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Research Article

Strong Convergence for Mixed Equilibrium Problems of Infinitely Nonexpansive Mappings

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We introduce a new iterative scheme for finding a common element of infinitely nonexpansive mappings, the set of solutions of a mixed equilibrium problems, and the set of solutions of the variational inequality for an α-inverse-strongly monotone mapping in a Hilbert Space. Then, the strong converge theorem is proved under some parameter controlling conditions. The results of this paper extend and improve the results of Jing Zhao and Songnian He(2009) and many others. Using this theorem, we obtain some interesting corollaries.

1. Introduction

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. And let C be a nonempty closed convex subset of H. Let $\varphi : C \to \mathbb{R}$ be a real-valued function and let $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction, that is, $\Theta(u,u) = 0$ for each $u \in C$. Ceng and Yao [1] considered the following mixed equilibrium problem.

Find $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{MEP}(\Theta,\varphi)$. It is easy to see that x^* is the solution of problem (1.1) and $x^* \in \operatorname{dom}\varphi = \{x \in \varphi(x) < +\infty\}$. In particular, if $\varphi \equiv 0$, the mixed equilibrium problem (1.1) reduced to the equilibrium problem.

Find $x^* \in C$ such that

$$\Theta(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $\mathrm{EP}(\Theta)$. If $\varphi \equiv 0$ and $\Theta(x, y) = \langle Ax, y - x\rangle$ for all $x, y \in C$, where A is a mapping from C to H, then the mixed equilibrium problem (1.1) becomes the following variational inequality.

Find $x^* \in C$ such that

$$\langle Ax^*, y - x^*\rangle, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $\mathrm{VI}(A, C)$.

The variational inequality and the mixed equilibrium problems which include fixed point problems, optimization problems, variational inequality problems have been extensively studied in literature. See, for example, [2–8].

In 1997, Combettes and Hirstoaga [9] introduced an iterative method for finding the best approximation to the initial data and proved a strong convergence theorem. Subsequently, Takahashi and Takahashi [7] introduced another iterative scheme for finding a common element of $\mathrm{EP}(\Theta)$ and the set of fixed points of nonexpansive mappings. Furthermore,Yao et al. [8, 10] introduced an iterative scheme for finding a common element of $\mathrm{EP}(\Theta)$ and the set of fixed points of finitely (infinitely) nonexpansive mappings.

Very recently, Ceng and Yao [1] considered a new iterative scheme for finding a common element of $\mathrm{MEP}(\Theta, \varphi)$ and the set of common fixed points of finitely many nonexpansive mappings in a Hilbert space and obtained a strong convergence theorem.

Now, we recall that a mapping $A : C \to H$ is said to be

(i) monotone if $\langle Au - Av, u - \mathrm{v}\rangle \geq 0$, for all $u, v \in C$,

(ii) L-Lipschitz if there exists a constant $L > 0$ such that $\|Au - Av\| \leq L\|u - v\|$, for all $u, v \in C$,

(iii) α-inverse strongly monotone if there exists a positive real number α such that $\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2$, for all $u, v \in C$.

It is obvious that any α-inverse strongly monotone mapping A is monotone and Lipscitz. A mapping $S : C \to C$ is called nonexpansive if $\|Su - Sv\| \leq \|u - v\|$, for all $u, v \in C$. We denote by $F(S) := \{x \in C : Sx = x\}$ the set of fixed point of S.

In 2006, Yao and Yao [11] introduced the following iterative scheme.

Let C be a closed convex subset of a real Hilbert space. Let A be an α-inverse strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap \mathrm{VI}(A, C) \neq \emptyset$. Suppose that $x_1 = u \in C$ and $\{x_n\}$ and $\{y_n\}$ are given by

$$\begin{aligned} y_n &= P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda_n Ay_n), \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequence in $[0, 1]$ and $\{\lambda_n\}$ is a sequence in $[0,2\lambda]$. They proved that the sequence $\{x_n\}$ defined by (1.4) converges strongly to a common element of $F(S) \cap \mathrm{VI}(A, C)$ under some parameter controlling conditions.

Moreover, Plubtieng and Punpaeng [12] introduced an iterative scheme (1.5) for finding a common element of the set of fixed point of nonexpansive mappings, the set of solutions of an equilibrium problems, and the set of solutions of the variational of inequality problem for an α-inverse strongly monotone mapping in a real Hilbert space. Suppose that $x_1 = u \in C$ and $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ are given by

$$\begin{gathered}
\Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = P_C(u_n - \lambda_n A u_n), \\
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda_n A y_n),
\end{gathered} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequence in $[0,1]$, $\{\lambda_n\}$ is a sequence in $[0,2\lambda]$, and $\{r_n\} \subset (0,\infty)$. Under some parameter controlling conditions, they proved that the sequence $\{x_n\}$ defined by (1.5) converges strongly to $P_{F(S)\cap \mathrm{VI}(A,C)\cap \mathrm{EP}(\Theta)} u$.

On the other hand, Yao et al. [8] introduced an iterative scheme (1.7) for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed point of infinitely many nonexpansive mappings in H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself and let $\{t_n\}_{n=1}^{\infty}$ be a sequence of real number in $[0,1]$. For each $n \geq 1$, define a mapping W_n of C into itself as follows:

$$\begin{gathered}
U_{n,n+1} = I, \\
U_{n,n} = t_n T_n U_{n,n+1} + (1 - t_n) I, \\
U_{n,n-1} = t_{n-1} T_{n-1} U_{n,n} + (1 - t_{n-1}) I, \\
\vdots \\
U_{n,k} = t_k T_k U_{n,k+1} + (1 - t_k) I, \\
U_{n,k-1} = t_{k-1} T_{k-1} U_{n,k} + (1 - t_{k-1}) I, \\
\vdots \\
U_{n,2} = t_2 T_2 U_{n,3} + (1 - t_2) I, \\
W_n = U_{n,1} = t_1 T_1 U_{n,2} + (1 - t_1) I.
\end{gathered} \tag{1.6}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $t_n, t_{n-1}, \ldots, t_1$. In [8], given $x_0 \in H$ arbitrarily, the sequences $\{x_n\}$ and $\{u_n\}$ are generated by

$$\begin{gathered}
\Theta(u_n, x) + \frac{1}{r_n}\langle x - u_n, u_n - x_n\rangle \geq 0, \quad \forall x \in C, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n u_n.
\end{gathered} \tag{1.7}$$

They proved that under some parameter controlling conditions, $\{x_n\}$ generated by (1.7) converges strongly to $z \in \cap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(\Theta)$, where $z = P_{\cap_{n=1}^{\infty} F(T_n)\cap \mathrm{EP}(\Theta)} f(z)$.

Subsequently, Ceng and Yao [13] introduced an iterative scheme by the viscosity approximation method:

$$
\begin{aligned}
&\Theta(u_n, x) + \frac{1}{r_n}\langle x - u_n, u_n - x_n\rangle \geq 0, \quad \forall x \in C,\\
&y_n = (1 - \gamma_n)x_n + \gamma_n W_n u_n,\\
&x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n f(y_n) + \beta_n W_n y_n,
\end{aligned}
\tag{1.8}
$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in (0,1) such that $\alpha_n + \beta_n \leq 1$. Under some parameter controlling conditions, they proved that the sequence $\{x_n\}$ defined by (1.8) converges strongly to $z \in \cap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(\Theta)$, where $z = P_{\cap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(\Theta)} f(z)$.

Recently, Zhao and He [14] introduced the following iterative process.

Suppose that $x_1 = u \in C$,

$$
\begin{aligned}
&\Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,\\
&y_n = s_n P_C(u_n - \lambda_n A u_n) + (1 - s_n)x_n,\\
&x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n W_n\left(P_C\left(y_n - \lambda_n A y_n\right)\right),
\end{aligned}
\tag{1.9}
$$

where $\{s_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\} \in [0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Under some parameter controlling conditions, they proved that the sequence $\{x_n\}$ defined by (1.9) converges strongly to $z \in \cap_{i=1}^{\infty} F(T_i) \cap \mathrm{VI}(A, C) \cap \mathrm{EP}(\Theta)$, where $z = P_{\cap_{i=1}^{\infty} F(T_i) \cap \mathrm{VI}(A,C) \cap \mathrm{EP}(\Theta)} u$.

Motivated by the ongoing research in this field, in this paper we suggest and analyze an iterative scheme for finding a common element of the set of fixed point of infinitely nonexpansive mappings, the set of solutions of an equilibrium problem and the set of solutions of the variational of inequality problem for an α-inverse strongly monotone mapping in a real Hilbert space. Under some appropriate conditions imposed on the parameters, we prove another strong convergence theorem and show that the approximate solution converges to a unique solution of some variational inequality which is the optimality condition for the minimization problem. The results of this paper extend and improve the results of Zhao and He [14] and many others. For some related works, we refer the readers to [15–22] and the references therein.

2. Preliminaries

Let H be a real Hilbert space and let C be a closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that

$$
\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C.
\tag{2.1}
$$

P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive mapping and satisfies

$$
\langle x - y, P_C x - P_C y\rangle \geq \left\|P_C x - P_C y\right\|^2, \quad \forall x, y \in H.
\tag{2.2}
$$

Moreover, P_C is characterized by the following properties: $P_c x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0,$$
$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H,\ y \in C. \tag{2.3}$$

It is clear that $u \in \mathrm{VI}(A, C) \Leftrightarrow u = P_C(u - \lambda A u), \lambda > 0$.

A space X is said to satisfy Opials condition if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \quad \forall y \in X, y \neq x. \tag{2.4}$$

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1 (see [23]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n \to \infty} \beta_n \leqslant \limsup_{n \to \infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n$ for all integer $n \geq 1$ and $\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leqslant 0$. Then $\lim_{n \to \infty} \|y_n - x_n\| = 0$.*

Lemma 2.2 (see [24]). *Let H be a real Hilbert space, let C be a closed convex subset of H, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $(I - T) x_n$ converge strongly to y, then $(I - T) x = y$.*

Lemma 2.3 (see [25]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \delta_n, \quad n \geq 0, \tag{2.5}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\lim_{n \to \infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(2) $\limsup_{n \to \infty} (\delta_n / \alpha_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

In this paper, for solving the mixed equilibrium problem, let us give the following assumptions for a bifunction Θ, φ and the set C:

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \leq 0$ for any $x, y \in C$;

(A3) Θ is upper-hemicontinuous, that is, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta\big(tz + (1 - t)x, y\big) \leq \Theta(x, y); \tag{2.6}$$

(A4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y) + \varphi(y_x) + \frac{1}{r_n}\langle y_x - z, z - x\rangle < \varphi(z), \tag{2.7}$$

(B2) C is a bounded set.

By a similar argument as in the proof of Lemma 2.3 in [26], we have the following result.

Lemma 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ be a bifunction from $C \times C \to \mathbb{R}$ that satisfies (A1)–(A4) and let $\varphi : C \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : \Theta(z, y) + \varphi(y) + \frac{1}{r}\langle (y - z, z - x)\rangle \geq \varphi(z), \forall y \in C \right\} \tag{2.8}$$

for all $x \in H$. Then, the following conditions hold:

(1) *for each $x \in H, T_r(x) \neq \emptyset$;*

(2) *T_r is single-valued;*

(3) *T_r is firmly nonexpansive, that is, for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*

(4) *$F(T_r) = \mathrm{MEP}(\Theta, \varphi)$;*

(5) *$\mathrm{MEP}(\Theta, \varphi)$ is closed and convex.*

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself, where C is a nonempty closed convex subset of a real Hilbert space H. Given a sequence $\{t_n\}_{n=1}^{\infty}$ in $[0,1]$, we define a sequence $\{W_n\}_{n=1}^{\infty}$ of self-mappings on C by (1.6). Then We have the following result.

Lemma 2.5 (see [27]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and let $\{t_n\}$ be a sequence in $(0, b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \geq 1$, $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 2.6 (see [8]). It can be shown from Lemma 2.5 that if D is a nonempty bounded subset of C, then for $\epsilon > 0$, there exists $n_0 \geq k$ such that for all $n > n_0$, $\sup_{x\in D}\|U_{n,k}x - U_k x\| \leq \epsilon$, where $U_k x = \lim_{n\to\infty} U_{n,k}x$.

Remark 2.7 (see [8]). Using Lemma 2.5, we define a mapping $W : C \to C$ as follows: $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, for all $x \in C$. W is called the W-mapping generated by $T_1, T_2, \ldots$ and $t_1, t_2, \ldots$.

Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive.

Indeed, for all $x, y \in C$, $\|W_x - W_y\| = \lim_{n\to\infty}\|W_n x - W_n y\| \leq \|x - y\|$.

If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \geq 0\}$. Hence it is clear from Remark 2.6 that for any arbitrary $\epsilon > 0$, there exists $n_0 \geq 1$ such that for all $n > n_0$, $\|W_n x_n - W x_n\| = \|U_{n,1}x_n - U_1 x_n\| \leq \sup_{x\in D}\|U_{n,1}x - U_1 x\| < \epsilon$.

This implies that $\lim_{n\to\infty}\|W_n x_n - W x_n\| = 0$.

Lemma 2.8 (see [27]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and let $\{t_n\}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Then $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

3. Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous and convex function. Let Θ be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4), let A be an α-inverse-strongly monotone mapping of C into H, and let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mapping on C such that $\cap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(A,C) \cap \mathrm{MEP}(\Theta,\varphi) \neq \emptyset$. Suppose that $\{s_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$,$\{\lambda_n\}$ is a sequence in $[0,2\alpha]$ such that $\lambda_n \in [a,b]$ for some a,b with $0 < a < b < 2\alpha$, and $\{r_n\} \subset (0,\infty)$ is a real sequence. Suppose that the following conditions are satisfied:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(iv) $0 < \liminf_{n\to\infty}s_n \le \limsup_{n\to\infty}s_n < 1/2$ *and* $\lim_{n\to\infty}|s_{n+1} - s_n| = 0$,

(v) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$,

(vi) $\liminf_{n\to\infty}r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Let f be a contraction of C into itself with coefficient $\beta \in (0,1)$. Assume that either (B1) or (B2) holds. Let the sequences $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be generated by, $x_1 \in C$ and

$$\begin{gathered}
\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\
y_n = s_n P_C(u_n - \lambda_n A u_n) + (1 - s_n)x_n, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n\left(P_C\left(y_n - \lambda_n A y_n\right)\right),
\end{gathered} \tag{3.1}$$

for all $n \in \mathbb{N}$, where W_n is defined by (1.6) and $\{t_n\}$ is a sequence in $(0,b]$, for some $b \in (0,1)$. Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \cap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(A,C) \cap \mathrm{MEP}(\Theta,\varphi)$, where $x^* = P_{\cap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(A,C) \cap \mathrm{MEP}(\Theta,\varphi)} f(x^*)$.

Proof. For any $x, y \in C$ and $\lambda_n \in [a,b] \subset (0,2\alpha)$, we note that

$$\begin{aligned}
\left\|(I - \lambda_n A)x - (I - \lambda_n A)y\right\|^2 &= \left\|x - y - \lambda_n(Ax - Ay)\right\|^2 \\
&= \left\|x - y\right\|^2 - 2\lambda_n\langle x - y, Ax - Ay\rangle + \lambda_n^2\left\|Ax - Ay\right\|^2 \\
&\le \left\|x - y\right\|^2 + \lambda_n(\lambda_n - 2\alpha)\left\|Ax - Ay\right\|^2 \\
&\le \left\|x - y\right\|^2,
\end{aligned} \tag{3.2}$$

which implies that $(I - \lambda_n A)$ is nonexpansive.

Let $\{T_{r_n}\}$ be a sequence of mappping defined as in Lemma 2.4 and let $x^* \in \cap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(A,C) \cap \mathrm{MEP}(\Theta,\varphi)$. Then $x^* = W_n x^*$ and $x^* = P_C(x^* - \lambda_n A x^*) = T_{r_n} x^*$. Put $v_n = P_C(y_n - \lambda_n A y_n)$. From (3.2) we have

$$\begin{aligned}
\|v_n - x^*\| &= \|P_C(y_n - \lambda_n A y_n) - P_C(x^* - \lambda_n A x^*)\| \\
&\le \|(y_n - \lambda_n A y_n) - (x^* - \lambda_n A x^*)\| \\
&\le \|y_n - x^*\| \\
&= \|s_n P_C(u_n - \lambda_n A u_n) + (1 - s_n)x_n - s_n P_C(x^* - \lambda_n A x^*) - (1 - s_n)x^*\| \\
&\le s_n \|P_C(u_n - \lambda_n A u_n) - P_C(x^* - \lambda_n A x^*)\| + (1 - s_n)\|x_n - x^*\| \\
&\le s_n \|u_n - x^*\| + (1 - s_n)\|x_n - x^*\| \\
&= s_n \|T_{r_n} x_n - T_{r_n} x^*\| + (1 - s_n)\|x_n - x^*\| \\
&\le s_n \|x_n - x^*\| + (1 - s_n)\|x_n - x^*\| \\
&= \|x_n - x^*\|.
\end{aligned} \tag{3.3}$$

Hence, we obtain that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) - \beta_n x_n - \gamma_n W_n v_n - x^*\| \\
&\le \alpha_n \|f(x_n) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|W_n v_n - x^*\| \\
&\le \alpha_n \|f(x_n) - f(x^*)\| + \alpha_n \|f(x^*) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|v_n - x^*\| \\
&\le \alpha_n \beta \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|x_n - x^*\| \\
&= (1-\beta)\alpha_n \frac{\|f(x^*) - x^*\|}{1-\beta} + \left[1 - (1-\beta)\alpha_n\right]\|x_n - x^*\| \\
&\le \max\left\{ \|x_n - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\beta} \right\} \\
&\le \max\left\{ \|x_0 - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\beta} \right\}.
\end{aligned} \tag{3.4}$$

Therefore $\{x_n\}$ is bounded. Consequently, $\{f(x_n)\}, \{u_n\}, \{y_n\}, \{v_n\}, \{W_n v_n\}, \{A u_n\}$, and $\{A y_n\}$ are also bounded.

Next, we claim that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Indeed, setting $x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$, for all $n \geq 1$, it follows that

$$
\begin{aligned}
z_{n+1} - z_n &= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} W_{n+1} v_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n W_n v_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} W_{n+1} v_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_{n+1} W_{n+1} v_n}{1-\beta_{n+1}} \\
&\quad + \frac{\gamma_{n+1} W_{n+1} v_n}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n W_n v_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1} f(x_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n)}{1-\beta_n} + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (W_{n+1} v_{n+1} - W_{n+1} v_n) \\
&\quad + \frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}} W_{n+1} v_n - \frac{1-\beta_n - \alpha_n}{1-\beta_n} W_n v_n \\
&= \frac{\alpha_{n+1} f(x_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n)}{1-\beta_n} + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (W_{n+1} v_{n+1} - W_{n+1} v_n) \\
&\quad + (w_{n+1} v_n - w_n v_n) + \frac{\alpha_n}{1-\beta_n} W_n v_n - \frac{\alpha_{n+1}}{1-\beta_{n+1}} W_{n+1} v_n.
\end{aligned}
\tag{3.5}
$$

Now, we estimate $\|W_{n+1} v_n - W_n v_n\|$ and $\|W_{n+1} v_{n+1} - W_{n+1} v_n\|$.

From the definition of $\{W_n\}$, (1.6), and since $T_i, U_{n,i}$ are nonexpansive, we deduce that, for each $n \geq 1$,

$$
\begin{aligned}
\|W_{n+1} v_n - W_n v_n\| &= \|t_1 T_1 U_{n+1,2} v_n - t_1 T_1 U_{n,2} v_n\| \\
&\leq t_1 \|U_{n+1,2} v_n - U_{n,2} v_n\| \\
&= t_1 \|t_2 T_2 U_{n+1,3} v_n - t_2 T_2 U_{n,3} v_n\| \\
&\leq t_1 t_2 \|U_{n+1,3} v_n - U_{n,3} v_n\| \\
&\ \vdots \\
&\leq \left(\prod_{i=1}^{n} t_i \right) \|U_{n+1,n+1} v_n - U_{n,n+1} v_n\| \\
&\leq M \prod_{i=1}^{n} t_i,
\end{aligned}
\tag{3.6}
$$

for some constant $M > 0$ such that $\sup\{\|U_{n+1,n+1} v_n - U_{n,n+1} v_n\|, n \geq 1\} \leq M$. And we note that

$$\begin{aligned}
\|W_{n+1}v_{n+1} - W_{n+1}v_n\| &\le \|(v_{n+1} - v_n)\| \\
&= \|P_C(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - P_C(y_n - \lambda_n Ay_n)\| \\
&\le \|(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - (y_n - \lambda_n Ay_n)\| \\
&\le \|(I - \lambda_{n+1}A)y_{n+1} - (I - \lambda_{n+1}A)y_n\| + |\lambda_n - \lambda_{n+1}|\|Ay_n\| \\
&\le \|y_{n+1} - y_n\| + |\lambda_n - \lambda_{n+1}|\|Ay_n\|,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|s_{n+1}P_C(u_{n+1} - \lambda_{n+1}Au_{n+1}) + (1 - s_{n+1})x_{n+1} \\
&\qquad -s_nP_C(u_n - \lambda_n Au_n) - (1 - s_n)x_n\| \\
&= \|s_{n+1}P_C(u_{n+1} - \lambda_{n+1}Au_{n+1}) - s_{n+1}P_C(u_n - \lambda_n Au_n) \\
&\qquad + (s_{n+1} - s_n)P_C(u_n - \lambda_n Au_n) + (1 - s_{n+1})x_{n+1} \\
&\qquad -(1 - s_{n+1} + s_{n+1} - s_n)x_n\| \\
&\le s_{n+1}\|(u_{n+1} - \lambda_{n+1}Au_{n+1}) - (u_n - \lambda_n Au_n)\| \\
&\quad + |s_{n+1} - s_n|\|u_n - \lambda_n Au_n\| + (1 - s_{n+1})\|x_{n+1} - x_n\| + |s_{n+1} - s_n|\|x_n\| \\
&\le s_{n+1}\{\|(u_{n+1} - \lambda_{n+1}Au_{n+1}) - (u_n - \lambda_n Au_n)\| \\
&\qquad\quad +|\lambda_n - \lambda_{n+1}|\|Au_n\|\} + |s_{n+1} - s_n|(\|u_n\| + \lambda_n\|Au_n\| + \|x_n\|) \\
&\qquad\quad + (1 - s_{n+1})\|x_{n+1} - x_n\| \\
&\le s_{n+1}\|u_{n+1} - u_n\| + s_{n+1}|\lambda_n - \lambda_{n+1}|\|Au_n\| \\
&\quad + |s_{n+1} - s_n|Q + (1 - s_{n+1})\|x_{n+1} - x_n\|,
\end{aligned} \tag{3.8}$$

where $Q = \sup\{\|u_n\|, \lambda_n\|Au_n\|, \|x_n\| : n \ge 1\}$.

Combining (3.7) and (3.8), we obtain

$$\begin{aligned}
\|v_{n+1} - v_n\| &\le s_{n+1}\|u_{n+1} - u_n\| + s_{n+1}|\lambda_n - \lambda_{n+1}|\|Au_n\| + |s_{n+1} - s_n|Q \\
&\quad + (1 - s_{n+1})\|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}|\|Ay_n\|.
\end{aligned} \tag{3.9}$$

On the other hand, from $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, we note that

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \tag{3.10}$$

$$\Theta(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0, \quad \forall y \in C. \tag{3.11}$$

Putting $y = u_{n+1}$ in (3.10) and $y = u_n$ in (3.11), we have

$$\begin{aligned}\Theta(u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0,\\ \Theta(u_{n+1}, u_n) + \varphi(u_n) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.\end{aligned} \tag{3.12}$$

So, from (A2) we get $\langle u_{n+1} - u_n, (u_n - x_n/r_n) - (u_{n+1} - x_{n+1})/r_{n+1}\rangle \geq 0$.

Hence $\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - (r_n/r_{n+1})(u_{n+1} - x_{n+1})\rangle \geq 0$.

Without loss of generality, we may assume that there exists a real number c such that $r_n > c > 0$, for all $n \geq 1$. Then we get

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1})\right\rangle\\ &\leq \|u_{n+1} - u_n\|\left\{\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\|\right\},\end{aligned} \tag{3.13}$$

and hence

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\|\\ &\leq \|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n|L,\end{aligned} \tag{3.14}$$

where $L = \sup\{\|u_n - x_n\| : n \geq 1\}$. Hence from (3.9) and (3.14), we have

$$\begin{aligned}\|W_{n+1}v_{n+1} - W_{n+1}v_n\| \leq{}& \|x_{n+1} - x_n\| + s_{n+1}\left(\frac{L}{c}|r_{n+1} - r_n| + |\lambda_n - \lambda_{n+1}|\|Au_n\|\right)\\ &+ |s_{n+1} - s_n|Q + |\lambda_n - \lambda_{n+1}|\|Ay_n\|.\end{aligned} \tag{3.15}$$

Combining (3.5), (3.6), and (3.15), we get

$$\begin{aligned}\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq{}& \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\left(\|f(x_{n+1})\| + \|W_{n+1}v_n\|\right) + \frac{\alpha_n}{1 - \beta_n}\left(\|f(x_n)\| + \|W_nv_n\|\right)\\ &+ \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\left\{\|x_{n+1} - x_n\| + s_{n+1}\left(\frac{L}{c}|r_{n+1} - r_n| + |\lambda_n - \lambda_{n+1}|\|Au_n\|\right)\right.\\ &\left.\qquad + |s_{n+1} - s_n|Q + |\lambda_n - \lambda_{n+1}|\|Ay_n\|\right\}\\ &+ M\prod_{i=1}^{n}t_i - \|x_{n+1} - x_n\|\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|f(x_{n+1})\| + \|W_{n+1}v_n\|\right) + \frac{\alpha_n}{1-\beta_n}\left(\|f(x_n)\| + \|W_n v_n\|\right) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\Bigg\{ s_{n+1}\left(\frac{L}{c}|r_{n+1}-r_n| + |\lambda_n - \lambda_{n+1}|\|Au_n\|\right) \\
&\qquad + |s_{n+1}-s_n|Q + |\lambda_n-\lambda_{n+1}|\|Ay_n\| \Bigg\} + M\prod_{i=1}^{n} t_i.
\end{aligned}
\tag{3.16}
$$

It follows from (3.16) and conditions (i)–(vi) and $0 < t_i \le b < 1$, for all $i \ge 1$ that

$$
\limsup_{n\to\infty}\left(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|\right) \le 0. \tag{3.17}
$$

By Lemma 2.1, we have $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Consequently,

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{3.18}
$$

From conditions (iv)–(vi), (3.7), (3.8), (3.14), and (3.18), we also get

$$
\lim_{n\to\infty}\|u_{n+1}-u_n\| = 0, \quad \lim_{n\to\infty}\|y_{n+1}-y_n\| = 0, \quad \lim_{n\to\infty}\|v_{n+1}-v_n\| = 0. \tag{3.19}
$$

Since $\alpha_n + \beta_n + \gamma_n = 1$ and from the definition of $\{x_n\}$, we have $x_{n+1} - x_n = \alpha_n(f(x_n) - x_n) + \gamma_n(W_n v_n - x_n)$. Then we have

$$
\|W_n v_n - x_n\| \le \frac{1}{\gamma_n}\left\{\|x_{n+1}-x_n\| + \alpha_n\|f(x_n)-x_n\|\right\} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.20}
$$

For $x^* \in \cap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(A,C) \cap \mathrm{MEP}(\Theta,\varphi)$, we have

$$
\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}x_n - T_{r_n}x^*\|^2 \\
&\le \langle T_{r_n}x_n - T_{r_n}x^*, x_n - x^*\rangle \\
&= \langle u_n - x^*, x_n - x^*\rangle \\
&= \frac{1}{2}\left(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2\right),
\end{aligned}
\tag{3.21}
$$

and hence $\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - u_n\|^2$.

From (3.3), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \left\|\alpha_n f(x_n)-\beta_n x_n-\gamma_n W_n v_n-x^*\right\|^2 \\
&\le \alpha_n\left\|f(x_n)-x^*\right\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|W_n v_n-x^*\|^2 \\
&\le \alpha_n\left\|f(x_n)-x^*\right\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|v_n-x^*\|^2 \\
&\le \alpha_n\left\|f(x_n)-x^*\right\|^2+\beta_n\|x_n-x^*\|^2 \\
&\quad+\gamma_n\left\{s_n\|x_n-x^*\|^2-s_n\|x_n-u_n\|^2+(1-s_n)\|x_n-x^*\|^2\right\} \\
&\le \alpha_n\left\|f(x_n)-x^*\right\|^2+(\beta_n+\gamma_n)\|x_n-x^*\|^2-\gamma_n s_n\|x_n-u_n\|^2 \\
&\le \alpha_n\left\|f(x_n)-x^*\right\|^2+\|x_n-x^*\|^2-\gamma_n s_n\|x_n-u_n\|^2.
\end{aligned}
\tag{3.22}
$$

That is,

$$
\begin{aligned}
\|x_n-u_n\|^2 &\le \frac{1}{\gamma_n s_n}\left\{\alpha_n\left\|f(x_n)-x^*\right\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\right\} \\
&\le \frac{1}{\gamma_n s_n}\left\{\alpha_n\left\|f(x_n)-x^*\right\|^2+\|x_{n+1}-x_n\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\right\}.
\end{aligned}
\tag{3.23}
$$

From (ii) and (3.18), we obtain

$$
\|x_n-u_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.24}
$$

From (3.2)-(3.3), we get

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \alpha_n\left\|f(x_n)-x^*\right\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|W_n v_n-x^*\|^2 \\
&\le \alpha_n\left\|f(x_n)-x^*\right\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|v_n-x^*\|^2 \\
&\le \alpha_n\left\|f(x_n)-x^*\right\|^2+\beta_n\|x_n-x^*\|^2 \\
&\quad+\gamma_n\left\|(y_n-\lambda_n Ay_n)-(x^*-\lambda_n Ax^*)\right\|^2 \\
&\le \alpha_n\left\|f(x_n)-x^*\right\|^2+\beta_n\|x_n-x^*\|^2 \\
&\quad+\gamma_n\left\{\left\|y_n-x^*\right\|^2+\lambda_n(\lambda_n-2\alpha)\left\|Ay_n-Ax^*\right\|^2\right\} \\
&\le \alpha_n\left\|f(x_n)-x^*\right\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2 \\
&\quad+\gamma_n\lambda_n(\lambda_n-2\alpha)\left\|Ay_n-Ax^*\right\|^2 \\
&\le \alpha_n\left\|f(x_n)-x^*\right\|^2+\|x_n-x^*\|^2+\gamma_n a(b-2\alpha)\left\|Ay_n-Ax^*\right\|^2.
\end{aligned}
\tag{3.25}
$$

Then we get,

$$\begin{aligned}-\gamma_n a(b-2\alpha)\|Ay_n - Ax^*\|^2 &\le \alpha_n\|f(x_n)-x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1}-x^*\|^2 \\ &\le \alpha_n\|f(x_n)-x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1}-x^*\|)(\|x_n - x_{n+1}\|).\end{aligned} \tag{3.26}$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$, we obtain

$$\|Ay_n - Ax^*\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.27}$$

We note that

$$\begin{aligned}\|v_n - x^*\|^2 &= \|P_C(y_n - \lambda_n Ay_n) - P_C(x^* - \lambda_n Ax^*)\|^2 \\ &\le \langle (y_n - \lambda_n Ay_n) - (x^* - \lambda_n Ax^*), v_n - x^*\rangle \\ &= \frac{1}{2}\Big\{ \|(y_n - \lambda_n Ay_n) - (x^* - \lambda_n Ax^*)\|^2 + \|v_n - x^*\|^2 \\ &\quad - \|(y_n - \lambda_n Ay_n) - (x^* - \lambda_n Ax^*) - (v_n - x^*)\|^2 \Big\} \\ &\le \frac{1}{2}\Big\{ \|y_n - x^*\|^2 + \|v_n - x^*\|^2 - \|(y_n - v_n) - \lambda_n(Ay_n - Ax^*)\|^2 \Big\} \\ &= \frac{1}{2}\Big\{ \|y_n - x^*\|^2 + \|v_n - x^*\|^2 - \|y_n - v_n\|^2 \\ &\quad + 2\lambda_n\langle y_n - v_n, Ay_n - Ax^*\rangle - \lambda_n^2\|Ay_n - Ax^*\|^2 \Big\}.\end{aligned} \tag{3.28}$$

Then we derive

$$\begin{aligned}\|v_n - x^*\|^2 &\le \|y_n - x^*\|^2 - \|y_n - v_n\|^2 + 2\lambda_n\langle y_n - v_n, Ay_n - Ax^*\rangle - \lambda_n^2\|Ay_n - Ax^*\|^2 \\ &\le \|x_n - x^*\|^2 - \|y_n - v_n\|^2 + 2\lambda_n\langle y_n - v_n, Ay_n - Ax^*\rangle.\end{aligned} \tag{3.29}$$

Hence

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \alpha_n\|f(x_n)-x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|W_n v_n - x^*\|^2 \\ &\le \alpha_n\|f(x_n)-x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|v_n - x^*\|^2 \\ &\le \alpha_n\|f(x_n)-x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\ &\quad + \gamma_n\Big\{ \|x_n - x^*\|^2 - \|y_n - v_n\|^2 + 2\lambda_n\langle y_n - v_n, Ay_n - Ax^*\rangle \Big\} \\ &\le \alpha_n\|f(x_n)-x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n\|y_n - v_n\|^2 \\ &\quad + 2\gamma_n\lambda_n\|y_n - v_n\|\|Ay_n - Ax^*\|,\end{aligned} \tag{3.30}$$

which imply that

$$\begin{aligned}\gamma_n\|y_n - v_n\|^2 &\le \alpha_n\left\|f(x_n) - x^*\right\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\quad + 2\gamma_n\lambda_n\|y_n - v_n\|\|Ay_n - Ax^*\| \\ &\le \alpha_n\left\|f(x_n) - x^*\right\|^2 + 2\gamma_n\lambda_n\|y_n - v_n\|\|Ay_n - Ax^*\| \\ &\quad + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\|.\end{aligned} \tag{3.31}$$

From condition (ii), (3.18), and (3.27), we get

$$\lim_{n\to\infty}\|y_n - v_n\| = 0. \tag{3.32}$$

Since

$$\begin{aligned}\|u_n - y_n\| &\le \|u_n - x_n\| + \|x_n - y_n\| \\ &= \|u_n - x_n\| + \|s_nP_C(u_n - \lambda_nAu_n) - s_nx_n\| \\ &\le \|u_n - x_n\| + s_n\|u_n - y_n\| + s_n\|v_n - W_nv_n\| + s_n\|W_nv_n - x_n\|,\end{aligned} \tag{3.33}$$

we have

$$\|u_n - y_n\| \le \frac{1}{1-s_n}\|u_n - x_n\| + \frac{s_n}{1-s_n}\|v_n - W_nv_n\| + \frac{s_n}{1-s_n}\|W_nv_n - x_n\|, \tag{3.34}$$

and then we obtain

$$\begin{aligned}\|W_nv_n - v_n\| &\le \|W_nv_n - x_n\| + \|x_n - u_n\| + \|u_n - y_n\| + \|y_n - v_n\| \\ &\le \|W_nv_n - x_n\| + \|x_n - u_n\| + \frac{1}{1-s_n}\|u_n - x_n\| \\ &\quad + \frac{1}{1-s_n}\|v_n - W_nv_n\| + \frac{s_n}{1-s_n}\|W_nv_n - x_n\| + \|y_n - v_n\|.\end{aligned} \tag{3.35}$$

So we get

$$\frac{1-2s_n}{1-s_n}\|W_nv_n - v_n\| \le \frac{1}{1-s_n}\|W_nv_n - x_n\| + \frac{2-s_n}{1-s_n}\|u_n - x_n\| + \|y_n - v_n\|. \tag{3.36}$$

From condition (iv) and (3.20), (3.24), and (3.32), we have $\lim_{n\to\infty}\|W_nv_n - v_n\| = 0$. Moreover, from Remark 2.7 we get $\lim_{n\to\infty}\|Wv_n - v_n\| = 0$.

Next, we show that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \le 0, \tag{3.37}$$

where $x^* = P_{\cap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(A,C) \cap \mathrm{MEP}(\Theta,\varphi)} f(x^*)$. Indeed, we choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, Wv_n - x^* \rangle = \lim_{i\to\infty} \langle f(x^*) - x^*, Wv_{n_i} - x^* \rangle. \tag{3.38}$$

Since $\{v_{n_i}\}$ is bounded, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to z. Without loss of generality, we can assume that $v_{n_i} \rightharpoonup z$.

From $\|Wv_n - v_n\| \to 0$, we obtain $Wv_{n_i} \rightharpoonup z$.

Next, we show that $z \in \cap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(A,C) \cap \mathrm{MEP}(\Theta,\varphi)$.

First, we show that $z \in \mathrm{MEP}(\Theta,\varphi)$. In fact by $u_n = T_{r_n} x_n \in \operatorname{dom}\varphi$, we have

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C. \tag{3.39}$$

From (A2), we also have

$$\varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq \Theta(y, u_n), \quad \forall y \in C, \tag{3.40}$$

and hence

$$\varphi(y) - \varphi(u_n) + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq \Theta(y, u_{n_i}), \quad \forall y \in C. \tag{3.41}$$

From $\|u_n - x_n\| \to 0$, $\|x_n - Wv_n\| \to 0$ and $\|Wv_n - v_n\| \to 0$, we get $u_{n_i} \to z$. It follows from (A4) that $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$ and from the lower semicontinuity of φ that

$$\Theta(y, z) + \varphi(z) - \varphi(y) \leq 0, \quad \forall y \in C. \tag{3.42}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $\Theta(y_t, z) + \varphi(z) - \varphi(y_t) \leq 0$. So, from (A1) and (A4), we have

$$\begin{aligned}
0 &= \Theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&\leq t\Theta(y_t, y) + (1-t)\Theta(y_t, z) + t\varphi(y) + (1-t)\varphi(z) - \varphi(y_t) \\
&\leq t\left[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)\right].
\end{aligned} \tag{3.43}$$

Dividing by t, we have

$$\Theta(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0, \quad \forall y \in C. \tag{3.44}$$

Letting $t \to 0$, it follows from the weakly semicontinuity of φ that

$$\Theta(z,y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C. \tag{3.45}$$

Hence $z \in \mathrm{MEP}(\Theta,\varphi)$.

Second, we show that $z \in F(W) = \cap_{n=1}^{\infty} F(T_n)$. Assume $z \notin F(W)$. Since $u_{n_i} \rightharpoonup z$ and $z \neq Wz$, by Opial's condition, we have

$$\begin{aligned}
\liminf_{i\to\infty}\|u_{n_i} - z\| &< \liminf_{i\to\infty}\|u_{n_i} - Wz\| \\
&\leq \liminf_{i\to\infty}(\|u_{n_i} - Wu_{n_i}\| + \|Wu_{n_i} - Wz\|) \\
&\leq \liminf_{i\to\infty}\|u_{n_i} - z\|,
\end{aligned} \tag{3.46}$$

which derives a contradiction. Thus we have $z \in F(T)$.

Finally, by the same argument in the proof of [28, Theorem 3.1], we can show that $z \in \mathrm{VI}(A,C)$.

Hence $z \in \cap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(A,C) \cap \mathrm{MEP}(\Theta,\varphi)$.

Since $x^* = P_{\cap_{n=1}^{\infty} F(T_n)\cap \mathrm{VI}(A,C)\cap \mathrm{MEP}(\Theta,\varphi)} f(x^*)$ and $\|x_n - Wv_n\| \to 0$, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle &= \limsup_{n\to\infty}\langle f(x^*) - x^*, Wv_n - x^*\rangle \\
&= \lim_{i\to\infty}\langle f(x^*) - x^*, Wv_{n_i} - x^*\rangle \\
&= \langle f(x^*) - x^*, z - x^*\rangle \leq 0.
\end{aligned} \tag{3.47}$$

Therefore, (3.37) holds.

Finally, we show that $x_n \to x^*$. From definition of $\{x_n\}$, we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|\alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n v_n - x^*\right\|^2 \\
&= \left\langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n v_n - x^*, x_{n+1} - x^*\right\rangle \\
&= \alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle + \beta_n\langle x_n - x^*, x_{n+1} - x^*\rangle \\
&\quad + \gamma_n\langle W_n v_n - x^*, x_{n+1} - x^*\rangle \\
&\leq \alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle + \frac{1}{2}\beta_n\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) \\
&\quad + \frac{1}{2}\gamma_n\left(\|v_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) \\
&\leq \alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle + \frac{1}{2}\beta_n\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) \\
&\quad + \frac{1}{2}\gamma_n\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle + \frac{1}{2}(1-\alpha_n)\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \right) \\
&\le \alpha_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle + \frac{1}{2}(1-\alpha_n)\|x_n - x^*\|^2 + \frac{1}{2}\|x_{n+1} - x^*\|^2,
\end{aligned}
\tag{3.48}
$$

which implies that

$$
\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle. \tag{3.49}
$$

By (3.47) and Lemma 2.3, we get that $\{x_n\}$ converges strongly to x^*.
This completes the proof. □

Setting $f(x_n) \equiv u$ and $\varphi = 0$ in Theorem 3.1., we have the following result.

Corollary 3.2 (see [14, Theorem 2.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4), let A be an α-inverse-strongly monotone mapping of C into H, and let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mapping on C such that $\cap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(A,C) \cap \mathrm{EP}(\Theta) \neq \emptyset$. Suppose that $x_1 = u \in C$, $\{s_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$,$\{\lambda_n\}$ is a sequence in $[0,2\alpha]$ such that $\lambda_n \in [a,b]$ for some a,b with $0 < a < b < 2\alpha$ and $\{r_n\} \subset (0,\infty)$ is a real sequence. Suppose that the following conditions are satisfied:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(iv) $0 < \liminf_{n\to\infty}s_n \le \limsup_{n\to\infty}s_n < 1/2$ *and* $\lim_{n\to\infty}|s_{n+1} - s_n| = 0$,

(v) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$,

(vi) $\liminf_{n\to\infty}r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Let the sequence $\{x_n\}$ be generated by,

$$
\begin{gathered}
\Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in C, \\
y_n = s_n P_C(u_n - \lambda_n A u_n) + (1 - s_n)x_n, \\
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n W_n\left(P_C\left(y_n - \lambda_n A y_n\right)\right),
\end{gathered}
\tag{3.50}
$$

for all $n \in \mathbb{N}$, where W_n is defined by (1.6) and $\{t_n\}$ is a sequence in $(0,b]$, for some $b \in (0,1)$. Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \cap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(A,C) \cap \mathrm{EP}(\Theta)$, where $x^* = P_{\cap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(A,C) \cap \mathrm{EP}(\Theta)} u$.

Setting $\varphi = 0$ in Theorem 3.1, we have the following result.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4), let A be an α-inverse-strongly monotone mapping of C into H, and let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mapping on C such that $\cap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(A, C) \cap \mathrm{EP}(\Theta) \neq \emptyset$. Suppose that $\{s_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$,$\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$ such that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$, and $\{r_n\} \subset (0, \infty)$ is a real sequence. Suppose that the following conditions are satisfied:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(iv) $0 < \liminf_{n\to\infty}s_n \leq \limsup_{n\to\infty}s_n < 1/2$ *and* $\lim_{n\to\infty}|s_{n+1} - s_n| = 0$,

(v) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$,

(vi) $\liminf_{n\to\infty}r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Let f be a contraction of C into itself with coefficient $\beta \in (0, 1)$ and let the sequence $\{x_n\}$ be generated by $x_1 \in C$ and

$$\begin{gathered}\Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = s_n P_C(u_n - \lambda_n A u_n) + (1 - s_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n\big(P_C\big(y_n - \lambda_n A y_n\big)\big),\end{gathered} \tag{3.51}$$

for all $n \in \mathbb{N}$, where W_n is defined by (1.6) and $\{t_n\}$ is a sequence in $(0, b]$, for some $b \in (0, 1)$. Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \cap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(A, C) \cap \mathrm{EP}(\Theta)$, where $x^* = P_{\cap_{n=1}^{\infty} F(T_n)\cap \mathrm{VI}(A,C)\cap \mathrm{EP}(\Theta)} f(x^*)$.

By Theorem 3.1, we obtain some interesting strong convergence theorems.

Setting $T_n x = x$ then we have $W_n x = x$ in Theorem 3.1, and we have the following result.

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous and convex function. Let Θ be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4), and let A be an α-inverse-strongly monotone mapping of C into H such that $\mathrm{VI}(A, C) \cap \mathrm{MEP}(\Theta, \varphi) \neq \emptyset$. Suppose that $\{s_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$,$\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$ such that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$ and $\{r_n\} \subset (0, \infty)$ is a real sequence. Suppose that the following conditions are satisfied:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(iv) $0 < \liminf_{n\to\infty}s_n \leq \limsup_{n\to\infty}s_n < 1/2$ *and* $\lim_{n\to\infty}|s_{n+1} - s_n| = 0$,

(v) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$,

(vi) $\liminf_{n\to\infty}r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Let f be a contraction of C into itself with coefficient $\beta \in (0,1)$. Assume that either (B1) or (B2) holds. Then the sequences $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ generated by, $x_1 \in C$ and

$$\begin{gathered}
\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = s_n P_C(u_n - \lambda_n A u_n) + (1 - s_n)x_n, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n\big(P_C(y_n - \lambda_n A y_n)\big)
\end{gathered} \tag{3.52}$$

converge strongly to a point $x^* \in \mathrm{VI}(A,C) \cap \mathrm{MEP}(\Theta,\varphi)$, where $x^* = P_{\mathrm{VI}(A,C)\cap\mathrm{MEP}(\Theta,\varphi)} f(x^*)$.

Setting $\Theta = 0, \varphi = 0$ and $r_n = 1$ then we have $u_n = P_C x_n = x_n$ in Theorem 3.1, and we have the following result.

Corollary 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mapping on C such that $\cap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(A,C) \neq \emptyset$. Suppose that $\{s_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$,$\{\lambda_n\}$ is a sequence in $[0,2\alpha]$ such that $\lambda_n \in [a,b]$ for some a,b with $0 < a < b < 2\alpha$. Suppose that the following conditions are satisfied:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(iv) $0 < \liminf_{n\to\infty}s_n \leq \limsup_{n\to\infty}s_n < 1/2$ *and* $\lim_{n\to\infty}|s_{n+1} - s_n| = 0$,

(v) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$.

Let f be a contraction of C into itself with coefficient $\beta \in (0,1)$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_1 \in C$ and

$$\begin{gathered}
y_n = s_n P_C(x_n - \lambda_n A x_n) + (1 - s_n)x_n, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n\big(P_C(y_n - \lambda_n A y_n)\big),
\end{gathered} \tag{3.53}$$

for all $n \in \mathbb{N}$, where W_n defined by (1.6) *and $\{t_n\}$ is a sequence in $(0,b]$, for some $b \in (0,1)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to a point $x^* \in \cap_{n=1}^{\infty} F(T_n) \cap \mathrm{VI}(A,C)$, where $x^* = P_{\cap_{n=1}^{\infty} F(T_n)\cap\mathrm{VI}(A,C)} f(x^*)$.*

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