Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 983802, 14 pages
doi:10.1155/2010/983802

Research Article

Common Fixed Point Theorem for Four Non-Self Mappings in Cone Metric Spaces

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Received 13 June 2009; Revised 1 March 2010; Accepted 18 April 2010

Academic Editor: Lai Jiu Lin

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We extend a common fixed point theorem of Radenovic and Rhoades for four non-self-mappings in cone metric spaces.

1. Introduction and Preliminaries

Recently, Huang and Zhang [1] generalized the concept of a metric space, replacing the set of real numbers by ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. Subsequently, the study of fixed point theorems in such spaces is followed by some other mathematicians; see [2–8]. The aim of this paper is to prove a common fixed point theorem for four non-self-mappings on cone metric spaces in which the cone need not be normal. This result generalizes the result of Radenović and Rhoades [5].

Consistent with Huang and Zhang [1], the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone if and only if

(a) P is closed, nonempty and $P \neq \{\theta\}$;

(b) $a, b \in R$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$;

(c) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \quad \text{implies } \|x\| \leq K\|y\|. \tag{1.1}$$

The least positive number satisfying the above inequality is called the normal constant of P, while $x \ll y$ stands for $y - x \in \operatorname{int} P$ (interior of P).

Definition 1.1 (see [1]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(d1) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

Definition 1.2 (see [1]). Let (X, d) be a cone metric space. One says that $\{x_n\}$ is

(e) a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n, m > N$, $d(x_n, x_m) \ll c$;

(f) a Convergent sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n > N$, $d(x_n, x) \ll c$ for some fixed $x \in X$.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to \theta$ as $n \to \infty$. It is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta(n, m \to \infty)$.

Remark 1.3 (see [9]). Let E be an ordered Banach (normed) space. Then c is an interior point of P if and only if $[-c, c]$ is a neighborhood of θ.

Corollary 1.4 (see [10]). (1) *If $a \leq b$ and $b \ll c$, then $a \ll c$.*
Indeed, $c - a = (c - b) + (b - a) \geq c - b$ implies $[-(c - a), c - a] \supseteq [-(c - b), c - b]$.

(2) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*
Indeed, $c - a = (c - b) + (b - a) \geq c - b$ implies $[-(c - a), c - a] \supseteq [-(c - b), c - b]$.

(3) *If $\theta \leq u \ll c$ for each $c \in \operatorname{int} P$, then $u = \theta$.*

Remark 1.5 (see [5, 11]). If $c \in \operatorname{int} P$, $\theta \leq a_n$, and $a_n \to \theta$, then there exists an n_0 such that for all $n > n_0$ we have $a_n \ll c$.

Remark 1.6 (see [6, 10]). If E is a real Banach space with cone P and if $a \leq ka$ where $a \in P$ and $0 < k < 1$, then $a = \theta$.

We find it convenient to introduce the following definition.

Definition 1.7 (see [5]). Let (X, d) be a complete cone metric space and C a nonempty closed subset of X, and $f, g : C \to X$ satisfying

$$d(fx, fy) \leq \lambda u, \tag{1.2}$$

where

$$u \in \left\{ \frac{d(gx,gy)}{2}, d(fx,gx), d(fy,gy), \frac{d(fx,gy)+d(fy,gx)}{q} \right\}, \tag{1.3}$$

for all $x, y \in C$, $0 < \lambda < 1/2$, $q \geq 2 - \lambda$, then f is called a generalized g-contractive mapping of C into X.

Definition 1.8 (see [2]). Let f and g be self-maps on a set X (i.e., $f, g : X \to X$). If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. Self-maps f and g are said to be weakly compatible if they commute at their coincidence point; that is, if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

2. Main Result

The following theorem is Radenović and Rhoades [5] generalization of Imdad and Kumar's [12] result in cone metric spaces.

Theorem 2.1. *Let (X, d) be a complete cone metric space and C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ (the boundary of C) such that*

$$d(x,z) + d(z,y) = d(x,y). \tag{2.1}$$

Suppose that $f, g : C \to X$ are such that f is a generalized g-contractive mapping of C into X, and

(i) *$\partial C \subseteq gC$, $fC \cap C \subseteq gC$,*

(ii) *$gx \subseteq \partial C \Rightarrow fx \in C$,*

(iii) *gC is closed in X.*

Then the pair (f, g) has a coincidence point. Moreover, if pair (f, g) is weakly compatible, then f and g have a unique common fixed point.

The purpose of this paper is to extend the above theorem for four non-self-mappings in cone metric spaces. We begin with the following definition.

Definition 2.2. Let (X, d) be a complete cone metric space and C a nonempty closed subset of X, and $F, G, S, T : C \to X$ satisfying

$$d(Fx, Gy) \leq \lambda u, \tag{2.2}$$

where

$$u \in \left\{ \frac{d(Tx,Sy)}{2}, d(Tx,Fx), d(Sy,Gy), \frac{d(Tx,Gy)+d(Sy,Fx)}{q} \right\}, \tag{2.3}$$

for all $x, y \in C$, $0 < \lambda < 1/2$, $q \geq 2 - \lambda$, then (F, G) is called a generalized (T, S)-contractive mappings pair of C into X.

Notice that by setting $G = F = f$ and $T = S = g$ in (2.2), one deduces the slightly generalized form of (1.3).

We state and prove our main result as follows.

Theorem 2.3. *Let (X, d) be a complete cone metric space and C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ (the boundary of C) such that*

$$d(x,z) + d(z,y) = d(x,y). \tag{2.4}$$

Suppose that $F, G, S, T : C \to X$ are such that (F, G) is a generalized (T, S)-contractive mappings pair of C into X, and

(I) $\partial C \subseteq SC \cap TC, FC \cap C \subseteq SC, GC \cap C \subseteq TC$,

(II) $Tx \subseteq \partial C \Rightarrow Fx \in C, Sx \subseteq \partial C \Rightarrow Gx \in C$,

(III) *SC and TC (or FC and GC) are closed in X.*
Then

(IV) *(F, T) has a point of coincidence,*

(V) *(G, S) has a point of coincidence.*

Moreover, if (F, T) and (G, S) are weakly compatible pairs, then F, G, S, and T have a unique common fixed point.

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way.

Let $x \in \partial C$ be arbitrary. Then (due to $\partial C \subseteq TC$) there exists a point $x_0 \in C$ such that $x = Tx_0$. Since $Tx \subseteq \partial C \Rightarrow Fx \in C$, one concludes that $Fx_0 \in FC \cap C \subseteq SC$. Thus, there exists $x_1 \in C$ such that $y_1 = Sx_1 = Fx_0 \in C$. Since $y_1 = Fx_0$ there exists a point $y_2 = Gx_1$ such that

$$d(y_1, y_2) = d(Fx_0, Gx_1). \tag{2.5}$$

Suppose that $y_2 \in C$. Then $y_2 \in GC \cap C \subseteq TC$ which implies that there exists a point $x_2 \in C$ such that $y_2 = Tx_2$. Otherwise, if $y_2 \notin C$, then there exists a point $p \in \partial C$ such that

$$d(Sx_1, p) + d(p, y_2) = d(Sx_1, y_2). \tag{2.6}$$

Since $p \in \partial C \subseteq TC$ there exists a point $x_2 \in C$ with $p = Tx_2$, so that

$$d(Sx_1, Tx_2) + d(Tx_2, y_2) = d(Sx_1, y_2). \tag{2.7}$$

Let $y_3 = Fx_2$ be such that $d(y_2, y_3) = d(Gx_1, Fx_2)$. Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(a) $y_{2n} = Gx_{2n-1}$, $y_{2n+1} = Fx_{2n}$,

(b) $y_{2n} \in C \Rightarrow y_{2n} = Tx_{2n}$ or $y_{2n} \notin C \Rightarrow Tx_{2n} \in \partial C$,

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n}). \tag{2.8}$$

(c) $y_{2n+1} \in C \Rightarrow y_{2n+1} = Sx_{2n+1}$ or $y_{2n+1} \notin C \Rightarrow Sx_{2n+1} \in \partial C$,

$$d(Tx_{2n}, Sx_{2n+1}) + d\big(Sx_{2n+1}, y_{2n+1}\big) = d\big(Tx_{2n}, y_{2n+1}\big). \tag{2.9}$$

We denote that

$$\begin{aligned} P_0 &= \big\{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} = y_{2i}\big\}, \\ P_1 &= \big\{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} \neq y_{2i}\big\}, \\ Q_0 &= \big\{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} = y_{2i+1}\big\}, \\ Q_1 &= \big\{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} \neq y_{2i+1}\big\}. \end{aligned} \tag{2.10}$$

Note that $(Tx_{2n}, Sx_{2n+1}) \notin P_1 \times Q_1$, as if $Tx_{2n} \in P_1$, then $y_{2n} \neq Tx_{2n}$, and one infers that $Tx_{2n} \in \partial C$ which implies that $y_{2n+1} = Fx_{2n} \in C$. Hence $y_{2n+1} = Sx_{2n+1} \in Q_0$. Similarly, one can argue that $(Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1$.

Now, we distinguish the following three cases. □

Case 1. If $(Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_0$, then from (2.2)

$$d(Tx_{2n}, Sx_{2n+1}) = d(Fx_{2n}, Gx_{2n-1}) \leq \lambda u_{2n-1}, \tag{2.11}$$

where

$$\begin{aligned} u_{2n-1} &\in \left\{\frac{d(Sx_{2n-1}, Tx_{2n})}{2}, d(Sx_{2n-1}, Gx_{2n-1}), d(Tx_{2n}, Fx_{2n}), \frac{d(Tx_{2n}, Gx_{2n-1}) + d(Sx_{2n-1}, Fx_{2n})}{q}\right\} \\ &= \left\{\frac{d\big(y_{2n-1}, y_{2n}\big)}{2}, d\big(y_{2n-1}, y_{2n}\big), d\big(y_{2n}, y_{2n+1}\big), \frac{d\big(y_{2n-1}, y_{2n+1}\big)}{q}\right\}. \end{aligned} \tag{2.12}$$

Clearly, there are infinite many n such that at least one of the following four cases holds:

(1)

$$d(Tx_{2n}, Sx_{2n+1}) \leq \lambda \frac{d\big(y_{2n-1}, y_{2n}\big)}{2} \leq \lambda d(Sx_{2n-1}, Tx_{2n}), \tag{2.13}$$

(2)

$$d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d\big(y_{2n-1}, y_{2n}\big) = \lambda d(Sx_{2n-1}, Tx_{2n}), \tag{2.14}$$

(3)

$$d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(y_{2n}, y_{2n+1}) \Longrightarrow d(Tx_{2n}, Sx_{2n+1}) = \theta \leq \lambda d(Sx_{2n-1}, Tx_{2n}), \quad (2.15)$$

(4)

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) &\leq \lambda \frac{d(y_{2n-1}, y_{2n+1})}{q} \\ &\leq \lambda \frac{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})}{q} \\ &= \lambda \frac{d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, Sx_{2n+1})}{q}, \end{aligned} \quad (2.16)$$

which implies $(1 - \lambda/q)d(Tx_{2n}, Sx_{2n+1}) \leq (\lambda/q)d(Sx_{2n-1}, Tx_{2n})$, that is,

$$d(Tx_{2n}, Sx_{2n+1}) \leq \frac{\lambda}{q-\lambda} d(Sx_{2n-1}, Tx_{2n}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}). \quad (2.17)$$

From (1), (2), (3), and (4) it follows that

$$d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}). \quad (2.18)$$

Similarly, if $(Sx_{2n+1}, Tx_{2n+2}) \in Q_0 \times P_0$, we have

$$d(Sx_{2n+1}, Tx_{2n+2}) = d(Fx_{2n}, Gx_{2n+1}) \leq \lambda d(Tx_{2n}, Sx_{2n+1}). \quad (2.19)$$

If $(Sx_{2n-1}, Tx_{2n}) \in Q_0 \times P_0$, we have

$$d(Sx_{2n-1}, Tx_{2n}) = d(Fx_{2n-2}, Gx_{2n-1}) \leq \lambda d(Tx_{2n-2}, Sx_{2n-1}). \quad (2.20)$$

Case 2. If $(Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_1$, then $Sx_{2n+1} \in Q_1$ and

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}) \quad (2.21)$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \leq d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1}) \quad (2.22)$$

and hence

$$d(Tx_{2n}, Sx_{2n+1}) \leq d(y_{2n}, y_{2n+1}) = d(Fx_{2n}, Gx_{2n-1}). \quad (2.23)$$

Now, proceeding as in Case 1, we have that (2.18) holds.

If $(Sx_{2n+1}, Tx_{2n+2}) \in Q_1 \times P_0$, then $Tx_{2n} \in P_0$. We show that

$$d(Sx_{2n+1}, Tx_{2n+2}) \leq \lambda d(Tx_{2n}, Sx_{2n-1}). \tag{2.24}$$

Using (2.21), we get

$$\begin{aligned} d(Sx_{2n+1}, Tx_{2n+2}) &\leq d(Sx_{2n+1}, y_{2n+1}) + d(y_{2n+1}, Tx_{2n+2}) \\ &= d(Tx_{2n}, y_{2n+1}) - d(Tx_{2n}, Sx_{2n+1}) + d(y_{2n+1}, Tx_{2n+2}). \end{aligned} \tag{2.25}$$

By noting that $Tx_{2n+2}, Tx_{2n} \in P_0$, one can conclude that

$$\begin{aligned} d(y_{2n+1}, Tx_{2n+2}) &= d(y_{2n+1}, y_{2n+2}) = d(Fx_{2n}, Gx_{2n+1}) \leq \lambda d(Tx_{2n}, Sx_{2n+1}), \\ d(Tx_{2n}, y_{2n+1}) &= d(y_{2n}, y_{2n+1}) = d(Fx_{2n}, Gx_{2n-1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}), \end{aligned} \tag{2.26}$$

in view of Case 1.

Thus,

$$d(Sx_{2n+1}, Tx_{2n+2}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}) - (1-\lambda) d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}), \tag{2.27}$$

and we proved (2.24).

Case 3. If $(Tx_{2n}, Sx_{2n+1}) \in P_1 \times Q_0$, then $Sx_{2n-1} \in Q_0$. We show that

$$d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n-2}). \tag{2.28}$$

Since $Tx_{2n} \in P_1$, then

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n}). \tag{2.29}$$

From this, we get

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) &\leq d(Tx_{2n}, y_{2n}) + d(y_{2n}, Sx_{2n+1}) \\ &= d(Sx_{2n-1}, y_{2n}) - d(Sx_{2n-1}, Tx_{2n}) + d(y_{2n}, Sx_{2n+1}). \end{aligned} \tag{2.30}$$

By noting that $Sx_{2n+1}, Sx_{2n-1} \in Q_0$, one can conclude that

$$\begin{aligned} d(y_{2n}, Sx_{2n+1}) &= d(y_{2n}, y_{2n+1}) = d(Fx_{2n}, Gx_{2n-1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}), \\ d(Sx_{2n-1}, y_{2n}) &= d(y_{2n-1}, y_{2n}) = d(Fx_{2n-2}, Gx_{2n-1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n-2}), \end{aligned} \tag{2.31}$$

in view of Case 1.

Thus,

$$d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n-2}) - (1-\lambda)d(Sx_{2n-1}, Tx_{2n}) \leq \lambda d(Sx_{2n-1}, Tx_{2n-2}), \quad (2.32)$$

and we proved (2.28).

Similarly, if $(Sx_{2n+1}, Tx_{2n+2}) \in Q_0 \times P_1$, then $Tx_{2n+2} \in P_1$, and

$$d(Sx_{2n+1}, Tx_{2n+2}) + d\big(Tx_{2n+2}, y_{2n+2}\big) = d\big(Sx_{2n+1}, y_{2n+2}\big). \quad (2.33)$$

From this, we have

$$\begin{aligned} d(Sx_{2n+1}, Tx_{2n+2}) &\leq d\big(Sx_{2n+1}, y_{2n+2}\big) + d\big(y_{2n+2}, Tx_{2n+2}\big) \\ &\leq d\big(Sx_{2n+1}, y_{2n+2}\big) + d\big(Sx_{2n+1}, y_{2n+2}\big) - d(Sx_{2n+1}, Tx_{2n+2}) \\ &= 2d\big(Sx_{2n+1}, y_{2n+2}\big) - d(Sx_{2n+1}, Tx_{2n+2}) \Longrightarrow d(Sx_{2n+1}, Tx_{2n+2}) \\ &\leq d\big(Sx_{2n+1}, y_{2n+2}\big). \end{aligned} \quad (2.34)$$

By noting that $Sx_{2n+1} \in Q_0$, one can conclude that

$$d(Sx_{2n+1}, Tx_{2n+2}) \leq d\big(Sx_{2n+1}, y_{2n+2}\big) = d(Fx_{2n}, Gx_{2n+1}) \leq \lambda d(Tx_{2n}, Sx_{2n+1}), \quad (2.35)$$

in view of Case 1.

Thus, in all Cases 1–3, there exists $w_{2n} \in \{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n-2}, Sx_{2n-1})\}$ such that

$$d(Tx_{2n}, Sx_{2n+1}) \leq \lambda w_{2n}, \quad (2.36)$$

and there exists $w_{2n+1} \in \{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1})\}$ such that

$$d(Sx_{2n+1}, Tx_{2n+2}) \leq \lambda w_{2n+1}. \quad (2.37)$$

Following the procedure of Assad and Kirk [13], it can easily be shown by induction that, for $n \geq 1$, there exists $w_2 \in \{d(Tx_0, Sx_1), d(Sx_1, Tx_2)\}$ such that

$$d(Tx_{2n}, Sx_{2n+1}) \leq \lambda^{n-1/2} w_2, \qquad d(Sx_{2n+1}, Tx_{2n+2}) \leq \lambda^n w_2. \quad (2.38)$$

From (2.38) and by the triangle inequality, for $n > m$, we have

$$\begin{aligned} d(Tx_{2n}, Sx_{2m+1}) &\leq d(Tx_{2n}, Sx_{2n-1}) + d(Sx_{2n-1}, Tx_{2n-2}) + \cdots + d(Tx_{2m+2}, Sx_{2m+1}) \\ &\leq \Big(\lambda^m + \lambda^{m+1/2} + \cdots + \lambda^{n-1}\Big) w_2 \leq \frac{\lambda^m}{1-\sqrt{\lambda}} w_2 \longrightarrow \theta, \quad \text{as } m \longrightarrow \infty. \end{aligned} \quad (2.39)$$

From Remark 1.5 and Corollary 1.4(1), $d(Tx_{2n}, Sx_{2m+1}) \ll c$.

Thus, the sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, \ldots, Sx_{2n-1}, Tx_{2n}, Sx_{2n-1}, \ldots\}$ is a Cauchy sequence. Then, as noted in [14], there exists at least one subsequence $\{Tx_{2n_k}\}$ or $\{Sx_{2n_k+1}\}$ which is contained in P_0 or Q_0, respectively, and finds its limit $z \in C$. Furthermore, subsequences $\{Tx_{2n_k}\}$ and $\{Sx_{2n_k+1}\}$ both converge to $z \in C$ as C is a closed subset of complete cone metric space (X, d). We assume that there exists a subsequence $\{Tx_{2n_k}\} \subseteq P_0$ for each $k \in N$, then $Tx_{2n_k} = y_{2n_k} = Gx_{2n_k-1} \in C \cap GC \subseteq TC$. Since TC as well as SC are closed in X, and $\{Tx_{2n_k}\}$ is Cauchy in TC, it converges to a point $z \in TC$. Let $w \in T^{-1}z$, then $Tw = z$. Similarly, $\{Sx_{2n_k+1}\}$ a subsequence of Cauchy sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, \ldots, Sx_{2n-1}, Tx_{2n}, Sx_{2n-1}, \ldots\}$ also converges to z as SC is closed. Using (2.2), one can write

$$d(Fw, z) \leq d(Fw, Gx_{2n_k-1}) + d(Gx_{2n_k-1}, z) \leq \lambda u_{2n_k-1} + d(Gx_{2n_k-1}, z), \tag{2.40}$$

where

$$\begin{aligned} u_{2n_k-1} \in {} & \left\{ \frac{d(Tw, Sx_{2n_k-1})}{2}, d(Tw, Fw), d(Sx_{2n_k-1}, Gx_{2n_k-1}), \frac{d(Tw, Gx_{2n_k-1}) + d(Fw, Sx_{2n_k-1})}{q} \right\} \\ = {} & \left\{ \frac{d(z, Sx_{2n_k-1})}{2}, d(z, Fw), d(Sx_{2n_k-1}, Gx_{2n_k-1}), \frac{d(z, Gx_{2n_k-1}) + d(Fw, Sx_{2n_k-1})}{q} \right\}. \end{aligned} \tag{2.41}$$

Let $\theta \ll c$. Clearly at least one of the following four cases holds for infinitely many n:

(1)

$$d(Fw, z) \leq \lambda \frac{d(z, Sx_{2n_k-1})}{2} + d(Gx_{2n_k-1}, z) \ll \lambda \frac{c}{2\lambda} + \frac{c}{2} = c; \tag{2.42}$$

(2)

$$\begin{aligned} d(Fw, z) \leq \lambda d(z, Fw) + d(Gx_{2n_k-1}, z) &\Longrightarrow d(Fw, z) \\ &\leq \frac{1}{1-\lambda} d(Gx_{2n_k-1}, z) \ll \frac{1}{1-\lambda}(1-\lambda)c = c; \end{aligned} \tag{2.43}$$

(3)

$$\begin{aligned} d(Fw, z) &\leq \lambda d(Sx_{2n_k-1}, Gx_{2n_k-1}) + d(Gx_{2n_k-1}, z) \\ &\leq \lambda (d(Sx_{2n_k-1}, z) + d(z, Gx_{2n_k-1})) + d(Gx_{2n_k-1}, z) \\ &\leq (\lambda + 1) d(Gx_{2n_k-1}, z) + \lambda d(Sx_{2n_k-1}, z) \\ &\ll (\lambda + 1) \frac{c}{2(\lambda + 1)} + \lambda \frac{c}{2\lambda} = c; \end{aligned} \tag{2.44}$$

(4)

$$\begin{aligned} d(Fw,z) &\leq \lambda \frac{d(z,Gx_{2n_k-1}) + d(Fw,Sx_{2n_k-1})}{q} + d(Gx_{2n_k-1},z) \\ &\leq \lambda \frac{d(z,Gx_{2n_k-1}) + d(Fw,z) + d(z,Sx_{2n_k-1})}{q} + d(Gx_{2n_k-1},z) \Longrightarrow d(Fw,z) \\ &\leq \frac{q+\lambda}{q-\lambda} d(Gx_{2n_k-1},z) + \frac{\lambda}{q-\lambda} d(z,Sx_{2n_k-1}) \\ &\ll \frac{q+\lambda}{q-\lambda} \frac{c}{2((q+\lambda)/(q-\lambda))} + \frac{\lambda}{q-\lambda} \frac{c}{2(\lambda/(q-\lambda))} = c. \end{aligned} \tag{2.45}$$

In all cases we obtain $d(Fw,z) \ll c$ for each $c \in \operatorname{int} P$. Using Corollary 1.4(3) it follows that $d(Fw,z) = \theta$ or $Fw = z$. Thus, $Fw = z = Tw$, that is, z is a coincidence point of F, T.

Further, since Cauchy sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, \ldots, Sx_{2n-1}, Tx_{2n}, Sx_{2n-1}, \ldots\}$ converges to $z \in C$ and $z = Fw$, $z \in FC \cap C \subseteq SC$, there exists $v \in C$ such that $Sv = z$. Again using (2.2), we get

$$d(Sv,Gv) = d(z,Gv) = d(Fw,Gv) \leq \lambda u, \tag{2.46}$$

where

$$\begin{aligned} u &\in \left\{ \frac{d(Tw,Sv)}{2}, d(Tw,Fw), d(Sv,Gv), \frac{d(Tw,Gv) + d(Fw,Sv)}{q} \right\} \\ &= \left\{ \theta, \theta, d(Sv,Gv), \frac{d(z,Gv) + \theta}{q} \right\} \\ &= \left\{ \theta, d(Sv,Gv), \frac{d(Sv,Gv)}{q} \right\}. \end{aligned} \tag{2.47}$$

Hence, we get the following cases:

$$d(Sv,Gv) \leq \lambda\theta = \theta, \qquad d(Sv,Gv) \leq \lambda d(Sv,Gv), \qquad d(Sv,Gv) \leq \frac{\lambda}{q} d(Sv,Gv). \tag{2.48}$$

Since $\lambda/q \leq \lambda/(2-\lambda) = \lambda/(1+(1-\lambda)) < \lambda$, using Remark 1.6 and Corollary 1.4(3), it follows that $Sv = Gv$; therefore, $Sv = z = Gv$, that is, z is a coincidence point of (G,S).

In case FC and GC are closed in X, $z \in FC \cap C \subseteq SC$ or $z \in GC \cap C \subseteq TC$. The analogous arguments establish (IV) and (V). If we assume that there exists a subsequence $\{Sx_{2n_k+1}\} \subseteq Q_0$ with TC as well SC being closed in X, then noting that $\{Sx_{2n_k+1}\}$ is a Cauchy sequence in SC, foregoing arguments establish (IV) and (V).

Suppose now that (F,T) and (G,S) are weakly compatible pairs, then

$$z = Fw = Tw \Longrightarrow Fz = FTw = TFw = Tz, \qquad z = Gv = Sv \Longrightarrow Gz = GSv = SGv = Sz. \tag{2.49}$$

Then, from (2.2),

$$d(Fz,z) = d(Fz,Gv) \le \lambda u, \tag{2.50}$$

where

$$\begin{aligned} u &\in \left\{ \frac{d(Sv,Tz)}{2}, d(Tz,Fz), d(Sv,Gv), \frac{d(Tz,Gv)+d(Sv,Fz)}{q} \right\} \\ &= \left\{ \frac{d(z,Fz)}{2}, d(Fz,Fz), d(z,z), \frac{d(Fz,z)+d(z,Fz)}{q} \right\} \\ &= \left\{ \frac{d(z,Fz)}{2}, \theta, \frac{2d(z,Fz)}{q} \right\}. \end{aligned} \tag{2.51}$$

Hence, we get the following cases:

$$d(Fz,z) \le \lambda \frac{d(z,Fz)}{2}, \qquad d(Fz,z) \le \lambda\theta = \theta \qquad \text{and } d(Fz,z) \le \frac{2\lambda d(z,Fz)}{q}, \tag{2.52}$$

Since $2\lambda/q \le 2\lambda/(2-\lambda) = 2\lambda/(1+(1-\lambda)) < 2\lambda < 1$, using Remark 1.6 and Corollary 1.4(3), it follows that $Fz = z$. Thus, $Fz = z = Tz$.

Similarly, we can prove that $Gz = z = Sz$. Therefore $z = Fz = Gz = Sz = Tz$, that is, z is a common fixed point of F, G, S, and T.

Uniqueness of the common fixed point follows easily from (2.2).

The following example shows that in general F, G, S, and T satisfying the hypotheses of Theorem 2.3 need not have a common coincidence justifying two separate conclusions (IV) and (V).

Example 2.4. Let $E = C^1([0,1], R)$, $P = \{\varphi \in E : \varphi(t) \ge 0,\ t \in [0,1]\}$, $X = [0,+\infty)$, $C = [0,2]$, and $d : X \times X \to E$ defined by $d(x,y) = |x-y|\varphi$, where $\varphi \in P$ is a fixed function, for example, $\varphi(t) = e^t$. Then (X,d) is a complete cone metric space with a nonnormal cone having the nonempty interior. Define F, G, S, and $T : C \to X$ as

$$Fx = x + \frac{4}{5}, \quad Gx = x^2 + \frac{4}{5}, \quad Tx = 5x, \quad Sx = 5x^2, \quad x \in C. \tag{2.53}$$

Since $\partial C = \{0,2\}$. Clearly, for each $x \in C$ and $y \notin C$ there exists a point $z = 2 \in \partial C$ such that $d(x,z) + d(z,y) = d(x,y)$. Further, $SC \cap TC = [0,20] \cap [0,10] = [0,10] \supset \{0,2\} = \partial C$, $FC \cap C = [4/5, 14/5] \cap [0,2] = [4/5, 2] \subset SC$, $GC \cap C = [4/5, 24/5] \cap [0,2] = [4/5, 2] \subset TC$, and SC, TC, FC, and GC are closed in X.

Also,

$$T0 = 0 \in \partial C \Longrightarrow F0 = \frac{4}{5} \in C, \qquad S0 = 0 \in \partial C \Longrightarrow G0 = \frac{4}{5} \in C,$$
$$T\left(\frac{2}{5}\right) = 2 \in \partial C \Longrightarrow F\left(\frac{2}{5}\right) = \frac{6}{5} \in C, \qquad S\left(\sqrt{\frac{2}{5}}\right) = 2 \in \partial C \Longrightarrow G\left(\sqrt{\frac{2}{5}}\right) = \frac{6}{5} \in C. \tag{2.54}$$

Moreover, for each $x, y \in C$,

$$d(Fx, Gy) = \left|x - y^2\right|\varphi = \frac{2}{5}\left(\frac{1}{2}d(Tx, Sy)\right), \tag{2.55}$$

that is, (2.2) is satisfied with $\lambda = 2/5$.

Evidently, $1 = T(1/5) = F(1/5) \neq 1/5$ and $1 = S(1/\sqrt{5}) = G(1/\sqrt{5}) \neq 1/\sqrt{5}$. Notice that two separate coincidence points are not common fixed points as $FT(1/5) \neq TF(1/5)$ and $SG(1/\sqrt{5}) \neq GS(1/\sqrt{5})$, which shows necessity of weakly compatible property in Theorem 2.3.

Next, we furnish an illustrate example in support of our result. In doing so, we are essentially inspired by Imdad and Kumar [12].

Example 2.5. Let $E = C^1([0,1], R)$, $P = \{\varphi \in E : \varphi(t) \geq 0, t \in [0,1]\}$, $X = [1, +\infty)$, $C = [1,3]$, and $d : X \times X \to E$ defined by $d(x,y) = |x - y|\varphi$, where $\varphi \in P$ is a fixed function, for example, $\varphi(t) = e^t$. Then (X, d) is a complete cone metric space with a nonnormal cone having the nonempty interior. Define F, G, S, and $T : C \to X$ as

$$Fx = \begin{cases} x^2 & \text{if } 1 \leq x \leq 2, \\ 2 & \text{if } 2 < x \leq 3, \end{cases} \qquad Tx = \begin{cases} 4x^4 - 3 & \text{if } 1 \leq x \leq 2, \\ 13 & \text{if } 2 < x \leq 3, \end{cases}$$
$$Gx = \begin{cases} x^3 & \text{if } 1 \leq x \leq 2, \\ 2 & \text{if } 2 < x \leq 3, \end{cases} \qquad Sx = \begin{cases} 4x^6 - 3 & \text{if } 1 \leq x \leq 2, \\ 13 & \text{if } 2 < x \leq 3. \end{cases} \tag{2.56}$$

Since $\partial C = \{1, 3\}$. Clearly, for each $x \in C$ and $y \notin C$ there exists a point $z = 3 \in \partial C$ such that $d(x,z) + d(z,y) = d(x,y)$. Further, $SC \cap TC = [1,253] \cap [1,61] = [1,61] \supset \{1,3\} = \partial C$, $FC \cap C = [1,4] \cap [1,3] = [1,3] \subset SC$, and $GC \cap C = [1,8] \cap [1,3] = [1,3] \subset TC$.

Also,

$$T1 = 1 \in \partial C \Longrightarrow F1 = 1 \in C, \qquad S1 = 1 \in \partial C \Longrightarrow G1 = 1 \in C,$$
$$T\left(\sqrt[4]{\frac{3}{2}}\right) = 3 \in \partial C \Longrightarrow F\left(\sqrt[4]{\frac{3}{2}}\right) = \sqrt{\frac{3}{2}} \in C, \qquad S\left(\sqrt[6]{\frac{3}{2}}\right) = 3 \in \partial C \Longrightarrow G\left(\sqrt[6]{\frac{3}{2}}\right) = \sqrt{\frac{3}{2}} \in C. \tag{2.57}$$

Moreover, if $x \in [1,2]$ and $y \in [2,3]$, then

$$d(Fx,Gy) = \left|x^2 - 2\right|\varphi = \frac{\left|x^4-4\right|}{|x^2+2|}\varphi = \frac{4\left|x^4-4\right|/2}{2|x^2+2|}\varphi = \frac{1}{2(x^2+2)}\frac{d(Tx,Sy)}{2}. \tag{2.58}$$

Next, if $x, y \in (2,3]$, then

$$d(Fx,Gy) = 0 = \lambda \cdot \frac{d(Tx,Sy)}{2}. \tag{2.59}$$

Finally, if $x, y \in [1,2]$, then

$$d(Fx,Gy) = \left|x^2 - y^3\right|\varphi = \frac{\left|x^4-y^6\right|}{\left|x^2+y^3\right|}\varphi = \frac{4\left|x^4-y^6\right|/2}{2\left|x^2+y^3\right|}\varphi = \frac{1}{2(x^2+y^3)}\frac{d(Tx,Sy)}{2}. \tag{2.60}$$

Therefore, condition (2.2) is satisfied if we choose $\lambda = \max\{1/2(x^2+2), 1/2(x^2+y^3)\} \in (0,1/2)$. Moreover 1 is a point of coincidence as $T1 = F1$ as well as $S1 = G1$ whereas both the pairs (F,T) and (G,S) are weakly compatible as $TF1 = 1 = FT1$ and $SG1 = 1 = GS1$. Also, SC, TC, FC, and GC are closed in X. Thus, all the conditions of Theorem 2.3 are satisfied and 1 is the unique common fixed point of F, G, S, and T. One may note that 1 is also a point of coincidence for both the pairs (F,T) and (G,S).

Remark 2.6. (1) Setting $G = F = f$ and $T = S = g$ in Theorem 2.3, one deduces Theorem 2.1 due to [5].

(2) Setting $G = F = f$ and $T = S = I_X$ in Theorem 2.3, we obtain the following result.

Corollary 2.7. *Let (X,d) be a complete cone metric space and C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ (the boundary of C) such that*

$$d(x,z) + d(z,y) = d(x,y). \tag{2.61}$$

Suppose that $f : C \to X$ satisfies the condition

$$d(fx,fy) \leq \lambda u(x,y), \tag{2.62}$$

where

$$u(x,y) \in \left\{\frac{d(x,y)}{2}, d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{q}\right\} \tag{2.63}$$

for all $x, y \in C$, $0 < \lambda < 1/2$, $q \geq 2 - \lambda$, and f has the additional property that for each $x \in \partial C$, $fx \in C$, f has a unique fixed point.

Acknowledgments

The authors would like to express their sincere appreciation to the referees for their very helpful suggestions and many kind comments. This project was supported by the National Natural Science Foundation of China (10461007 and 10761007) and supported partly by the Provincial Natural Science Foundation of Jiangxi, China (2008GZS0076 and 2009GZS0019).

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