on metric spaces as follows.

Definition 2.3. A mapping $T : X \to X$ is said to be a generalized Lipschitzian (in short G2-Lipschitzian) mapping if for each integer $n \geq 1$ there exists a constant $k_n > 0$ (depending on n) such that

$$d(T^n x, T^n y) \leq k_n \max\left\{ d(x, y), \frac{1}{2} d(x, T^n x), \frac{1}{2} d(y, T^n y) \right\} \tag{2.3}$$

for every $x, y \in X$. If $k_n = k$ for all $n \geq 1$, then T is called uniformly G2-Lipschitzian.

In the following (motivated by Khan and Imdad [11]), we define yet another class of generalized Lipschitzian mappings on metric spaces.

Definition 2.4. A mapping $T : X \to X$ is said to be a generalized Lipschitzian (in short G3-Lipschitzian) mapping if for each integer $n \geq 1$ there exists a constant $k_n > 0$ (depending on n) such that

$$d(T^n x, T^n y) \leq k_n \max\left\{d(x,y), \frac{1}{2}d(x,T^n x), \frac{1}{2}d(y,T^n y), \frac{1}{2}d(x,T^n y), \frac{1}{2}d(y,T^n x)\right\} \tag{2.4}$$

for every $x, y \in X$. If $k_n = k$ for all $n \geq 1$, then T is called uniformly G3-Lipschitzian.

Definition 2.5 (see [12]). A mapping $T : X \to X$ is called asymptotically regular, if

$$\lim_{n\to\infty} d\left(T^{n+1}x, T^n x\right) = 0 \quad \forall\, x \in X. \tag{2.5}$$

Let G be a sub-semigroup of $[0,\infty)$ with addition "+" such that

$$t - h \in G \quad \forall\, t, h \in G \text{ with } t \geq h. \tag{2.6}$$

Notice that the foregoing condition is satisfied if we take $G = [0,\infty)$ or $G = Z^+$, the set of nonnegative integers.

Let $\Im = \{T(t) : t \in G\}$ be a family of self mappings on X. Then $\Im$ is called a (one-parameter) semigroup on X if the following conditions are satisfied:

(i) $T(0)x = x$ for all $x \in X$;

(ii) $T(s+t)x = T(s)(T(t)x)$ for all $s, t \in G$ and $x \in X$;

(iii) for all $x \in X$, a mapping $t \to T(t)x$ from G into X is continuous when G has the relative topology of $[0,\infty)$;

(iv) for each $t \in G$, $T(t) : X \to X$ is continuous.

A semigroup $\Im = \{T(t) : t \in G\}$ on X is said to be asymptotically regular at a point $x \in X$ if

$$\lim_{t\to\infty} d(T(t+h)x, T(t)x) = 0 \quad \forall\, h \in G. \tag{2.7}$$

If $\Im$ is asymptotically regular at each $x \in X$, then $\Im$ is called an asymptotically regular semigroup on X.

Definition 2.6. A semigroup $\Im = \{T(t) : t \in G\}$ of self mappings defined on X is called a uniformly G1-Lipschitzian semigroup if

$$\begin{aligned} d(T(t)x, T(t)y) \leq{}& a(t)d(x,y) + b(t)(d(x,T(t)x) + d(y,T(t)y) \\ &+ c(t)(d(x,T(t)y) + d(y,T(t)x), \end{aligned} \tag{2.8}$$

for each $x, y \in X$, where $a(t), b(t),$ and $c(t)$ are nonnegative constants, $b(t)+c(t) < 1$, $\sup\{a(t) : t \in G\} = a < \infty$, $\sup\{b(t) : t \in G\} = b < \infty$, and $\sup\{c(t) : t \in G\} = c < \infty$ with $b + c < 1$.

The simplest uniformly G1-Lipschitzian semigroup is a semigroup of iterates of a mapping $T : X \to X$ whenever $\sup\{a_n : n \in N\} = a < \infty$, $\sup\{b_n : n \in N\} = b < \infty$ and $\sup\{c_n : n \in N\} = c < \infty$ with $b + c < 1$.

The following definition is introduced by Soliman [9].

Definition 2.7. A semigroup $\Im = \{T(t) : t \in G\}$ of self mappings defined on X is called a uniformly G2-Lipschitzian semigroup if

$$\sup\{k(t) : t \in G\} = k < \infty, \tag{2.9}$$

whenever

$$d\big(T(t)x, T(t)y\big) \leq k(t) \max\left\{ d(x,y), \frac{1}{2}d(x,T(t)x), \frac{1}{2}d\big(y,T(t)y\big) \right\}, \tag{2.10}$$

for each $x, y \in X$ and $\max\{d(x,y), (1/2)d(x,T(t)x), (1/2)d(y,T(t)y)\} \neq 0$.

Definition 2.8. A semigroup $\Im = \{T(t) : t \in G\}$ of self mappings defined on X is called a uniformly G3-Lipschitzian semigroup if

$$\sup\{k(t) : t \in G\} = k < \infty, \tag{2.11}$$

whenever

$$d\big(T(t)x, T(t)y\big) \leq k(t) M(x,y) \tag{2.12}$$

for each $x, y \in X$ and $M(x,y) = \max\{d(x,y), (1/2)d(x,T(t)x), (1/2)d(y,T(t)y), (1/2)d(x,T(t)y), (1/2)d(y,T(t)x)\} \neq 0$.

The simplest uniformly G3-Lipschitzian semigroup is a semigroup of iterates of a mapping $T : X \to X$ with $\sup\{k_n : n \in N\} = k < \infty$.

Here it may be pointed out that the different terms, namely, uniformly k-Lipschitzian semigroups of self mappings, uniformly G1-Lipschitzian semigroups of self mappings, uniformly G2-Lipschitzian semigroups of self mappings, uniformly G3-Lipschitzian semigroups of self mappings, and uniformly k-Lipschitzian semigroups of self mappings are adopted to facilitate the statements of our results.

Remark 2.9. Notice that the class of uniformly G3-Lipschitzian semigroups is relatively larger than the other classes, namely, uniformly G1-Lipschitzian semigroups, uniformly G2-Lipschitzian semigroups, and also uniformly k-Lipschitzian semigroups.

In a metric space (X,d), let F stand for a nonempty family of subsets of X. Following Khamsi [1], we say that F defines a convexity structure on X if F is stable under intersection. Also, we say that F has *Property (R)* if any decreasing sequence $\{C_n\}$ of closed bounded nonempty subsets of X with $C_n \in F$ has a nonvoid intersection. Recall that a subset of X is said to be admissible (cf. [13]) if it can be expressed as an intersection of closed balls. We denote by $A(X)$ the family of all admissible subsets of X. It is obvious that $A(X)$ defines a convexity structure on X. Throughout this paper any other convexity structure F on X is always assumed to contain $A(X)$. Let M be a bounded subset of X whereas $B[x,r]$ stands for the closed ball centered at x with radius r. Following Lim and Xu [3], we will adopt the following conventions and notations:

$$\begin{gathered} r(x, M) = \sup\{d(x, y) : y \in M\} \quad \text{for } x \in X, \\ \delta(M) = \sup\{r(x, M) : x \in M\}, \\ R(M) = \inf\{r(x, M) : x \in M\}. \end{gathered} \tag{2.13}$$

For a bounded subset A of X, we define the admissible hull of A as the intersection of all those admissible subsets of X which contain A and is denoted by $\mathrm{ad}(A)$, that is,

$$\mathrm{ad}(A) = \bigcap\{B : A \subseteq B \subseteq X \text{ with } B \text{ admissible}\}. \tag{2.14}$$

Proposition 2.10 (see [3]). *For a point $x \in X$ and a bounded subset A of X, one has*

$$r(x, \mathrm{ad}(A)) = r(x, A). \tag{2.15}$$

Definition 2.11 (see [1]). A metric space (X, d) is said to have normal (resp., uniform normal) structure if there exists a convexity structure F on X such that $R(A) < \delta(A)$ (resp., $R(A) \leq c\delta(A)$ for some constant $c \in (0, 1)$) for all $A \in F$ which is bounded and consists of more than one point. In this case F is said to be normal (resp., uniformly normal) in X.

We define the normal structure coefficient $\widetilde{N}(X)$ of X (with respect to a given convexity structure F) as the number

$$\sup\left\{\frac{R(A)}{\delta(A)}\right\}, \tag{2.16}$$

where the supremum is taken over all bounded $A \in F$ with $\delta(A) > 0$. It is said that X has uniform normal structure if and only if $\widetilde{N}(X) < 1$.

Khamsi [1] proved the following result which will be very handy in the proof of our main theorem.

Proposition 2.12 (see [1]). *Let X be a complete bounded metric space and let F be a convexity structure of X with uniform normal structure. Then F has the property (R).*

Definition 2.13 (see [3]). A metric space (X, d) is said to have property (P) if given any two bounded sequences $\{x_n\}$ and $\{z_n\}$ in X, one can find some $z \in \bigcap_{n=1}^{\infty} \mathrm{ad}\{z_j : j \geq n\}$ such that $\limsup_{n\to\infty} d(z, x_n) \leq \limsup_{j\to\infty} \limsup_{n\to\infty} d(z_j, x_n)$.

The following lemma due to Lim and Xu [3] will be utilized in proving our results.

Lemma 2.14 (see [3]). *Let (X, d) be a complete bounded metric space equipped with uniform normal structure and the property (P). Then for any sequence $\{x_n\} \in X$ and constant $\overline{c} > \widetilde{N}(X)$, the normal* *structure coefficient with respect to a given convexity structure F, there exists some $z \in X$ satisfying the following properties:*

(i) $\limsup_{n\to\infty} d(z, x_n) \leq \overline{c} \cdot \delta(\{x_n\})$;

(ii) $d(z, y) \leq \lim_{n\to\infty} \sup\, d(x_n, y)$ *for all* $y \in X$.

Definition 2.15 (see [5]). Let (X, d) be a metric space and $\Im = \{T(t) : t \in G\}$ a semigroup of self mappings on X. Let one write the set

$$w(\infty) = \{\{t_n\} : \{t_n\} \subset G,\ t_n \longrightarrow \infty\}. \tag{2.17}$$

Definition 2.16 (see [5]). Let (X, d) be a complete bounded metric space and $\Im = \{T(t) : t \in G\}$ a semigroup of self mappings defined on X. Then $\Im$ is said to have the property $(*)$ if for each $x \in X$ and each $\{t_n\} \in w(\infty)$, the following conditions are satisfied:

(a) the sequence $\{T(t_n)x\}$ is bounded;

(b) for any sequence $\{z_n\}$ in $\mathrm{ad}\{T(t_n)x : n \geq 1\}$ there exists some $z \in \bigcap_{n=1}^{\infty} \mathrm{ad}\{z_j : j \geq n\}$ such that

$$\limsup_{n\to\infty} d(z, T(t_n)x) \leq \limsup_{j\to\infty} \limsup_{n\to\infty} d\left(z_j, T(t_n)x\right). \tag{2.18}$$

Yao and Zeng [5] proved the following result which will be used in the proof of our results.

Lemma 2.17. *Let (X, d) be a complete bounded metric space with uniform normal structure and $\Im = \{T(t) : t \in G\}$ a semigroup of self mappings defined on X equipped with property $(*)$. Then for each $x \in X$, each $\{t_n\} \in w(\infty)$, and any constant $\widetilde{N}(X) < \overline{c}$ (where $\widetilde{N}(X)$ stands for the normal structure coefficient with respect to the given convexity structure F), there exists some $z \in \bigcap_{n=1}^{\infty} \mathrm{ad}\{z_j : j \geq n\}$ satisfying the following properties:*

(I) $\limsup_{n\to\infty} d(z, T(t_n)x) \leq \overline{c} \cdot A(\{T(t_n)x\})$, *where* $A(\{T(t_n)x\}) = \limsup_{n\to\infty} \{d(T(t_i)x, T(t_j)x) : i, j \geq n\}$;

(II) $d(z, y) \leq \limsup_{n\to\infty} d(T(t_n)x, y)$ *for all* $y \in X$.

3. Common Fixed Point Theorems

Our first result is a fixed point theorem for uniformly G1-Lipschitzian semigroups of self mappings defined on bounded metric spaces with uniform normal structure.

Theorem 3.1. *Let (X, d) be a complete bounded metric space equipped with uniform normal structure. If $\Im = \{T(t) : t \in G\}$ is an asymptotically regular and uniformly G1-Lipschitzian semigroup of self mappings on X which satisfies the property $(*)$ with $\beta < 1/\sqrt{\widetilde{N}(X)}$, then there exists some $z \in X$ such that $T(t)z = z$ for all $t \in G$.*

Proof. Choose a constant $\overline{c}$ such that $\widetilde{N}(X) < \overline{c} < 1$ and $\beta < 1/\sqrt{\overline{c}}$. We can pick a sequence $\{t_n\} \in w(\infty)$ such that $\{t_{n+1} - t_n\} \in w(\infty)$. Observe that

$$\{d(T(t_j)x, T(t_i)x) : i, j \geq n\} = \{d(T(t_j)x, T(t_i)x) : j > i \geq n\} \cup \{0\} \tag{3.1}$$

for each $n \in N$ and $x \in X$.

Now fix an $x_0 \in X$. Then by Lemma 2.17, we can inductively construct a sequence $\{x_l\}_{l=1}^{\infty} \subset X$ such that $x_{l+1} \in \bigcap_{n=1}^{\infty} \operatorname{ad}\{T(t_i)x_l : i \geq n\}$; for each integer $l \geq 0$,

(III) $\limsup_{n\to\infty} d(T(t_n)x_l, x_{l+1}) \leq \overline{c} \cdot A(\{T(t_n)x_l\})$, where $A(\{T(t_n)x_l\}) = \limsup_{n\to\infty}\{d(T(t_i)x_l, T(t_j)x_l) : i, j \geq n\}$;

(IV) $d(x_{l+1}, y) \leq \limsup_{n\to\infty} d(T(t_n)x_l, y)$ for all $y \in X$.

Let

$$D_l = \limsup_{n\to\infty} d(x_{l+1}, T(t_n)x_l) \quad \forall\, l \geq 0, \qquad h = \overline{c} \cdot \beta < 1. \tag{3.2}$$

Now for each $i > j \geq 1$, using (III) and the asymptotic regularity of T on X, we have

$$\begin{aligned} d(T(t_i)x_l, T(t_j)x_l) &= d(T(t_j)x_l, T(t_j)T(t_i - t_j)x_l) \\ &\leq a(t_j)d(x_l, T(t_i - t_j)x_l) + b(t_j) \\ &\quad \times \left[d(x_l, T(t_j)x_l) + d(T(t_i - t_j)x_l, T(t_j)T(t_i - t_j)x_l)\right] \\ &\quad + c(t_j)\left[d(x_l, T(t_i)x_l) + d(T(t_i - t_j)x_l, T(t_j)x_l)\right], \end{aligned} \tag{3.3}$$

whereas

$$\begin{aligned} d(x_l, T(t_j)x_l) &\leq \limsup_{n\to\infty} d(T(t_n)x_{l-1}, T(t_j)x_l) \\ &\leq \limsup_{n\to\infty} d(T(t_n)x_{l-1}, T(t_n + t_j)x_{l-1}) + \limsup_{n\to\infty} d(T(t_n + t_j)x_{l-1}, T(t_j)x_l) \\ &\leq \limsup_{n\to\infty} d(T(t_j)(T(t_n)x_{l-1}), T(t_j)x_l) \\ &\leq a(t_j)\limsup_{n\to\infty} d(T(t_n)x_{l-1}, x_l) \\ &\quad + b(t_j)\left[\limsup_{n\to\infty} d(T(t_n)x_{l-1}, T(t_n + t_j)x_{l-1}) + \limsup_{n\to\infty} d(x_l, T(t_j)x_l)\right] \\ &\quad + c(t_j)\left[\limsup_{n\to\infty} d(T(t_n)x_{l-1}, T(t_j)x_l) + \limsup_{n\to\infty} d(x_l, T(t_n + t_j)x_{l-1})\right] \\ &\leq a(t_j)D_{l-1} + \left[b(t_j) + c(t_j)\right]d(x_l, T(t_j)x_l) + 2c(t_j)D_{l-1}, \end{aligned} \tag{3.4}$$

or

$$d(x_l, T(t_j)x_l)\left[1 - (b(t_j) + c(t_j))\right] \leq \left[a(t_j) + 2c(t_j)\right]D_{l-1}, \tag{3.5}$$

so that

$$d(x_l, T(t_j)x_l) \le \frac{a(t_j) + 2c(t_j)}{1 - (b(t_j) + c(t_j))} D_{l-1}. \tag{3.6}$$

Similarly, one can also show that

$$\begin{gathered} d(x_l, T(t_i - t_j)x_l) \le \frac{a(t_i - t_j) + 2c(t_i - t_j)}{1 - (b(t_i - t_j) + c(t_i - t_j))} D_{l-1}, \\ d(x_l, T(t_i)x_l) \le \frac{a(t_i) + 2c(t_i)}{1 - (b(t_i) + c(t_i))} D_{l-1}. \end{gathered} \tag{3.7}$$

Now making use of (3.6) and (3.7) in (3.3), we have

$$\begin{aligned} &d(T(t_i)x_l, T(t_j)x_l) \\ &= d(T(t_j)x_l, T(t_j)T(t_i - t_j)x_l) \le a(t_j)\frac{a(t_i - t_j) + 2c(t_i - t_j)}{1 - (b(t_i - t_j) + c(t_i - t_j))} D_{l-1} + [b(t_j) + c(t_j)] \\ &\quad \times \left[\frac{a(t_j) + 2c(t_j)}{1 - (b(t_j) + c(t_j))} D_{l-1} + \frac{a(t_i - t_j) + 2c(t_i - t_j)}{1 - (b(t_i - t_j) + c(t_i - t_j))} D_{l-1} + \frac{a(t_i) + 2c(t_i)}{1 - (b(t_i) + c(t_i))} D_{l-1}\right] \\ &\le \Bigg\{ [a(t_j) + b(t_j) + c(t_j)] \frac{a(t_i - t_j) + 2c(t_i - t_j)}{1 - (b(t_i - t_j) + c(t_i - t_j))} \\ &\qquad + [b(t_j) + c(t_j)] \left[\frac{a(t_j) + 2c(t_j)}{1 - (b(t_j) + c(t_j))} + \frac{a(t_i) + 2c(t_i)}{1 - (b(t_i) + c(t_i))}\right]\Bigg\} D_{l-1}, \end{aligned} \tag{3.8}$$

which implies that for each $n \ge 1$,

$$\begin{aligned} &\sup\{d(T(t_i)x_l, T(t_j)x_l) : i, j \ge n\} \\ &\le \Bigg\{ [\sup\, a(t_j) + \sup\, b(t_j) + \sup\, c(t_j)] \sup \frac{a(t_i - t_j) + 2c(t_i - t_j)}{1 - (b(t_i - t_j) + c(t_i - t_j))} \\ &\qquad + [\sup\, b(t_j) + \sup\, c(t_j)] \left[\sup \frac{a(t_j) + 2c(t_j)}{1 - (b(t_j) + c(t_j))} + \sup \frac{a(t_i) + 2c(t_i)}{1 - (b(t_i) + c(t_i))}\right]\Bigg\} D_{l-1} \\ &\le \Bigg\{ [\sup\, a(t_j) + \sup\, b(t_j) + \sup\, c(t_j)] \frac{\sup\, a(t_i - t_j) + 2\sup\, c(t_i - t_j)}{1 - (\sup\, b(t_i - t_j) + \sup\, c(t_i - t_j))} \\ &\qquad + [\sup\, b(t_j) + \sup\, c(t_j)] \left[\frac{\sup\, a(t_j) + 2\sup\, c(t_j)}{1 - (\sup\, b(t_j) + \sup\, c(t_j))} + \frac{\sup\, a(t_i) + 2\sup\, c(t_i)}{1 - (\sup\, b(t_i) + \sup\, c(t_i))}\right]\Bigg\} D_{l-1}. \end{aligned} \tag{3.9}$$

By taking the limit of both the sides of (3.9) as $n \to \infty$ with each $(i > j \geq n)$, we have

$$\limsup_{n\to\infty}\{d(T(t_i)x_l, T(t_j)x_l) : i > j \geq n\}$$
$$\leq \left\{[a_1 + b_1 + c_1]\frac{a_3 + 2c_3}{1 - (b_3 + c_3)} + [b_1 + c_1]\left[\frac{a_1 + 2c_1}{1 - (b_1 + c_1)} + \frac{a_2 + 2c_2}{1 - (b_2 + c_2)}\right]\right\}D_{l-1} \leq \beta D_{l-1}, \tag{3.10}$$

where $\limsup_{n\to\infty} a(t_j) = a_1$, $\limsup_{n\to\infty} a(t_i) = a_2$, $\limsup_{n\to\infty} a(t_i - t_j) = a_3$, $\limsup_{n\to\infty} b(t_j) = b_1$, $\limsup_{n\to\infty} b(t_i) = b_2$, $\limsup_{n\to\infty} b(t_i - t_j) = b_3$, $\limsup_{n\to\infty} c(t_j) = c_1$, $\limsup_{n\to\infty} c(t_i) = c_2$, $\limsup_{n\to\infty} c(t_i - t_j) = c_3$, and

$$\beta = \left\{[a_1 + b_1 + c_1]\frac{a_3 + 2c_3}{1 - (b_3 + c_3)} + [b_1 + c_1]\left[\frac{a_1 + 2c_1}{1 - (b_1 + c_1)} + \frac{a_2 + 2c_2}{1 - (b_2 + c_2)}\right]\right\}. \tag{3.11}$$

Hence by using (III) and (3.8), we have

$$\begin{aligned} D_l &= \limsup_{n\to\infty} d(x_{l+1}, T(t_n)x_l) \\ &\leq \overline{c}\limsup_{n\to\infty}\{d(T(t_i)x_l, T(t_j)x_l) \\ &\leq \beta\overline{c}D_{l-1} \leq (\overline{c}\beta)^2 D_{l-2} \leq \cdots \leq h^l D_0. \end{aligned} \tag{3.12}$$

Thus $\lim_{l\to\infty} D_l = 0$ and henceforth

$$\begin{aligned} d(x_{l+1}, x_l) &\leq \limsup_{l\to\infty} d(T(t_n)x_l, x_l) \\ &\leq \frac{a(t_n) + 2c(t_n)}{1 - [b(t_n) + c(t_n)]} D_{l-1}. \end{aligned} \tag{3.13}$$

By taking the limit of both the sides of (3.13) as $l \to \infty$, we have

$$\lim_{l\to\infty} d(x_{l+1}, x_l) = 0, \tag{3.14}$$

which shows that $\{x_l\}$ is a Cauchy sequence and is convergent as X is complete. Let $z = \lim_{l\to\infty} x_l$. Then we have

$$\begin{aligned} d(z, T(t_n)z) &= \lim_{l\to\infty}\limsup_{n\to\infty} d(x_l, T(t_n)x_l) \\ &\leq \lim_{l\to\infty} D_{l-1} \leq \lim_{l\to\infty} h^{l-1}D_0 = 0, \end{aligned} \tag{3.15}$$

that is, $\lim_{n\to\infty} d(z, T(t_n)z) = 0$. Hence for each $s \in G$, we deduce that

$$\begin{aligned} d(z, T(s)z) &= \lim_{l\to\infty} d(x_l, T(s)x_l) \\ &\le \lim_{l\to\infty} \limsup_{n\to\infty} d(x_l, T(t_n + s)x_{l-1}) \\ &\le \lim_{l\to\infty} \limsup_{n\to\infty} d(x_l, T(t_n)x_{l-1}) + \lim_{l\to\infty} \limsup_{n\to\infty} d(T(t_n)x_{l-1}, T(t_n + s)x_{l-1}) \\ &\le \lim_{l\to\infty} D_{l-1} \le \lim_{l\to\infty} h^{l-1} D_0 = 0, \end{aligned} \tag{3.16}$$

yielding thereby $d(z, T(s)z) = 0$, that is, $T(s)z = z$ for each $s \in G$. This concludes the proof. □

Our second result is a fixed point theorem for uniformly G3-Lipschitzian semigroups of self mappings defined on bounded metric spaces equipped with uniform normal structure.

Theorem 3.2. *Let (X, d) be a complete bounded metric space equipped with uniform normal structure. If $\Im = \{T(t) : t \in G\}$ is an asymptotically regular and uniformly G3-Lipschitzian semigroup of self mappings defined on X with $\widetilde{k} < 1/\sqrt{\widetilde{N}(X)}$ which also satisfies the property $(*)$, then there exists some $z \in X$ such that $T(t)z = z$ for all $t \in G$.*

Proof. Choose a constant $\bar{c}$ such that $\widetilde{N}(X) < \bar{c} < 1$ and $\widetilde{k} < 1/\sqrt{\bar{c}}$. We can select a sequence $\{t_n\} \in w(\infty)$ such that $\{t_{n+1} - t_n\} \in w(\infty)$ and $\lim_{n\to\infty} k(t_n) = \widetilde{k}$, where $\widetilde{k} > 0$. Observe that

$$\left\{d\left(T\left(t_j\right)x, T\left(t_i\right)x\right) : i, j \ge n\right\} = \left\{d\left(T\left(t_j\right)x, T\left(t_i\right)x\right) : j > i \ge n\right\} \cup \{0\} \tag{3.17}$$

for each $n \in N$ and $x \in X$.

Now fix an $x_0 \in X$. Then by Lemma 2.17, we can inductively construct a sequence $\{x_l\}_{l=1}^{\infty} \subset X$ such that $x_{l+1} \in \bigcap_{n=1}^{\infty} \operatorname{ad}\{T(t_i)x_l : i \ge n\}$; for each integer $l \ge 0$,

(III) $\limsup_{n\to\infty} d(T(t_n)x_l, x_{l+1}) \le \bar{c} \cdot A(\{T(t_n)x_l\})$, where $A(\{T(t_n)x_l\}) = \limsup_{n\to\infty}\{d(T(t_i)x_l, T(t_j)x_l) : i, j \ge n\}$;

(IV) $d(x_{l+1}, y) \le \limsup_{n\to\infty} d(T(t_n)x_l, y)$ for all $y \in X$.

Let

$$D_l = \limsup_{n\to\infty} d(x_{l+1}, T(t_n)x_l) \quad \forall\, l \ge 0, \qquad h = \bar{c} \cdot \widetilde{k} < 1. \tag{3.18}$$

Observe that for each $i > j \geq 1$, using (IV) and the asymptotic regularity of $\mathfrak{I}$ on X, we have

$$\begin{aligned}
d(x_l, T(t_i - t_j)x_l) &\leq \limsup_{n\to\infty} d(x_l, T(t_n + t_i - t_j)x_{l-1}) \\
&\leq \limsup_{n\to\infty} d(x_l, T(t_n)x_{l-1}) + \limsup_{n\to\infty} d(T(t_n)x_{l-1}, T(t_n + t_i - t_j)x_{l-1}) \\
&\leq D_{l-1}, \\
d(x_l, T(t_j)x_l) &\leq \limsup_{n\to\infty} d(x_l, T(t_n + t_j)x_{l-1}) \\
&\leq \limsup_{n\to\infty} d(x_l, T(t_n)x_{l-1}) + \limsup_{n\to\infty} d(T(t_n)x_{l-1}, T(t_n + t_j)x_{l-1}) \\
&\leq D_{l-1}, \\
d(T(t_i - t_j)x_l, T(t_i)x_l) &\leq \limsup_{n\to\infty} d(T(t_n + t_i - t_j)x_{l-1}, T(t_i)x_l) \\
&\leq \limsup_{n\to\infty} d(T(t_n + t_i - t_j)x_{l-1}, T(t_n)x_{l-1}) + \limsup_{n\to\infty} d(T(t_n)x_{l-1}, x_l) \\
&\quad + d(x_l, T(t_i)x_l) \leq 2D_{l-1},
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
d(T(t_i)x_l, T(t_j)x_l) &= d(T(t_j)x_l, T(t_j)T(t_i - t_j)x_l) \\
&\leq k(t_j)\max\Big\{ d(x_l, T(t_i - t_j)x_l), \frac{1}{2}d(x_l, T(t_j)x_l), \\
&\qquad \frac{1}{2}d(T(t_i - t_j)x_l, T(t_i)x_l), \frac{1}{2}d(x_l, T(t_i)x_l), \\
&\qquad \frac{1}{2}d(T(t_i - t_j)x_l, T(t_j)x_l)\Big\}.
\end{aligned} \tag{3.20}$$

Now making use of (3.19) in (3.20), we get

$$d(T(t_i)x_l, T(t_j)x_l) \leq k(t_j)\max\left\{ D_{l-1}, \frac{1}{2}D_{l-1}, D_{l-1}, \frac{1}{2}D_{l-1}, D_{l-1} \right\} = k(t_j)D_{l-1}, \tag{3.21}$$

which implies that for each $n \geq 1$,

$$\begin{aligned}
\sup\{d(T(t_i)x_l, T(t_j)x_l) : i, j \geq n\} &\leq \sup\{k(t_j)D_{l-1} : i > j \geq n\} \\
&\leq D_{l-1} \cdot \sup\{k(t_j) : j \geq n\}.
\end{aligned} \tag{3.22}$$

Hence by using (III) and (3.22), we have

$$\begin{aligned} D_l &= \limsup_{n\to\infty} d(x_{l+1}, T(t_n)x_l) \le \overline{c}A(\{T(t_n)x_l\}) \le \overline{c}\limsup_{n\to\infty}\{d(T(t_i)x_l, T(t_j)x_l) : i,j \ge n\} \\ &\le \overline{c}\cdot D_{l-1}\cdot \limsup_{n\to\infty} k(t_n) \\ &\le \overline{c}\cdot \lim_{n\to\infty} k(t_n)\cdot D_{l-1} = \overline{c}\cdot\widetilde{k}\cdot D_{l-1} = hD_{l-1} \le h^2 D_{l-2} \le \cdots \le h^l D_0. \end{aligned} \tag{3.23}$$

Hence by the asymptotic regularity of $\mathfrak{I}$ on X, we have for each integer $n \ge 1$,

$$\begin{aligned} d(x_{l+1}, x_l) &\le \limsup_{n\to\infty} d(T(t_n)x_l, x_l) \\ &\le \limsup_{n\to\infty}\limsup_{m\to\infty} d(x_l, T(t_m+t_n)x_{l-1}) \\ &\le \limsup_{m\to\infty} d(x_l, T(t_m)x_{l-1}) + \limsup_{n\to\infty}\limsup_{m\to\infty} d(T(t_m)x_{l-1}, T(t_m+t_n)x_{l-1}) \\ &\le D_{l-1}. \end{aligned} \tag{3.24}$$

It follows from (3.23) that

$$d(x_{l+1}, x_l) \le D_{l-1} \le h^{l-1}D_0. \tag{3.25}$$

Thus, we have $\sum_{l=0}^{\infty} d(x_{l+1}, x_l) \le D_0 \sum_{l=0}^{\infty} h^{l-1} < \infty$. Consequently $\{x_l\}$ is Cauchy and hence convergent as X is complete. Let $z = \lim_{l\to\infty} x_l$. Then we have

$$\begin{aligned} \limsup_{n\to\infty} d(z, T(t_n)z) &= \lim_{l\to\infty}\limsup_{n\to\infty} d(x_l, T(t_n)x_l) \\ &\le \lim_{l\to\infty} D_{l-1} \le \lim_{l\to\infty} h^{l-1}D_0 = 0, \end{aligned} \tag{3.26}$$

that is, $\lim_{n\to\infty} d(z, T(t_n)z) = 0$. Hence for each $s \in G$, we deduce that

$$\begin{aligned} d(z, T(s)z) &= \lim_{l\to\infty} d(x_l, T(s)x_l) \le \lim_{l\to\infty}\limsup_{n\to\infty} d(x_l, T(t_n+s)x_{l-l}) \\ &\le \lim_{l\to\infty}\limsup_{n\to\infty} d(x_l, T(t_n)x_{l-l}) + \lim_{l\to\infty}\limsup_{n\to\infty} d(T(t_n)x_{l-1}, T(t_n+s)x_{l-l}) \\ &\le \lim_{l\to\infty} D_{l-1} \le \lim_{l\to\infty} h^{l-1}D_0 = 0. \end{aligned} \tag{3.27}$$

Then we have $d(z, T(s)z) = 0$, that is, $T(s)z = z$ for each $s \in G$. □

Since the class of uniformly k-Lipschitzian semigroups of self mappings is contained in the class of uniformly G2-Lipschitzian semigroups of self mappings, therefore Theorem 3.1 yields the following.

Corollary 3.3 (see [5]). *Let (X, d) be a complete bounded metric space with uniform normal structure. If $\Im = \{T(t) : t \in G\}$ is an asymptotically regular and uniformly k-Lipschitzian semigroup of self mappings defined on X equipped with the property $(*)$ which also satisfies*

$$\left(\liminf_{t\to\infty} k(t)\right) \cdot \left(\limsup_{t\to\infty} k(t)\right) < \widetilde{N}(X)^{-1/2}, \tag{3.28}$$

then there exists some $z \in X$ such that $T(t)z = z$ for all $t \in G$.

Again, as the class of uniformly G3-Lipschitzian semigroups is larger than the class of uniformly G2-Lipschitzian semigroups, therefore using Theorem 3.2, one immediately derives the following result due to Soliman [9].

Corollary 3.4 (see [9]). *Let (X, d) be a complete bounded metric space with uniform normal structure. If $\Im = \{T(t) : t \in G\}$ is an asymptotically regular and uniformly G2-Lipschitzian semigroup of self mappings defined on X with $\widetilde{k} < 1/\sqrt{\widetilde{N}(X)}$ which also satisfies the property $(*)$. Then there exists some $z \in X$ such that $T(t)z = z$ for all $t \in G$.*

If one replaces the respective one parameter semigroups of generalized Lipschitzian mappings in Theorems 3.1 and 3.2 with respective semigroups of iterates of generalized Lipschitzian mappings, then one can immediately derive the following two corollaries.

Corollary 3.5. *Let (X, d) be a complete bounded metric space equipped with uniform normal structure and the property (P). If $T : X \to X$ is a self-mapping whose set of iterates is an asymptotically regular semigroup of G1-Lipschitzian mappings satisfying the condition* (2.2)*, then there exists some $z \in X$ such that $Tz = z$.*

Corollary 3.6. *Let (X, d) be a complete bounded metric space equipped with uniform normal structure and the property (P). If $T : X \to X$ is a self-mapping whose set of iterates is an asymptotically regular semigroup of G3-Lipschitzian mappings satisfying the condition* (2.4)*, then there exists some $z \in X$ such that $Tz = z$.*

Remark 3.7. It will be interesting to establish Theorems 3.1 and 3.2 for left reversible semigroup of self mappings defined on a complete bounded metric space equipped with uniform normal structure following the lines of Holmes and Lau [14], Lau and Takahashi [15], and Lau [16].

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References

[1] M. A. Khamsi, "On metric spaces with uniform normal structure," *Proceedings of the American Mathematical Society*, vol. 106, no. 3, pp. 723–726, 1989.

[2] E. Maluta, "Uniformly normal structure and related coefficients," *Pacific Journal of Mathematics*, vol. 111, no. 2, pp. 357–369, 1984.

[3] T.-C. Lim and H. K. Xu, "Uniformly Lipschitzian mappings in metric spaces with uniform normal structure," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 25, no. 11, pp. 1231–1235, 1995.

[4] E. Casini and E. Maluta, "Fixed points of uniformly Lipschitzian mappings in spaces with uniformly normal structure," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 9, no. 1, pp. 103–108, 1985.

[5] J.-C. Yao and L.-C. Zeng, "Fixed point theorem for asymptotically regular semigroups in metric spaces with uniform normal structure," *Journal of Nonlinear and Convex Analysis*, vol. 8, no. 1, pp. 153–163, 2007.

[6] S. Atsushiba and W. Takahashi, "Weak and strong convergence theorems for nonexpansive semigroups in Banach spaces," *Fixed Point Theory and Applications*, vol. 2005, no. 3, pp. 343–354, 2005.

[7] H. He and R. Chen, "Strong convergence theorems of the CQ method for nonexpansive semigroups," *Fixed Point Theory and Applications*, vol. 2007, Article ID 59735, 8 pages, 2007.

[8] S. Saejung, "Strong convergence theorems for nonexpansive semigroups without Bochner integrals," *Fixed Point Theory and Applications*, vol. 2008, Article ID 745010, 7 pages, 2008.

[9] A. H. Soliman, "A fixed point theorem for semigroups on metric spaces with uniform normal structure," *Scientiae Mathematicae Japonicae*, vol. 69, no. 3, pp. 323–328, 2009.

[10] J. S. Jung and B. S. Thakur, "Fixed point theorems for generalized Lipschitzian semigroups," *International Journal of Mathematics and Mathematical Sciences*, vol. 28, no. 1, pp. 41–50, 2001.

[11] M. S. Khan and M. Imdad, "Fixed points of certain involutions in Banach spaces," *Journal of the Australian Mathematical Society. Series A*, vol. 37, no. 2, pp. 169–177, 1984.

[12] W. A. Kirk, "A fixed point theorem for mappings which do not increase distances," *The American Mathematical Monthly*, vol. 72, pp. 1004–1006, 1965.

[13] N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, vol. 7 of *Pure and Applied Mathematics*, Interscience, New York, NY, USA, 1958.

[14] R. D. Holmes and A. T. Lau, "Non-expansive actions of topological semigroups and fixed points," *Journal of the London Mathematical Society*, vol. 5, pp. 330–336, 1972.

[15] A. T. Lau and W. Takahashi, "Invariant submeans and semigroups of nonexpansive mappings on Banach spaces with normal structure," *Journal of Functional Analysis*, vol. 142, no. 1, pp. 79–88, 1996.

[16] A. T. Lau, "Invariant means on almost periodic functions and fixed point properties," *The Rocky Mountain Journal of Mathematics*, vol. 3, no. 1, pp. 69–75, 1973.