AN EXPLICIT CONSTRUCTION OF SUNNY NONEXPANSIVE RETRACTIONS IN BANACH SPACES

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Received 22 March 2005

An explicit algorithmic scheme for constructing the unique sunny nonexpansive retraction onto the common fixed point set of a nonlinear semigroup of nonexpansive mappings in a Banach space is analyzed and a proof of convergence is given.

1. Introduction

Throughout this paper all vector spaces are real and we denote by $\mathbb{N}$ and $\mathbb{R}_+$ the set of nonnegative integers and nonnegative real numbers, respectively. Let $(X, \|\cdot\|)$ be a Banach space and let X^* be its dual. The value of $y \in X^*$ at $x \in X$ will be denoted by $\langle x, y\rangle$. We also denote by $J : X \to 2^{X^*}$ the normalized duality map from X into the family of nonempty (by the Hahn-Banach theorem) weak-star compact convex subsets of X^*, which is defined by $Jx = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$ for all $x \in X$. The Banach space X is said to be smooth or to have a Gâteaux differentiable norm if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists for each $x, y \in X$ with $\|x\| = \|y\| = 1$. The space X is said to have a uniformly Gâteaux differentiable norm if, for each $y \in X$ with $\|y\| = 1$, the limit (1.1) is attained uniformly in $x \in X$ with $\|x\| = 1$. It is known [12, Lemma 2.2] that if the norm of X is uniformly Gâteaux differentiable, then the duality map is single-valued and norm to weak star uniformly continuous on each bounded subset of X. Let C be a nonempty, closed and convex subset of X and E be a nonempty subset of C. A mapping $Q : C \to X$ is *nonexpansive* if $\|Qx - Qy\| \le \|x - y\|$ for all $x, y \in C$. A mapping $Q : C \to E$ is called a *retraction* from C onto E if $Qx = x$ for all $x \in E$. A retraction Q from C onto E is called *sunny* if Q has the following property: $Q(Qx + t(x - Qx)) = Qx$ for all $x \in C$ and $t \ge 0$ with $Qx + t(x - Qx) \in C$. It is known [6, Lemma 13.1] that in a smooth Banach space X, a retraction Q from C onto E is both sunny and nonexpansive if and only if

$$\langle x - Qx, J(y - Qx)\rangle \le 0 \tag{1.2}$$

for all $x \in C$ and $y \in E$. Hence, there is at most one sunny nonexpansive retraction from C onto E. For example, if E is a nonempty, closed and convex subset of a Hilbert space

Fixed Point Theory and Applications 2005:3 (2005) 295–305
DOI: 10.1155/FPTA.2005.295

H, then the nearest point projection P_E from H onto E is the unique sunny nonexpansive retraction of H onto E. This is not true for all Banach spaces, since outside Hilbert space, nearest point projections, although sunny, are no longer nonexpansive. On the other hand, sunny nonexpansive retractions do sometimes play a similar role in Banach spaces to that of nearest point projections in a Hilbert space. So an interesting problem arises: for which subsets of a Banach space does a sunny nonexpansive retraction exist? If it does exist, how can one find it? It is known [6, Theorem 13.2] that if C is a closed convex subset of a uniformly smooth Banach space and $T : C \to C$ is nonexpansive, then the fixed point set of T is a sunny nonexpansive retract of C. More generally, Bruck [3, Theorem 2] proves that if C is a closed convex subset of a reflexive Banach space every bounded, closed and convex subset of which has the fixed point property for nonexpansive mappings and $T : C \to C$ is nonexpansive, then its fixed point set is a nonexpansive retract of C. (It is still an open question whether all bounded, closed and convex subsets of reflexive Banach spaces have this fixed point property.) For a weak sufficient condition on the underlying space which guarantees that nonexpansive retracts are, in fact, sunny nonexpansive retracts see [10, Theorem 4.1]. In the present paper we show that if F is the nonempty common fixed point set of a commuting family of nonexpansive self-mappings of closed convex subsets C of certain Banach spaces X, satisfying an asymptotic regularity condition, then it is possible to construct the sunny nonexpansive retraction Q of C onto F in an explicit iterative way. The origin of our work lies in a recent publication by Domínguez Benavides, López Acedo and Xu [5] who attempted to construct sunny nonexpansive retractions using both implicit and explicit iterative schemes (cf. the discussion in [1]). Our work improves, corrects and generalizes some of the results obtained in the above paper. It is also related to a result of Reich [11], where the case of a single mapping is dealt with. In this connection we would also like to refer the interested reader to the results obtained by Suzuki [14] who deals with an implicit scheme for constructing the sunny nonexpansive retraction onto the common fixed point set of some one-parameter semigroups of nonexpansive mappings. For related results in Hilbert space see Aleyner and Censor [1], Bauschke [2], Deutsch and Yamada [4], Halpern [7], Lions [8], and Wittmann [15].

2. Preliminaries and notations

Let l^∞ denote the real Banach space of all bounded sequences $a = (a_1, a_2, \ldots)$ with the norm defined by $\|a\| = \sup_n |a_n|$. A continuous linear functional LIM on l^∞ is called a *Banach limit* when LIM satisfies $\mathrm{LIM}(a) \geq 0$ if $a_n \geq 0$, $n = 1, 2, \ldots$, $\mathrm{LIM}(\{a_n\}) = \mathrm{LIM}(\{a_{n+1}\})$ and $\|\mathrm{LIM}\| = \mathrm{LIM}(1) = 1$. To prove our theorem, we need the following two propositions [13, Propositions 1 and 2], which can be deduced from the arguments in the proof of [9, Theorem 1]. We sketch their proofs for the sake of completeness.

Proposition 2.1. *Let α be a real number and let $a = (a_1, a_2, \ldots) \in l^\infty$. Then* $\mathrm{LIM}(a) \leq \alpha$ *for all Banach limits LIM if and only if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\frac{a_k + a_{k+1} + \cdots + a_{k+n-1}}{n} < \alpha + \varepsilon \tag{2.1}$$

for all $n \geq n_0$ and $k \in \mathbb{N}$.

Proof. First we prove the necessity of (2.1). Assume $\mathrm{LIM}(a) \le \alpha$ for all Banach limits LIM. Define a sublinear functional β from l^∞ into the real line $\mathbb{R}$ by

$$\beta((b_1,b_2,\dots)) = \limsup_{n\to\infty} \sup_{k\in\mathbb{N}} \frac{1}{n} \sum_{i=k}^{k+n-1} b_i, \tag{2.2}$$

where $(b_1,b_2,\dots) \in l^\infty$. By the Hahn-Banach theorem, there exists a linear functional μ from l^∞ into $\mathbb{R}$ such that $\mu \le \beta$ and $\mu(a) = \beta(a)$. It is not difficult to see that μ is a Banach limit. Since $\mu(a) \le \alpha$, there exists, for each $\varepsilon > 0$, a natural number $n_0 \in \mathbb{N}$ which satisfies (2.1). Next we prove that (2.1) is sufficient. Let μ be a Banach limit and let $\varepsilon > 0$. By the hypothesis, there exists n_0 such that (2.1) is satisfied. Hence we have

$$\mu(a) = \mu\left(\frac{a_k + a_{k+1} + \cdots + a_{k+n_0-1}}{n_0}\right) \le \alpha + \varepsilon. \tag{2.3}$$

Since ε is an arbitrary positive number, we see that $\mu(a) \le \alpha$. □

Proposition 2.2. *Let α be a real number and let $a = (a_1,a_2,\dots) \in l^\infty$ be such that*

$$\mathrm{LIM}(a) \le \alpha \tag{2.4}$$

for all Banach limits LIM, and

$$\limsup_{n\to\infty} (a_{n+1} - a_n) \le 0. \tag{2.5}$$

Then

$$\limsup_{n\to\infty} a_n \le \alpha. \tag{2.6}$$

Proof. Let $\varepsilon > 0$. By Proposition 2.1, there exists $n \ge 2$ such that

$$\frac{a_k + a_{k+1} + \cdots + a_{k+n-1}}{n} < \alpha + \frac{\varepsilon}{2} \tag{2.7}$$

for all $k \in \mathbb{N}$. Choose $k_0 \in \mathbb{N}$ such that $a_{k+1} - a_k < \varepsilon/(n-1)$ for all $k \ge k_0$. Let $k \ge k_0 + n$. Then we have

$$a_k = a_{k-i} + (a_{k-i+1} - a_{k-i}) + (a_{k-i+2} - a_{k-i+1}) + \cdots + (a_k - a_{k-1}) \le a_{k-i} + \frac{i\varepsilon}{n-1} \tag{2.8}$$

for each $i = 0,1,2,\dots,n-1$. So we obtain

$$a_k \le \frac{a_k + a_{k-1} + \cdots + a_{k-n+1}}{n} + \frac{1}{n} \cdot \frac{n(n-1)}{2} \cdot \frac{\varepsilon}{n-1} \le \alpha + \varepsilon. \tag{2.9}$$

Hence we have

$$\limsup_{k\to\infty} a_k \le \alpha + \varepsilon. \tag{2.10}$$

Since ε is an arbitrary positive number, the proposition is proved. □

3. Convergence theorem

Let X be a Banach space, C a nonempty, closed and convex subset of X, G an unbounded subset of $\mathbb{R}_+$ such that

$$\begin{gathered} t+h \in G \quad \forall t,h \in G, \\ t-h \in G \quad \forall t,h \in G \text{ with } t \geq h, \end{gathered} \tag{3.1}$$

and $\Gamma = \{T_t : t \in G\}$ a family of nonexpansive self-mappings of C such that the set F of the common fixed points of Γ is nonempty. We make the following assumptions.

Assumptions on the space. X is a reflexive Banach space with a uniformly Gâteaux differentiable norm such that each nonempty, bounded, closed and convex subset K of X has the common fixed point property for nonexpansive mappings; that is, any family of commuting nonexpansive self-mappings of K has a common fixed point. Note that all these assumptions are fulfilled whenever X is uniformly smooth.

Assumptions on the mappings. Γ is a uniformly asymptotically regular semigroup on bounded subsets of C, that is,

$$T_{s+t}x = T_s T_t x \tag{3.2}$$

for all $t,s \in G$, $x \in C$, and for all bounded subsets K of C there holds

$$\lim_{r\to\infty} \sup_K \|T_s T_r x - T_r x\| = 0, \tag{3.3}$$

uniformly for all $s \in G$. Note that both these assumptions hold when the trajectories of the semigroup Γ converge uniformly on bounded subsets of X.

Assumptions on the parameters. $\{\lambda_n\}$ is a sequence of numbers in $[0,1)$ with the following properties:

$$\lambda_n \longrightarrow 0, \tag{3.4}$$

$$\prod_{n=0}^{\infty}(1-\lambda_n) = 0; \quad \text{equivalently, } \sum_{n=0}^{\infty} \lambda_n = \infty, \tag{3.5}$$

$$\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty. \tag{3.6}$$

Observe that given points $f \in F$, $u, x_0 \in C$, and the bounded subset $D = \{x \in C : \|x - f\| \leq \max(\|x_0 - f\|, \|u - f\|)\}$, there exists a sequence $\{r_n\} \subseteq G$ such that

$$r_0 < r_1 < r_2 < \cdots < r_n < \cdots, \quad \lim_{n\to\infty} r_n = \infty, \tag{3.7}$$

$$\sum_{n=0}^{\infty} \sup_D \|T_s T_{r_n} x - T_{r_n} x\| < \infty, \tag{3.8}$$

uniformly for all $s \in G$. We now define the sequence $\{x_n\}$ by

$$x_{n+1} = \lambda_n u + (1-\lambda_n) T_{r_n} x_n, \tag{3.9}$$

where $n \geq 0$; we say that $\{x_n\}$ has anchor u and initial point x_0.

Theorem 3.1. *If the above assumptions on the space, mappings and parameters hold, then the sequence generated by (3.9) converges in norm to Qu, where Q is the unique sunny nonexpansive retraction from C onto F.*

Proof. We first prove the result for the special case $x_0 = u$ and then extend it to the general case. We divide our proof into a sequence of separate claims.

Claim 3.2. For all $n \geq 0$ and every $f \in F$,

$$\|x_n - f\| \leq \|u - f\|. \tag{3.10}$$

We proceed by induction on n. Fix $f \in F$. Clearly, (3.10) holds for $n = 0$. If $\|x_n - f\| \leq \|u - f\|$, then

$$\begin{aligned} \|x_{n+1} - f\| &\leq \lambda_n \|u - f\| + (1-\lambda_n)\|T_{r_n}x_n - f\| \\ &\leq \lambda_n \|u - f\| + (1-\lambda_n)\|x_n - f\| \\ &\leq \|u - f\|, \end{aligned} \tag{3.11}$$

as required.

Claim 3.3. The following strong convergence holds:

$$x_{n+1} - T_{r_n}x_n \longrightarrow 0. \tag{3.12}$$

This is true because (3.10) guarantees that $\{x_n\}$ is bounded, which, in turn, implies that $\{T_{r_n}x_n\}$ is also bounded. The boundedness of $\{T_{r_n}x_n\}$ together with (3.4) imply, in view of (3.9), our assertion.

Claim 3.4. The differences of consecutive iterates strongly converge to zero, namely,

$$x_{n+1} - x_n \longrightarrow 0. \tag{3.13}$$

Indeed, it follows from (3.10) that $x_n \in D$ for all $n \geq 0$. By the boundedness of $\{x_n\}$ and $\{T_{r_n}x_n\}$ there exists some constant $L \geq 0$ such that $\|x_{n+1} - x_n\| \leq L$ and $\|u - T_{r_n}x_n\| \leq L$ for all $n \geq 0$. Therefore, for all $n \geq 1$ we get

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(\lambda_n - \lambda_{n-1})(u - T_{r_{n-1}}x_{n-1}) + (1-\lambda_n)(T_{r_n}x_n - T_{r_{n-1}}x_{n-1})\| \\ &\leq \|(\lambda_n - \lambda_{n-1})(u - T_{r_{n-1}}x_{n-1})\| + \|(1-\lambda_n)(T_{r_n}x_n - T_{r_n}x_{n-1})\| \\ &\quad + \|(1-\lambda_n)(T_{r_n}x_{n-1} - T_{r_{n-1}}x_{n-1})\| \\ &\leq |\lambda_n - \lambda_{n-1}|\,\|u - T_{r_{n-1}}x_{n-1}\| + (1-\lambda_n)\|x_n - x_{n-1}\| \\ &\quad + \|T_{r_n}x_{n-1} - T_{r_{n-1}}x_{n-1}\| \\ &\leq |\lambda_n - \lambda_{n-1}|\,L + (1-\lambda_n)\|x_n - x_{n-1}\| \\ &\quad + \|T_{r_n}x_{n-1} - T_{r_{n-1}}x_{n-1}\|. \end{aligned} \tag{3.14}$$

Since Γ is a semigroup, we are able to rewrite the last term as follows:

$$||T_{r_n}x_{n-1} - T_{r_{n-1}}x_{n-1}|| = ||T_{r_n - r_{n-1}}T_{r_{n-1}}x_{n-1} - T_{r_{n-1}}x_{n-1}||. \tag{3.15}$$

Thus

$$\begin{aligned} ||x_{n+1} - x_n|| \le & \; |\lambda_n - \lambda_{n-1}|L + (1-\lambda_n)||x_n - x_{n-1}|| \\ & + ||T_{r_n - r_{n-1}}T_{r_{n-1}}x_{n-1} - T_{r_{n-1}}x_{n-1}|| \end{aligned} \tag{3.16}$$

for all $n \ge 1$. Hence, inductively,

$$\begin{aligned} ||x_{n+1} - x_n|| \le & \; L\sum_{k=m}^{n} |\lambda_k - \lambda_{k-1}| + ||x_m - x_{m-1}||\prod_{k=m}^{n}(1-\lambda_k) \\ & + \sum_{k=m}^{n} ||T_{r_k - r_{k-1}}T_{r_{k-1}}x_{k-1} - T_{r_{k-1}}x_{k-1}||, \end{aligned} \tag{3.17}$$

for all $n \ge m \ge 1$. Taking now the limit as n tends to $+\infty$, we obtain

$$\begin{aligned} & \limsup_{n\to\infty} ||x_{n+1} - x_n|| \\ & \quad \le L\sum_{k=m}^{\infty} |\lambda_k - \lambda_{k-1}| + L\prod_{k=m}^{\infty}(1-\lambda_k) + \sum_{k=m}^{\infty}\sup_D ||T_{r_k - r_{k-1}}T_{r_{k-1}}x - T_{r_{k-1}}x|| \\ & \quad = L\sum_{k=m}^{\infty} |\lambda_k - \lambda_{k-1}| + \sum_{k=m}^{\infty}\sup_D ||T_{r_k - r_{k-1}}T_{r_{k-1}}x - T_{r_{k-1}}x|| \end{aligned} \tag{3.18}$$

by (3.5). On the other hand, conditions (3.6) and (3.8) imply that

$$\begin{aligned} \lim_{m\to\infty}\sum_{k=m}^{\infty} |\lambda_k - \lambda_{k-1}| &= 0, \\ \lim_{m\to\infty}\sum_{k=m}^{\infty}\sup_D ||T_{r_k - r_{k-1}}T_{r_{k-1}}x - T_{r_{k-1}}x|| &= 0. \end{aligned} \tag{3.19}$$

Altogether, by letting m tend to ∞, we conclude that $x_{n+1} - x_n \to 0$, as claimed.

Claim 3.5. For each fixed $s \in G$,

$$T_s x_n - x_n \longrightarrow 0. \tag{3.20}$$

Indeed, let $s \in G$. Then

$$\begin{aligned} ||T_s x_n - x_n|| &\le ||T_s x_n - T_s T_{r_n}x_n|| + ||T_s T_{r_n}x_n - T_{r_n}x_n|| + ||T_{r_n}x_n - x_n|| \\ &\le 2||x_n - T_{r_n}x_n|| + \sup_D ||T_s T_{r_n}x - T_{r_n}x|| \\ &\le 2(||x_n - x_{n+1}|| + ||x_{n+1} - T_{r_n}x_n||) + \sup_D ||T_s T_{r_n}x - T_{r_n}x||. \end{aligned} \tag{3.21}$$

Combining (3.12), (3.13), and (3.8), we see that $T_s x_n - x_n \to 0$, as asserted.

Let LIM be a Banach limit and let $\{\alpha_s\}_{s\in G}$ be a net in the interval $(0,1)$ such that $\lim_{s\to\infty}\alpha_s = 0$. By Banach's fixed point theorem, for each $s\in G$, there exists a unique point $z_s\in C$ satisfying the equation $z_s=\alpha_s u+(1-\alpha_s)T_s z_s$. Since the following claim is essentially proved in [5], we include only a sketch of its proof.

Claim 3.6.

$$z_s \longrightarrow Qu, \tag{3.22}$$

where $Q: C\to F$ is the unique sunny nonexpansive retraction from C onto F.

Indeed, let $\{s_n\}$ be a subsequence of G such that $\lim_{n\to\infty} s_n=\infty$. Since $\{z_{s_n}\}$ is bounded, we can define a functional g on C by

$$g(x)=\mathrm{LIM}\left(\left\{\|z_{s_n}-x\|^2\right\}\right). \tag{3.23}$$

We have for each $r\in G$,

$$\begin{aligned} g(T_r x) &= \mathrm{LIM}\left(\left\{\|z_{s_n}-T_r x\|^2\right\}\right)=\mathrm{LIM}\left(\left\{\|T_r T_{s_n} z_{s_n}-T_r x\|^2\right\}\right)\\ &\le \mathrm{LIM}\left(\left\{\|T_{s_n} z_{s_n}-x\|^2\right\}\right)\\ &= \mathrm{LIM}\left(\left\{\|z_{s_n}-x\|^2\right\}\right), \end{aligned} \tag{3.24}$$

by (3.3). In other words,

$$g(T_r x)\le g(x) \tag{3.25}$$

for all $r\in G$ and $x\in C$. Let $K=\{x\in C: g(x)=\min_C g\}$. Since g is convex and continuous, $\lim_{\|x\|\to\infty} g(x)=\infty$ and X is reflexive, K is a nonempty, closed, bounded and convex subset of C. From (3.25) we see that K is invariant under each T_r; that is, $T_r(K)\subset K$, $r\in G$. Hence K contains a common fixed point of Γ. Let $q\in K\bigcap F$ be such a common fixed point. Since q is a minimizer of g over C, it follows that for each $x\in C$,

$$\begin{aligned} 0 &\le \lim_{\lambda\to 0^+}\frac{1}{\lambda}\big(g(q+\lambda(x-q))-g(q)\big)\\ &= \mathrm{LIM}\left(\left\{\lim_{\lambda\to 0^+}\frac{1}{\lambda}\left(\|(z_{s_n}-q)+\lambda(q-x)\|^2-\|z_{s_n}-q\|^2\right)\right\}\right)\\ &= \mathrm{LIM}\left(\left\{2\langle q-x, J(z_{s_n}-q)\rangle\right\}\right). \end{aligned} \tag{3.26}$$

Thus,

$$\mathrm{LIM}\left(\left\{\langle x-q, J(z_{s_n}-q)\rangle\right\}\right)\le 0 \tag{3.27}$$

for all $x\in C$. On the other hand, for any $f\in F$,

$$z_{s_n}-f=(1-\alpha_{s_n})(T_{s_n}z_{s_n}-f)+\alpha_{s_n}(u-f). \tag{3.28}$$

It follows that

$$\begin{aligned}||z_{s_n}-f||^2 &= (1-\alpha_{s_n})\langle T_{s_n}z_{s_n}-f,J(z_{s_n}-f)\rangle+\alpha_{s_n}\langle u-f,J(z_{s_n}-f)\rangle \\ &\le (1-\alpha_{s_n})||z_{s_n}-f||^2+\alpha_{s_n}\langle u-f,J(z_{s_n}-f)\rangle.\end{aligned} \tag{3.29}$$

Hence

$$||z_{s_n}-f||^2 \le \langle u-f,J(z_{s_n}-f)\rangle. \tag{3.30}$$

Combining (3.27) and (3.30), we get

$$\text{LIM}\left(\left\{||z_{s_n}-q||^2\right\}\right) \le 0. \tag{3.31}$$

Hence there is a subsequence $\{z_{r_j}\}$ of $\{z_{s_n}\}$ such that $\lim_{j\to\infty}\|z_{r_j}-q\|=0$. Assume that there exists another subsequence $\{z_{p_k}\}$ of $\{z_{s_n}\}$ such that $\lim_{k\to\infty}\|z_{p_k}-\tilde{q}\|=0$, where $\tilde{q}\in K\bigcap F$. Then (3.30) implies that

$$||q-\tilde{q}||^2 \le \langle u-\tilde{q},J(q-\tilde{q})\rangle. \tag{3.32}$$

Similarly we have

$$||\tilde{q}-q||^2 \le \langle u-q,J(\tilde{q}-q)\rangle. \tag{3.33}$$

Adding up (3.32) and (3.33) we obtain $q=\tilde{q}$. Therefore $\{z_s\}$ converges in norm to a point in F. Now we define $Q: C\to F$ by $Qu=\lim_{s\to\infty}z_s$. Then Q is a retraction from C onto F. Moreover, by (3.30) we get for all $f\in F$,

$$||Qu-f||^2 \le \langle u-f,J(Qu-f)\rangle. \tag{3.34}$$

That is,

$$\langle u-Qu,J(f-Qu)\rangle \le 0 \tag{3.35}$$

for all $f\in F$. Therefore Q is the unique sunny nonexpansive retraction from C onto F (see (1.2)).

Claim 3.7.

$$\limsup_{n\to\infty}\langle u-Qu,J(x_n-Qu)\rangle \le 0. \tag{3.36}$$

Since T_s is nonexpansive, (3.20) implies that

$$\text{LIM}\left(\left\{||x_n-T_sz_s||^2\right\}\right)=\text{LIM}\left(\left\{||T_sx_n-T_sz_s||^2\right\}\right)\le \text{LIM}\left(\left\{||x_n-z_s||^2\right\}\right). \tag{3.37}$$

Since $(1-\alpha_s)(x_n - T_s z_s) = (x_n - z_s) - \alpha_s(x_n - u)$, we have

$$\begin{aligned}(1-\alpha_s)^2 \|x_n - T_s z_s\|^2 &\geq \|x_n - z_s\|^2 - 2\alpha_s \langle x_n - u, J(x_n - z_s)\rangle \\ &= \|x_n - z_s\|^2 - 2\alpha_s \langle x_n - z_s + z_s - u, J(x_n - z_s)\rangle \\ &= \|x_n - z_s\|^2 - 2\alpha_s \langle x_n - z_s, J(x_n - z_s)\rangle - 2\alpha_s \langle z_s - u, J(x_n - z_s)\rangle \\ &= (1-2\alpha_s)\|x_n - z_s\|^2 + 2\alpha_s \langle u - z_s, J(x_n - z_s)\rangle.\end{aligned} \tag{3.38}$$

Therefore

$$\begin{aligned}&(1-\alpha_s)^2 \operatorname{LIM}\left(\left\{\|x_n - z_s\|^2\right\}\right) \\ &\quad \geq (1-2\alpha_s)\operatorname{LIM}\left(\left\{\|x_n - z_s\|^2\right\}\right) + 2\alpha_s \operatorname{LIM}\left(\left\{\langle u - z_s, J(x_n - z_s)\rangle\right\}\right)\end{aligned} \tag{3.39}$$

for each $n \geq 0$. These inequalities yield

$$\frac{\alpha_s}{2}\operatorname{LIM}\left(\left\{\|x_n - z_s\|^2\right\}\right) \geq \operatorname{LIM}\left(\left\{\langle u - z_s, J(x_n - z_s)\rangle\right\}\right). \tag{3.40}$$

Since

$$\begin{aligned}&\langle u - z_s, J(x_n - z_s)\rangle - \langle u - Qu, J(x_n - Qu)\rangle \\ &\quad = \langle u - z_s - (u - Qu), J(x_n - z_s)\rangle + \langle u - Qu, J(x_n - z_s) - J(x_n - Qu)\rangle,\end{aligned} \tag{3.41}$$

we obtain by letting s tend to ∞ that

$$0 \geq \operatorname{LIM}\left(\left\{\langle u - Qu, J(x_n - Qu)\rangle\right\}\right) \tag{3.42}$$

because X has a uniformly Gâteaux differentiable norm and (3.22) holds. On the other hand, we have

$$\lim_{n\to\infty} \left|\langle u - Qu, J(x_{n+1} - Qu)\rangle - \langle u - Qu, J(x_n - Qu)\rangle\right| = 0 \tag{3.43}$$

by (3.13). Hence we obtain by Proposition 2.2,

$$\limsup_{n\to\infty} \langle u - Qu, J(x_n - Qu)\rangle \leq 0, \tag{3.44}$$

as claimed.

Now we can conclude the proof for the special case $x_0 = u$.

Claim 3.8.

$$x_n \longrightarrow Qu. \tag{3.45}$$

Indeed, since

$$(1-\lambda_n)(T_{r_n} x_n - Qu) = (x_{n+1} - Qu) - \lambda_n(u - Qu), \tag{3.46}$$

we have

$$\|(1-\lambda_n)(T_{r_n}x_n - Qu)\|^2 \geq \|x_{n+1} - Qu\|^2 - 2\lambda_n\langle u - Qu, J(x_{n+1} - Qu)\rangle. \tag{3.47}$$

Hence

$$\|x_{n+1} - Qu\|^2 \leq (1-\lambda_n)\|x_n - Qu\|^2 + 2(1-(1-\lambda_n))\langle u - Qu, J(x_{n+1} - Qu)\rangle \tag{3.48}$$

for each $n \geq 0$. Let $\varepsilon > 0$ be given. By (3.36), there exists $m \geq 0$ such that

$$\langle u - Qu, J(x_n - Qu)\rangle \leq \frac{\varepsilon}{2} \tag{3.49}$$

for all $n \geq m$. Therefore

$$\|x_{n+m} - Qu\|^2 \leq \left(\prod_{k=m}^{n+m-1}(1-\lambda_k)\right)\|x_m - Qu\|^2 + \left(1 - \prod_{k=m}^{n+m-1}(1-\lambda_k)\right)\varepsilon \tag{3.50}$$

for all $n \geq 1$. Hence by (3.5) we get

$$\limsup_{n\to\infty}\|x_n - Qu\|^2 = \limsup_{n\to\infty}\|x_{n+m} - Qu\|^2 \leq \varepsilon. \tag{3.51}$$

Since ε is an arbitrary positive real number, we conclude that $\{x_n\}$ converges strongly to Qu; that is, the special case is verified.

Finally, we extend the proof to the general case. Let $\{x_n\}$ be the sequence generated by (3.9) with an initial point x_0 (possibly different from u) and let $\{y_n\}$ be another sequence generated by (3.9) with an initial point $y_0 = u$. On the one hand, by the special case,

$$y_n \longrightarrow Qu. \tag{3.52}$$

On the other hand, it is easily checked that

$$\|x_n - y_n\| \leq \|x_0 - y_0\|\prod_{k=0}^{n-1}(1-\lambda_k) \tag{3.53}$$

for all $n \geq 1$. Thus, $x_n - y_n \to 0$ and, altogether, $x_n \to Qu$. □

Note added in proof. We are now able to prove Theorem 3.1 under much weaker assumptions on the mappings and the parameters. We expect the details to be part of a forthcoming paper.

Acknowledgments

The research of the second author was partially supported by the Israel Science Foundation founded by the Israel Academy of Sciences and Humanities (Grant 592/00), by the Fund for the Promotion of Research at the Technion, and by the Technion VPR Fund.

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