A CONTRACTION THEOREM IN FUZZY METRIC SPACES

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A fixed point theorem is proved. Moreover, fuzzy Edelstein's contraction theorem is described. Finally, the existence of at least one periodic point is shown.

1. Introduction

After Zadeh pioneering's paper [15], where the Theory of Fuzzy Sets was introduced, hundreds of examples have been supplied where the nature of uncertainty in the behavior of a given system possesses fuzzy rather than stochastic nature. Non-stationary fuzzy systems described by fuzzy processes look as their natural extension into the time domina. From different viewpoints they were carefully studied.

Fixed-point theory for contraction type mappings in fuzzy metric space is closely related to the fixed-point theory for the same type of mappings in probabilistic metric space of Menger type (see [10, 13]). The concept of fuzzy metric spaces recently have been introduced in different ways by many authors [1, 2, 8]. George and Veeramani [3, 4] modified the concept of fuzzy metric space which has been introduced by Kramosil and Michálek [9] and obtained a Hausdorff topology for this kind of fuzzy metric space.

Here, we claim that if $(X,M,*)$ is a fuzzy metric space, and A a contractive mapping of X into itself such that there exists a point $x \in X$ whose sequence of iterates $(A^n(x))$ contains a convergent subsequence $(A^{n_i}(x))$; then $\xi = \lim_{i\to\infty} A^{n_i}(x) \in X$ is a unique fixed point. In addition, we can prove fuzzy Edelstein's contraction theorem. Note that this happen when we consider the fuzzy metric space in the George and Veeramani's sense. In addition, it is claimed that fuzzy Edelstein's contraction theorem is true whenever we consider the fuzzy metric space in the Kramosil and Michálek's sense. Finally, the existence of at least one periodic point will be proved and two question would arise. In order to do this, we recall some concepts and results that will be required in the sequel.

Definition 1.1 [12]. A binary operation $* : [0,1 \times [0,1] \to [0,1]$ is a continuous t-norm if $([0,1],*)$ is a topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a,b,c,d \in [0,1]$.

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Definition 1.2 [3]. The 3-tuple $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times]0,\infty[$ satisfying the following conditions: for all $x,y,z\in X$ and $t,s>0$,

(i) $M(x,y,t)>0$,
(ii) $M(x,y,t)=1$ if and only if $x=y$,
(iii) $M(x,y,t)=M(y,x,t)$,
(iv) $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$,
(v) $M(x,y,\cdot):]0,\infty[\to[0,1]$ is continuous.

Lemma 1.3 [5]. *$M(x,y,\cdot)$ is nondecreasing for all $x,y\in X$.*

In order to introduce a Hausdorff topology on the fuzzy metric space, the following definitions are needed.

Definition 1.4 [3]. Let $(X,M,*)$ be a fuzzy metric space. The open ball $B(x,r,t)$ for $t>0$ with center $x\in X$ and radius r, $0<r<1$, is defined as $B(x,r,t)=\{y\in X: M(x,y,t)>1-r\}$. The family $\{B(x,r,t): x\in X,\ 0<r<1,\ t>0\}$ is a neighborhood's system for a Hausdorff topology on X, that we call induced by the fuzzy metric M.

Definition 1.5 [3]. In a metric space (X,d), the 3-tuple $(X,M_d,*)$ where $M_d(x,y,t)=t/t+d(x,y)$ and $a*b=ab$, is a fuzzy metric space. This M_d is called the standard fuzzy metric induced by d.

The topologies induced by the standard fuzzy metric and the corresponding metric are the same.

Theorem 1.6 [3]. *A sequence (x_n) in a fuzzy metric space $(X,M,*)$ converges to x if and only if $M(x_n,x,t)\to 1$ as $n\to\infty$.*

Definition 1.7 [3]. A sequence (x_n) in a fuzzy metric space $(X,M,*)$ is a Cauchy sequence if and only if for each $\varepsilon\in(0,1)$ and each $t>0$, there exists $n_0\in\mathbb{N}$ such that $M(x_n,x_m,t)>1-\varepsilon$ for all $n,m\geq n_0$.

A fuzzy metric space in which every Cauchy sequence is convergent is called a complete fuzzy metric space.

Lemma 1.8 [7]. *In a fuzzy metric space $(X,M,*)$, for any $r\in(0,1)$ there exists an $s\in(0,1)$ such that $s*s\geq r$.*

Lemma 1.9 [11]. *Let $(X,M,*)$ be a fuzzy metric space. Then M is a continuous function on $X\times X\times(0,+\infty)$.*

George and Veeramani [3] proved that every fuzzy metric M on X generates a topology τ_M on X. In addition, they showed that (X,τ_M) is a Housdorff first countable topological space. Moreover, if (X,d) is a metric space, then the topology generated by d coincides with the topology τ_{M_d} generated by the induced fuzzy metric M_d.

Definition 1.10 [6]. A fuzzy metric space $(X,M,*)$ is called precompact if for each $0<r<1$, and each $t>0$, there is a finite subset A of X, such that $X=\bigcup_{a\in A}B(a,r,t)$. In this case, we say that M is a precompact fuzzy metric on X.

Theorem 1.11 [6]. *A fuzzy metric space is percompact if and only if every sequence has a Cauchy subsequence.*

Definition 1.12 [6]. A fuzzy metric space $(X,M,*)$ is called compact if (X,τ_M) is a compact topological space.

Theorem 1.13 [6]. *A fuzzy metric space is compact if and only if it is precompact and complete.*

Definition 1.14 [7]. Let $(X,M,*)$ be a fuzzy metric space. We call the mapping $f : X \to X$ fuzzy contractive mapping, if there exists $k \in (0,1)$ such that

$$\frac{1}{M(f(x),f(y),t)} - 1 \leq k\left(\frac{1}{M(x,y,t)} - 1\right), \tag{1.1}$$

for each $x \neq y \in X$ and $t > 0$, (k is called the contractive constant of f).

Proposition 1.15 [7]. *Let $(X,M,*)$ be a fuzzy metric space. If $f : X \to X$ is fuzzy contractive mapping then f is t-uniformly continuous.*

Grabiec [5] proved a fuzzy Banach contraction theorem whenever fuzzy metric space was considered in the sense of Kramosil and Michálek and was complete in Grabiec's sense. Then Vasuki [14] generalized Grabiec's result for common fixed point theorem for a sequence of mapping in a fuzzy metric space. Gregori and Sapena [7] gave fixed point theorems for complete fuzzy metric space in the sense of George and Veeramani, and also for Kramosil and Michálek's fuzzy metric space which are complete in Grabiec's sense. George and Veeramani [3] have pointed out that the definition of Cauchy sequence given by Grabiec is weaker and hence it is essential to modify that definition to get better results in fuzzy metric space. Finally, Žikić [16] proved that the fixed point theorem of Gregori and Sapena holds under more general conditions (theory of countable extension of a t-norm).

In the next section, we are concerned with the implications of modifications in the assumptions. Exactly, in the absence of completeness of the space, we obtain some information on the convergence of a sequence of iterates. Finally, fuzzy Edelstein's theorem is proved for the fuzzy metric space in the George and Veeramani's sense.

2. Fixed point under contractive map

In this section, the definition of contractive map is rewritten and an iterative theorem is proved. In fact, this theorem shows the existence of a fixed point of a contractive map. In order to do this, we recall Definition 1.14 as follows:

Definition 2.1. Let $(X,M,*)$ be a fuzzy metric space. We call the mapping $f : X \to X$ fuzzy contractive mapping, if

$$\frac{1}{M(f(x),f(y),t)} - 1 < \left(\frac{1}{M(x,y,t)} - 1\right), \tag{2.1}$$

for each $x \neq y \in X$ and $t > 0$, or we call $f : X \to X$ fuzzy contractive mapping, if

$$M(f(x), f(y), t) > M(x, y, t), \tag{2.2}$$

for each $x \neq y \in X$ and $t > 0$.

Theorem 2.2. *Let $(X, M, *)$ be a fuzzy metric space, and A a fuzzy contractive mapping of X into itself such that there exists a point $x \in X$ whose sequence of iterates $(A^n(x))$ contains a convergent subsequence $(A^{n_i}(x))$; then $\xi = \lim_{i\to\infty} A^{n_i}(x) \in X$ is a unique fixed point.*

Proof. Suppose $A(\xi) \neq \xi$ and consider the sequence $(A^{n_i+1}(x))$ which, it can easily be verified, converges to $A(\xi)$.

For any fixed $t \in (0, +\infty)$, the mapping $r(p, q)$ of $Y = X \times X$ into the real line defined by

$$r(p, q) = \frac{M(A(p), A(q), t)}{M(p, q, t)}. \tag{2.3}$$

Note that A is a fuzzy contractive mapping of X into itself, and also Lemma 1.9 shows that M is continuous on $X \times X \times (0, +\infty)$. Thus r is a continuous function on Y. This shows that there exists a neighborhood U of $(\xi, A(\xi)) \in Y$ such that $p, q \in U$ implies

$$1 < R < r(p, q). \tag{2.4}$$

Let $B_1 = B_1(\xi, \rho, t)$ and $B_2 = B_2(A(\xi), \rho, t)$ be open neighborhoods centered at ξ and $A(\xi)$, respectively, and of radius $\rho > 0$ small enough such that $B_1 \bigcap B_2 = \Phi$ and $B_1 \times B_2 \subset U$.

By the assumption there exists a positive integer N such that $i > N$ implies $A^{n_i}(x) \in B_1$ and hence by (2.2) also $A^{n_i+1}(x) \in B_2$. On the other hand, for such i, it follows from (2.3) and (2.4) that

$$M(A^{n_i+1}(x), A^{n_i+2}(x), t) > RM(A^{n_i}(x), A^{n_i+1}(x), t). \tag{2.5}$$

A repeated use of (2.5) for $l > j > N$ now gives

$$\begin{aligned} M(A^{n_l}(x), A^{n_l+1}(x), t) &\geq M(A^{n_{l-1}+1}(x), A^{n_{l-1}+2}(x), t) \\ &> RM(A^{n_{l-1}}(x), A^{n_{l-1}+1}(x), t) \\ &\geq \cdots \\ &> R^{l-j} M(A^{n_j}(x), A^{n_j+1}(x), t) \longrightarrow \infty, \quad \text{as } l \longrightarrow \infty. \end{aligned} \tag{2.6}$$

Which is contradiction with the property (v) of fuzzy metric $(M, *)$ in Definition 1.2. Thus $A(\xi) = \xi$ and this means that ξ is a fixed point of A.

In order to prove the uniqueness of ξ, suppose there is an $\eta \neq \xi$ with $A(\eta) = \eta$, then it follows that

$$M(\xi,\eta,t) = M(A(\xi),A(\eta),t) > M(\xi,\eta,t), \tag{2.7}$$

which is contradiction. This proves the uniqueness and, thus, accomplishes the proof of this theorem. □

Theorem 2.2 will imply some information on the convergence of sequence of iterates.

Remark 2.3. Let all assumptions of Theorem 2.2 hold. If $(A^n(x))$, $x \in X$, contains a convergent subsequence $(A^{n_i}(x))$, then $\lim_{n\to\infty} A^n(x)$ exists and coincides with the fixed point ξ.

Proof. By Theorem 2.2 we have $\lim_{i\to\infty} A^{n_i}(x) = \xi$. Given $1 > \delta > 0$ there exists, then, a positive integers N_0 such that $i > N_0$ implies $M(\xi, A^{n_i}(x), t) > 1 - \delta$. If $m = n_i + l$ (n_i fixed, l variable) is any positive integer $> n_i$ then

$$M(\xi,A^m(x),t) = M(A^l(\xi),A^{n_i+l}(x),t) > M(\xi,A^{n_i}(x),t) > 1-\delta, \tag{2.8}$$

which proves the above assertion. □

Due to Theorems 1.11 and 1.13, in the fuzzy compact spaces, the following condition

$$\begin{gathered}\text{there exists a point } x \in X \text{ whose sequence of iterates}\\ (A^n(x)) \text{ contains a convergent subsequence } (A^{n_i}(x))\end{gathered} \tag{2.9}$$

is always satisfied. Thus fuzzy Edelstein's contractive theorem is as follows.

Remark 2.4. If X is a fuzzy compact space and A is a contractive self-mapping on X then there exists a unique fixed point of A.

Proof. It is easy to see by Theorems 1.13 and 2.2. □

Note that Remark 2.4 is considered when $(X,M,*)$ is a compact fuzzy metric space in the sense of George and Veeramani [3]. Also we can state this remark when $(X,M,*)$ is a compact fuzzy metric space in the sense of Kramosil and Michálek [9]. In order to do this, we prove the next lemma.

Lemma 2.5. *If $(X,M,*)$ is a compact fuzzy metric space in the sense of George and Veeramani, then it can be considered in the sense of Kramosil and Michálek.*

Proof. Let $M' : X^2 \times [0,\infty) \to [0,1]$ defined by

$$M'(x,y,t) = \begin{cases} M(x,y,t) & \text{for } x,y \in X,\ t > 0, \\ 0 & \text{for } x,y \in X,\ t = 0. \end{cases} \tag{2.10}$$

Then $(X,M',*)$ is a compact fuzzy metric space in the sense of Kramosil and Michálek. □

In the next section, the concept of periodic points or eventually fixed points in a fuzzy metric spaces is defined. Then the existence of at least one periodic point of ε-contractive self-mapping f on X is proved. Finally, two questions would arise.

3. Periodic points

In this section, first, we define a periodic point or an eventually fixed point. Then we prove the existence of a periodic point in a fuzzy metric space.

Definition 3.1. Let $(X,M,*)$ be a fuzzy metric space, and f is a self-mapping of X. Then ξ is a periodic point or an eventually fixed point, if there exists a positive integer k such that $f^k(\xi) = \xi$.

Definition 3.2. Let $(X,M,*)$ be a fuzzy metric space, we say that the mapping $f : X \to X$ is a fuzzy ε-contractive if there exists $0 < \varepsilon < 1$, such that if

$$1 - \varepsilon < M(x,y,t) < 1, \tag{3.1}$$

then

$$M(f(x), f(y), t) > M(x,y,t), \tag{3.2}$$

for all $t > 0$, and $x, y \in X$.

Theorem 3.3. *Let $(X,M,*)$ be a fuzzy metric space, where the continuous t-norm $*$ is defined as $a * b = \min\{a,b\}$ for $a,b \in [0,1]$. Suppose f is a fuzzy ε-contractive self-mapping of X such that*

$$\begin{gathered}\text{there exists a point } x \in X \text{ whose sequence of iterates} \\ (f^n(x)) \text{ contains a convergent subsequence } (f^{n_i}(x)),\end{gathered} \tag{3.3}$$

then $\xi = \lim_{i\to\infty} f^{n_i}(x)$ is a periodic point.

Proof. By the condition (3.3), there exists a positive integer N_1 such that $i > N_1$ implies

$$M(f^{n_i}(x), \xi, t) > 1 - \varepsilon, \tag{3.4}$$

for each $\varepsilon \in (0,1)$ and each $t > 0$.

Notice that f is a fuzzy ε-contractive, this fact and inequality (3.4) will imply

$$M(f^{n_i+1}(x), f(\xi), t) > M(f^{n_i}(x), \xi, t) \tag{3.5}$$

and so $M(f^{n_i+1}(x), f(\xi), t) > 1 - \varepsilon$. After $n_{i+1} - n_i$ iterations we obtain:

$$M(f^{n_{i+1}}(x), f^{n_{i+1}-n_i}(\xi), t) > 1 - \varepsilon. \tag{3.6}$$

Note that

$$M(\xi, f^{n_{i+1}-n_i}(\xi), t) \geq M(\xi, f^{n_{i+1}}(x), t_0) * M(f^{n_{i+1}}(x), f^{n_{i+1}-n_i}(\xi), t_1) > (1-\varepsilon) * (1-\varepsilon), \tag{3.7}$$

where $t = t_0 + t_1$. Due to the definition of $*$ which is $a * b = \min\{a, b\}$, we obtain:

$$M(\xi, f^{n_{i+1}-n_i}(\xi), t) > 1 - \varepsilon. \tag{3.8}$$

Suppose that $\eta = f^{n_{i+1}-n_i}(\xi) \neq \xi$. Now, For any fixed $t \in (0, +\infty)$, the mapping $r(p, q)$ of $Y = X \times X$ into the real line defined by

$$r(p, q) = \frac{M(f(p), f(q), t)}{M(p, q, t)}. \tag{3.9}$$

Note that f is a fuzzy contractive mapping of X into itself, and also Lemma 1.9 shows that M is continuous on $X \times X \times (0, +\infty)$. Thus r is a continuous function on Y. With respect to this fact that $r(\xi, \eta) > 1$, it is easy to see there exists a neighborhood U of $(\xi, \eta) \in Y$ such that $p, q \in U$ implies

$$r(p, q) > R > 1. \tag{3.10}$$

Let $B_1 = B_1(\xi, \rho, t)$ and $B_2 = B_2(\eta, \rho, t)$ be open neighborhoods centered at ξ and η, respectively, and of radius $0 < \rho < 1$ small enough such that

$$B_1 \cap B_2 = \phi, \tag{3.11}$$

or

$$0 < \rho < \frac{1}{3} M(\xi, \eta, t) < 1, \tag{3.7}$$

and $B_1 \times B_2 \subset U$.

A positive integer N_2 can now be found with the property that $j > N_2$ implies

$$(f^{n_j}(x), f^{n_j+n_{i+1}-n_i}(x)) \in B_1 \times B_2. \tag{3.12}$$

Since $B_1 \times B_2 \subset U$, then (3.10) will imply

$$M(f^{n_j+1}(x), f^{n_j+n_{i+1}-n_i+1}(x), t) > RM(f^{n_j}(x), f^{n_j+n_{i+1}-n_i}(x), t). \tag{3.13}$$

Consider $l > j > N_2$, two cases happen.
Case 1. If $n_l = n_{l-1} + 1$ then

$$M(f^{n_l}(x), f^{n_l+n_{i+1}-n_i}(x), t) = M(f^{n_{l-1}+1}(x), f^{n_{l-1}+n_{i+1}-n_i+1}(x), t). \tag{3.14}$$

Case 2. If $n_l > n_{l-1} + 1$ then by (3.13), and this fact that $R > 1$

$$M(f^{n_l}(x), f^{n_l+n_{i+1}-n_i}(x), t) > M(f^{n_{l-1}+1}(x), f^{n_{l-1}+n_{i+1}-n_i+1}(x), t). \tag{3.15}$$

Thus from (3.14) and (3.15), we have:

$$M\big(f^{n_l}(x), f^{n_l+n_{i+1}-n_i}(x), t\big) \geq M\big(f^{n_{l-1}+1}(x), f^{n_{l-1}+n_{i+1}-n_i+1}(x), t\big). \tag{3.16}$$

Also (3.13) help us to prove:

$$\begin{aligned} M\big(f^{n_{l-1}+1}(x), f^{n_{l-1}+n_{i+1}-n_i+1}(x), t\big) &> RM\big(f^{n_{l-1}}(x), f^{n_{l-1}+n_{i+1}-n_i}(x), t\big) \\ &\geq \cdots \\ &> R^{l-j}M\big(f^{n_j}(x), f^{n_j+n_{i+1}-n_i}(x), t\big). \end{aligned} \tag{3.17}$$

Hence

$$M\big(f^{n_l}(x), f^{n_l+n_{i+1}-n_i}(x), t\big) > R^{l-j}M\big(f^{n_j}(x), f^{n_j+n_{i+1}-n_i}(x), t\big) \longrightarrow \infty, \quad l \nearrow \infty, \tag{3.18}$$

which is clearly incompatible with the property (v) of Definition 1.2. Hence, putting $k = n_{i+1} - n_i$, we have $f^k(\xi) = \xi$ as asserted. □

Corollary 3.4. *If $(X, M, *)$ is a compact fuzzy metric space and f is a fuzzy ε-contractive self-mapping of X then there exists at least one periodic point.*

Proof. It is easy to see by Theorems 1.11 and 3.3. □

Corollary 3.5. *If, in Theorem 3.3, $M(\xi, f(\xi), t) > 1 - \varepsilon$, then there is a contradiction.*

Proof. Note that $M(\xi, f(\xi), t) > 1 - \varepsilon$, and also f is a fuzzy ε-contractive, thus we have

$$M\big(f^2(\xi), f(\xi), t\big) > M\big(f(\xi), \xi, t\big). \tag{3.19}$$

After $k+1$ iterations we obtain

$$M\big(f^{k+1}(\xi), f^k(\xi), t\big) > M\big(f(\xi), \xi, t\big). \tag{3.20}$$

By putting $f^k(\xi) = \xi$ in the above inequality, we find

$$M\big(f(\xi), \xi, t\big) > M\big(f(\xi), \xi, t\big). \tag{3.21}$$

This is a contradiction. □

Question 3.6. It is natural to ask whether Theorem 2.2 would remain true if (3.4) is substituted by a localized version such as

$$p \neq q, p, q \in B\big(x, \varepsilon(x), t\big) \text{ implies } M\big(f(p), f(q), t\big) > M(p, q, t), \tag{3.22}$$

where $B(x, \varepsilon(x), t) = \{z \in X \mid M(z, x, t) > 1 - \varepsilon(x)\}$.

Question 3.7. It is natural to ask whether Theorem 3.3 would remain true if $*$ is replaced by an arbitrary t-norm.

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