FIXED POINT RESULTS AND THEIR APPLICATIONS TO MARKOV PROCESSES

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New existence and comparison results are proved for fixed points of increasing operators and for common fixed points of operator families in partially ordered sets. These results are then applied to derive existence and comparison results for invariant measures of Markov processes in a partially ordered Polish space.

1. Introduction

A. Tarski proved in his fundamental paper [18] that the set Fix(G) of fixed points of any increasing self-mapping G of a complete lattice is also a complete lattice. Davis completed this work by showing in [3] that a lattice is complete if each of its increasing self-mappings has a fixed point. As a generalization of this result Markowsky proved in [16] that each self-mapping G of a partially ordered set (poset) X has the least fixed point if and only if each chain of X, also the empty chain, has the supremum, and that in such a case each chain of Fix(G) has the supremum in Fix(G) (see also [2]).

In [9, 10] it is shown that if $G : X \to X$ is increasing, if nonempty well-ordered (w.o.) and inversely well-ordered (i.w.o.) subsets of $G[X]$ have supremums and infimums in X, and if for some $c \in X$ either supremums or infimums of $\{c,x\}$ exist for each $x \in X$, then G has maximal or minimal fixed points, and least or greatest fixed points in certain order intervals of X. Applications of these results to operator equations, as well as various types of explicit and implicit differential equations are presented, for example, in [1, 8, 9, 10]. To meet the demands of our applications to Markov processes we will prove in Section 2 similar fixed point results when the existence of supremums or infimums of $\{c,x\}$, $x \in X$, is replaced by weaker hypotheses. Results on the structure of the fixed point set are also derived. The proofs are based on a recursion principle introduced in [11].

In [18] existence of common fixed points is also proved for commutative families of increasing self-mappings of a complete lattice X. As for generalizations of these results, see, for example [4, 16, 19]. In Section 3 we derive existence results for common fixed points of a family of mappings $G_t : X \to X$, $t \in S$, where S is a nonempty set, in cases when for some $t_0 \in S$ results of Section 2 are applicable to $G = G_{t_0}$, when (i) $G_t G_{t_0} = G_{t_0} G_t$ for each $t \in S$, and when (ii) either $G_{t_0} x \le G_t x$ or $G_t x \le G_{t_0} x$ for all $t \in S$ and $x \in X$. For

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instance, if X is a closed ball in $\mathbb{R}^m$, ordered coordinatewise, a family $\{G_t\}_{t\in S}$ has a common fixed point if G_{t_0} is increasing and satisfies (i) and (ii) (cf. Example 3.4). The results of Section 2 can also be applied to prove the existence of increasing selectors for fixed points of an increasing family of increasing mappings (cf. Remarks 3.3 and Example 3.4).

The obtained results are then applied in Section 4 to prove existence and comparison results for invariant measures of Markov processes in a partially ordered Polish space E. Such results have applications in ergodic theory, in economics and in statistics (see, e.g., [13, 17, 20]). No compactness hypotheses are imposed on E.

Examples are given to demonstrate the obtained results. For instance, the example of Subsection 4.4 is constructed to justify the need of the new fixed point results derived in Section 2.

2. Fixed point results

In this section $X = (X, \leq)$ is a poset. Recall that a subset C of X is *well-ordered* (respectively *inversely well-ordered*) if each nonempty subset of C has the least (respectively greatest) element.

When $a, b \in X$, $a \leq b$, we denote

$$[a) = \{x \in X \mid a \leq x\}, \qquad (a] = \{x \in X \mid x \leq a\}, \qquad [a,b] = \{x \in X \mid a \leq x \leq b\}. \tag{2.1}$$

Given a subset Y of X we say that a mapping $G : X \to X$ is *increasing* in Y if $Gx \leq Gy$ whenever $x, y \in Y$ and $x \leq y$. We say that $x \in Y$ is the *least fixed point* of G in Y if $x = Gx$, and if $x \leq y$ whenever $y \in Y$ and $y = Gy$. The greatest fixed point of G in Y is defined similarly, by reversing the inequality. A fixed point x of G is called *maximal* if $x = y$ whenever $y = Gy$ and $x \leq y$, and *minimal* if $y = Gy$ and $y \leq x$ imply $x = y$.

A nonempty subset Y of X is called *relatively well-order complete* if each nonempty w.o. or i.w.o. subset of Y has supremums and infimums in X. If these supremums and infimums belong to Y, we say that Y is *well-order complete*. Denote

$$\begin{aligned} Y_+ &= \{y \in X \mid y = \sup W \text{ for some w.o. subset } W \text{ of } Y\}, \\ Y_- &= \{y \in X \mid y = \inf W \text{ for some i.w.o. subset } W \text{ of } Y\}. \end{aligned} \tag{2.2}$$

If there exists a $c \in X$ and an increasing mapping $f^c : Y \to X$ such that $f^c(y)$ is an upper bound of $\{c, y\}$ for all $y \in Y$, we say that f^c is an *up-map* of Y. If there exists a $c \in X$ and an increasing mapping $f_c : Y \to X$ such that $f_c(y)$ is a lower bound of $\{c, y\}$ for all $y \in Y$, we say that f_c is a *down-map* of Y.

For instance, if $\sup\{c, x\}$ exists for each $x \in Y$, the mapping $f^c(y) = \sup\{c, y\}$ is an up-map of Y. Similarly, $f_c(y) = \inf\{c, y\}$ is a down-map of Y if $\inf\{c, x\}$ exists for each $x \in Y$. If c is a lower bound of Y in X, then $f^c(y) \equiv y$ is an up-map of Y, and if c is an upper bound of Y in X, then $f_c(y) \equiv y$ is a down-map of Y.

A basis to our considerations is the following recursion principle. (Cf. [11, Lemma 1.1.1].)

Lemma 2.1. *Given a set $\mathcal{D}$ of subsets of X containing the empty set $\emptyset$, and a mapping $\mathcal{F} : \mathcal{D} \to X$, there is a unique well-ordered subset C of X with property*

(A) *$x \in C$ if and only if $x = \mathcal{F}(C^{<x})$, where $C^{<x} = \{y \in C \mid y < x\}$.*

If $\mathcal{F}(C)$ is defined, it is not a strict upper bound of C.

As an application of Lemma 2.1 we prove the following auxiliary result.

Lemma 2.2. *Assume that $G : X \to X$ is an increasing mapping.*

(a) *If all nonempty w.o. subsets of $G[X]$ have supremums, and if $G[X]_+$ has an up-map f^c, then equation $x = f^c(Gx)$ has the least solution $\overline{x}$, and*

$$\overline{x} = \min\{x \in X \mid f^c(Gx) \leq x\}. \tag{2.3}$$

(b) *If all nonempty i.w.o. subsets of $G[X]$ have infimums, and if $G[X]_-$ has a down-map f_c, then equation $x = f_c(Gx)$ has the greatest solution $\underline{x}$, and*

$$\underline{x} = \max\{x \in X \mid x \leq f_c(Gx)\}. \tag{2.4}$$

Moreover, both $\underline{x}$ and $\overline{x}$ are increasing with respect to G.

Proof. (a) Let $f^c : G[X]_+ \to X$ be an up-map of $G[X]_+$, and let $\mathcal{D}$ denote the family of all well-ordered subsets of X. Let $W \in \mathcal{D} \setminus \emptyset$ be given. To prove that $G[W]$ is well-ordered, let B be a nonempty subset of $G[W]$. Then $A = \{x \in W \mid Gx \in B\}$ is a nonempty subset of W. Because W is well-ordered, then $x_0 = \min A$ exists. If $y \in B$, then $y = Gx$ for some $x \in A$. Thus $x_0 \leq x$, whence $Gx_0 \leq Gx = y$ because G is increasing. Consequently, Gx_0 is the least element of B. This implies, by definition, that $G[W]$ is well-ordered. Thus $\sup G[W]$ exists, by a hypothesis.

The above result guarantees that we can define a mapping $\mathcal{F} : \mathcal{D} \to X$ by

$$\mathcal{F}(\emptyset) = c, \qquad \mathcal{F}(W) = f^c\big(\sup G[W]\big), \quad W \in \mathcal{D} \setminus \emptyset. \tag{2.5}$$

According to this definition condition (A) of Lemma 2.1 can be rewritten in the following form:

$$c = \min C, \qquad c < x \in C \quad \text{iff } x = f^c\big(\sup G[C^{<x}]\big). \tag{2.6}$$

By Lemma 2.1 there exists only one well-ordered set C in X satisfying (2.6). It is nonempty because $c \in C$. Since G is increasing, then $G[C]$ is a nonempty well-ordered subset of $G[X]$. Thus $\sup G[C]$ exists and belongs to $G[X]_+$, whence $\mathcal{F}(C) = f^c(\sup G[C])$ is defined. Since f^c is increasing, it follows from (2.6) that $\mathcal{F}(C)$ is an upper bound of C. This result and the last conclusion of Lemma 2.1 imply that $\overline{x} := \mathcal{F}(C) = \max C$. Since G is increasing, then $G\overline{x} = \max G[C] = \sup G[C]$, whence $\overline{x} = \mathcal{F}(C) = f^c(G\overline{x})$, that is, $x = \overline{x}$ is a solution of equation $x = f^c(Gx)$.

To prove that (2.3) holds, assume that $y \in X$ and $f^c(Gy) \leq y$. Since $f^c(Gy)$ is, by definition, an upper bound of $\{c, Gy\}$, then $\min C = c \leq f^c(Gy) \leq y$. If $c < x \in C$, and if y is an upper bound of $C^{<x}$, then Gy is an upper bound of $G[C^{<x}]$. Thus $\sup G[C^{<x}] \leq Gy$,

whence $x = f^c(\sup G[C^{<x}]) \le f^c(Gy) \le y$. Consequently, by transfinite induction, $x \le y$ for each $x \in C$, that is, $\overline{x} = \max C \le y$. This result implies that (2.3) holds, and that $x = \overline{x}$ is the least solution of equation $x = f^c(Gx)$.

The proof of (b) is dual to the above one, and the last conclusion of Lemma is a consequence of (2.3) and (2.4). □

Applying Lemma 2.2 we obtain existence and comparison results for fixed points of an increasing mapping $G : X \to X$ if one of the following hypotheses holds.

(Ga) $G[X]$ has a lower bound, and nonempty w.o. subsets of $G[X]$ have supremums in X.

(Gb) $G[X]$ has an upper bound, and nonempty i.w.o. subsets of $G[X]$ have infimums in X.

Corollary 2.3. *Let $G : X \to X$ be increasing.*

(a) *If (Ga) holds, then G has the least fixed point which is increasing with respect to G.*

(b) *If (Gb) holds, then G has the greatest fixed point which is increasing with respect to G.*

Proof. Assume that (Ga) holds, and let $c \in X$ be a lower bound of $G[X]$. Then $f^c(y) \equiv y$ is an up-map of $G[X]_+$, and equation $x = f^c(Gx)$ is reduced to the fixed point equation $x = Gx$. The first conclusion of (a) follows then from Lemma 2.2(a). Similarly, the first conclusion of (b) follows from Lemma 2.2(b), and the last conclusions of (a) and (b) follow from the last conclusion of Lemma 2.2. □

The next result is a consequence of Lemma 2.2 and Corollary 2.3.

Theorem 2.4. *Assume that $G : X \to X$ is increasing, that $G[X]$ is relatively well-order complete, and that $G[X]_+$ has an up-map f^c. Then*

(a) *G has the greatest fixed point x^* in $(\overline{x}]$, where $\overline{x} = \min\{x \in X \mid x = f^c(Gx)\}$;*

(b) *$\overline{x}$ and x^* are increasing with respect to G.*

Proof. The given hypotheses ensure by Lemma 2.2 that the least solution $\overline{x}$ of equation $x = f^c(Gx)$ exists, and that $\overline{x}$ is increasing with respect to G. Since G is increasing and $G\overline{x} \le f^c(G\overline{x}) = \overline{x}$, then $\overline{x}$ is an upper bound of $G[(\overline{x}]]$, whence Corollary 2.3, with $X = (\overline{x}]$, implies that G has the greatest fixed point x^* in $(\overline{x}]$, and that x^* is increasing with respect to G. □

The following result is a dual to Theorem 2.4.

Proposition 2.5. *Assume that $G : X \to X$ is increasing, that $G[X]$ is relatively well-order complete, and that $G[X]_-$ has a down-map f_c. Then*

(a) *G has the least fixed point x_* in $[\underline{x})$, where $\underline{x} = \max\{x \in X \mid x = f_c(Gx)\}$;*

(b) *x_* and $\underline{x}$ are increasing with respect to G.*

The next result gives some information on the structure of the fixed point set Fix(G).

Proposition 2.6. *If the hypotheses of Theorem 2.4 or Proposition 2.5 hold, then* Fix(G) *is well-order complete. In particular, G has minimal and maximal fixed points.*

Proof. Both Theorem 2.4 and Proposition 2.5 imply that Fix(G) is nonempty. Let W be a nonempty well-ordered subset of Fix(G). Since $G[W] = W$ is nonempty and well-ordered subset of $G[X]$, which is relatively well-order complete, then $x_0 = \sup W$ exists in X. Because $x \le x_0$ for each $x \in W$ and G is increasing, then $x = Gx \le Gx_0$ for each $x \in W$. Thus Gx_0 is an upper bound of W, and since $x_0 = \sup W$, then $x_0 \le Gx_0$. Thus G, restricted to $[x_0)$, satisfies the hypothesis (Ga), whence G has by Corollary 2.3(a) the least fixed point x_1 in $[x_0)$. Obviously, x_1 is the least upper bound of W in $\mathrm{Fix}(G)$.

The fact that each nonempty i.w.o. subset of $\mathrm{Fix}(G)$ has the greatest lower bound in $\mathrm{Fix}(G)$ follows similarly from Corollary 2.3(b). Thus $\mathrm{Fix}(G)$ is well-order complete.

Since each nonempty well-ordered subset of $\mathrm{Fix}(G)$ has an upper bound in $\mathrm{Fix}(G)$, it follows from the version of Zorn's Lemma due to Bourbaki (cf., e.g., [11, page 6]) that $\mathrm{Fix}(G)$ has a maximal element, which is a maximal fixed point of G. The dual reasoning shows that G has a minimal fixed point. □

As a consequence of Theorem 2.4 and Propositions 2.5 and 2.6 we obtain the following fixed point results, which generalize those of [10, Theorem 2.1 and Proposition 2.3].

Corollary 2.7. *Let $G : X \to X$ be an increasing mapping whose range is relatively well-order complete, and assume that $c \in X$.*

(a) *If $\sup\{c,x\}$ exist for all $x \in G[X]_+$, then $\mathrm{Fix}(G)$ is well-order complete, G has minimal and maximal fixed points, and the greatest fixed point x^* in $(\overline{x}]$, where $\overline{x}$ is the least solution of equation $x = \sup\{c, Gx\}$.*

(b) *If $\inf\{c,x\}$ exist for all $x \in G[X]_-$, then $\mathrm{Fix}(G)$ is well-order complete, G has minimal and maximal fixed points, and the least fixed point x_* in $[\underline{x})$, where $\underline{x}$ is the greatest solution of equation $x = \inf\{c, Gx\}$.*

(c) *x^*, x_*, $\underline{x}$ and $\overline{x}$ are increasing with respect to G.*

Proof. The hypotheses of Theorem 2.4 and Proposition 2.5 hold when $f^c(y) = \sup\{c, y\}$, $y \in G[X]_+$ in (a) and $f_c(y) = \inf\{c, y\}$, $y \in G[X]_-$ in (b). □

Example 2.8. Let l^1 be the space of all absolutely summable real sequences, ordered by $(x_n)_{n=0}^\infty \le (y_n)_{n=0}^\infty$ if and only if $x_n \le y_n$ for each $n \in \mathbb{N}$, and

$$Y = \left\{ (x_n)_{n=0}^\infty \in l^1 \mid \sum_{n=0}^\infty |x_n| \le 1 \right\}. \tag{2.7}$$

Show that the results of Corollary 2.7 hold for each increasing mapping $G : X \to X$ for which $G[X] \subseteq Y \subseteq X \subseteq l^1$.

Solution. Assume that $G : X \to X$ is increasing, and that $G[X] \subseteq Y \subseteq X \subseteq l^1$. It is well-known (cf., e.g., [11, Corollary 5.8.6]) that all nonempty w.o. and i.w.o. subsets of Y have supremums and infimums in X. Thus $G[X]$ is relatively well-order complete. Moreover, choosing $c = (0)_{n=0}^\infty$, it is easy to see that both $\sup\{c,x\}$ and $\inf\{c,x\}$ exist for all $x \in Y$. Since $G[X]_- \cup G[X]_+ \subseteq Y$, then G satisfies the hypotheses of Corollary 2.7. □

Example 2.9. Let l^∞ denote the space of all bounded real sequences, ordered as l^1 in Example 2.8. Denote $Y = \{(x_n)_{n=0}^\infty \in l^\infty \mid \sup_{n\in\mathbb{N}} |x_n| \le 1\}$. Show that each increasing mapping $G : X \to X$ for which $G[X] \subseteq Y \subseteq X \subseteq l^\infty$ has least and greatest fixed points.

Solution. The constant sequences $(-1)_{n=0}^{\infty}$ and $(1)_{n=0}^{\infty}$ are least and greatest elements of Y, whence they are lower and upper bounds of $G[X]$. Moreover, as in Example 2.8, one can show that $G[X]$ is relatively well-order complete. Thus the hypotheses of Corollary 2.3 hold. □

Remarks 2.10. Formula (2.6) is reduced to the generalized iteration method (I) introduced in [11, Theorem 1.1.1] when $f^c(y) \equiv y$. In particular, Corollary 2.3(a) is a special case of [11, Theorem 1.2.1], and Corollary 2.3(b) is a special cases of [11, Proposition 1.2.1].

It can be shown that the first elements of the well-ordered subset C which satisfies (2.6) are the following iterations: $x_0 = c$, $x_{n+1} = f^c(Gx_n)$, $n = 0, 1, \ldots$, as long as x_{n+1} exists and $x_n < x_{n+1}$. If $x_{n+1} = x_n$, then $\overline{x} = x_n$. If $(x_n)_{n=0}^{\infty}$ is strictly increasing, then $x_\omega = f^c(\sup\{Gx_n\}_{n\in\mathbb{N}})$ is the next element of C. Choosing $x_0 = x_\omega$ above we obtain the next possible elements of C, and so on.

Similarly, it is easy to show that if $G[X]$ in Theorem 2.4 and in Proposition 2.5 is finite, the fixed points x^* and x_* of G are the last elements of the finite sequences determined by the following algorithms:

(i) $x_0 = c$. For n from 0 while $x_n \neq Gx_n$, $x_{n+1} = Gx_n$ if $Gx_n < x_n$ else $x_{n+1} = f^c(Gx_n)$.

(ii) $x_0 = c_-$. For n from 0 while $x_n \neq Gx_n$, $x_{n+1} = Gx_n$ if $Gx_n > x_n$ else $x_{n+1} = f_c(Gx_n)$.

Compared to results of [6], no sequencibility hypotheses are needed in the proof of Proposition 2.6. Propositions 2.1 and 2.2 of [10] could also have been used to prove the existence of maximal and minimal fixed points of G in Proposition 2.6.

3. On common fixed points of mapping families

In this section we apply results of Section 2 to derive existence results for common fixed points of a family of mappings $G_t : X \to X$, $t \in S$, where X is a poset and S is a nonempty set. By a *common fixed point* of $\{G_t\}_{t\in S}$ we mean a point $x \in X$ for which $G_t x = x$ for each $t \in S$. Least, greatest, minimal and maximal common fixed points are defined as in the case of a single operator. We assume that for a fixed $t_0 \in S$

(G0) $G_t G_{t_0} = G_{t_0} G_t$ for all $t \in S$,

and that one of the following hypotheses is valid.

(G1) $G_t x \leq G_{t_0} x$ for all $t \in S$ and $x \in X$.

(G2) $G_{t_0} x \leq G_t x$ for all $t \in S$ and $x \in X$.

As an application of Proposition 2.6 we prove the following existence result for the existence of common fixed points of the operator family $\{G_t\}_{t\in S}$.

Theorem 3.1. *Assume that (G0) holds, that G_{t_0} is increasing, that $G_{t_0}[X]$ is relatively well-order complete, and that $G_{t_0}[X]_+$ has an up-map or $G_{t_0}[X]_-$ has a down-map.*

(a) *If (G1) holds, the family $\{G_t\}_{t\in S}$ has a minimal common fixed point.*

(b) *If (G2) holds, the family $\{G_t\}_{t\in S}$ has a maximal common fixed point.*

Proof. (a) The hypotheses assumed for G_{t_0} imply by Proposition 2.6 that G_{t_0} has a minimal fixed point x_-. Applying this result and the hypothesis (G0) we see that $G_{t_0} G_t x_- = G_t G_{t_0} x_- = G_t x_-$. Thus each $G_t x_-$ is also a fixed point of G_{t_0}. Hence, if (G1) holds, then $G_t x_- \leq G_{t_0} x_- = x_-$ for all $t \in S$. Since x_- is a minimal fixed point of G_{t_0}, then $G_t x_- = x_-$, whence x_- is a common fixed point of $\{G_t\}_{t\in S}$. To prove that x_- is minimal, assume that x is a common fixed point of $\{G_t\}_{t\in S}$, and that $x \le x_-$. Then x is a fixed point of G_{t_0} and x_- is its minimal fixed point, so that $x = x_-$. Thus x_- is a minimal common fixed point of $\{G_t\}_{t\in S}$.

(b) The result of (b) is similar consequence of Proposition 2.6. □

The next result is a consequence of Corollary 2.3.

Theorem 3.2. (a) *If (G0) and (G1) hold, if G_{t_0} is increasing, and if (Ga) holds for $G = G_{t_0}$, then $\{G_t\}_{t\in S}$ has the least common fixed point, and it is increasing with respect to G_{t_0}.*

(b) *If (G0) and (G2) hold, if G_{t_0} is increasing, and if (Gb) holds for $G = G_{t_0}$, then $\{G_t\}_{t\in S}$ has the greatest common fixed point, and it is increasing with respect to G_{t_0}.*

Proof. (a) The hypotheses (Ga) given for $G = G_{t_0}$ imply by Corollary 2.3 that G_{t_0} has the least fixed point x_*, and it is increasing with respect to G_{t_0}. This result and the hypotheses (G0) and (G1) imply that $G_t x_*$ is a fixed point of G_{t_0} and $G_t x_* \le G_{t_0} x_* = x_*$ for each $t \in S$. Since x_* is the least fixed point of G_{t_0}, then $G_t x_* = x_*$, whence x_* is a common fixed point of $\{G_t\}_{t\in S}$. If x is a common fixed point of $\{G_t\}_{t\in S}$, then x is a fixed point of G_{t_0}, and x_* is its least fixed point, whence $x_* \le x$. Thus x_* is the least common fixed point of $\{G_t\}_{t\in S}$.

The proof of (b) is similar to the above one. □

Remarks 3.3. The phrase: "increasing with respect to G" of a least (greatest) fixed point x_* (x^*) of $G : X \to X$ in some order interval means that if $\tilde{G} : X \to X$ satisfies the same hypotheses as G, and if $Gx \le \tilde{G}x$ for all $x \in X$, then $x \le \tilde{x}_*$ $(x^* \le \tilde{x}^*)$, where $\tilde{x}_*$ $(\tilde{x}^*)$ denotes the corresponding least (greatest) fixed point of $\tilde{G}$. In particular, the results of Section 2 can be applied to find increasing selectors for fixed points of increasing families of increasing mappings, as demonstrated in the last part of the next example. The first part of it proves a result stated in the Introduction.

Example 3.4. When $c = (c_1,\dots,c_m) \in \mathbb{R}^m$ and $R \in (0,\infty)$, the space

$$X = \left\{ x = (x_1,\dots,x_m) \in \mathbb{R}^m \mid \sum_{i=1}^{m} (x_i - c_i)^2 \le R^2 \right\}, \tag{3.1}$$

ordered coordinatewise, is well-order complete. The center $c = (c_1,\dots,c_m)$ of X is an order center of X, that is, $\sup\{c,x\}$ and $\inf\{c,x\}$ belong to X for each $x \in C$. Hence, if $\{G_t\}_{t\in S}$ satisfies (G0) and (G1), (respectively (G0) and (G2)), and if G_{t_0} is increasing, it follows from Theorem 3.1 that $\{G_t\}_{t\in S}$ has a minimal (respectively a maximal) common fixed point. Moreover, if S is a poset if $G : X \times S \to X$ is increasing with respect to the product ordering of $X \times S$, it follows from Corollary 2.7 that $G_t = G(\cdot,t)$ has the greatest fixed point x_t^* in $(\overline{x}_t]$, where $\overline{x}_t = \min\{x \in X \mid x = \sup\{c, G_t x\}\}$, and that the mapping $t \to x_t^*$ is increasing.

4. Applications to Markov processes

4.1. Preliminaries. Let $E = (E,d,\le)$ be a partially ordered (p.o.) Polish space, that is, a complete and separable metric space equipped with a partial ordering $\le$, which is closed in the sense that if $x_n \to x$, $y_n \to y$ and $x_n \le y_n$ for each n, then $x \le y$. For instance, closed subsets of separable ordered Banach spaces are p.o. Polish spaces. Let $\mathcal{B}$ be a family of all Borel subsets of E, and let $\mathcal{M}$ denote the space of probability measures on E, that is, the space of all countably additive functions $p : \mathcal{B} \to [0,1]$ for which $p(E) = 1$.

We say that a sequence (p_n) of $\mathcal{M}$ *converges weakly* to $p \in \mathcal{M}$ if $\lim_{n\to\infty} \int_E f(x)p_n(dx) = \int_E f(x)p(dx)$ for each bounded and continuous function $f : E \to \mathbb{R}$.

In view of [5, Theorem 11.3.3] the weak convergence is equivalent to the convergence in the *Prohorov metric* ρ on $\mathcal{M}$ defined by

$$\rho(p,q) := \inf\{\epsilon > 0 \mid p(A) \le q(\{x \in E \mid d(x,A) < \epsilon\}) + \epsilon \ \forall A \in \mathcal{B}\}, \quad p,q \in \mathcal{M}. \tag{4.1}$$

This result and [14, Theorem 2] imply that relation: $p \preceq q$ if and only if $p(A) \le q(A)$ for each $A \in \mathcal{B}$ which is increasing, that is, $y \in A$ whenever $x \in A$ and $x \le y$, defines a closed partial ordering on $(\mathcal{M},\rho)$.

Lemma 4.1. *Let B be a nonempty closed set in E, and $\mathcal{P} := \{p \in \mathcal{M} \mid p(B) = 1\}$.*

(a) *If increasing sequences of B converge, then each nonempty w.o. subset W of $\mathcal{P}$ contains an increasing sequence which converges to* $\sup W$.

(b) *If decreasing sequences of B converge, then each nonempty i.w.o. subset W of $\mathcal{P}$ contains a decreasing sequence which converges to* $\inf W$.

(c) *If monotone sequences of B converge, then $\mathcal{P}$ is relatively well-order complete in $\mathcal{M}$.*

(d) *$\mathcal{P}$ is a closed subset of $(\mathcal{M},\rho)$.*

Proof. (a) Assume that each increasing sequence of B converges, and let $(p_n)_{n=0}^\infty$ be an increasing sequence in $\mathcal{P}$. By [15, Proposition 4] there exists an a.s. increasing sequence $\{Y_n \mid n \in \mathbb{N}\}$ of B-valued random variables, defined on a common probability space (Ω,μ) such that $\mu(Y_n^{-1}(A)) = p_n(A)$ for all $n \in \mathbb{N}$ and $A \in \mathcal{B}$. Since increasing sequences of B converge, then $(Y_n)_{n=0}^\infty$ converges a.s. in Ω, and the limit is by [5, Theorem 4.2.2] a random variable on (Ω,μ). Thus, by [15, Theorem 6] the sequence $(p_n)_{n=0}^\infty$ converges weakly, and hence in $(\mathcal{M},\rho)$. The assertion of (a) follows then from [11, Proposition 1.1.5].

The proof in the case (b) is dual to the above one, and (c) follows from (a) and (b).

(d) If $(p_n)_{n=0}^\infty$ is a sequence in $\mathcal{P}$ which converges to p in $(\mathcal{M},\rho)$, then $p_n \to p$ weakly by [5, Theorem 11.3.3]. Thus, by [5, Theorem 11.1.1], $p(B) \ge \limsup_{n\to\infty} p_n(B) = 1$, whence $p \in \mathcal{P}$. This result implies that $\mathcal{P}$ is a closed subset of $(\mathcal{M},\rho)$. □

4.2. Existence results for invariant measures of a Markov process. Let $S = (S,+)$ be a commutative groupoid. For instance, S can be a set of all nonnegative or positive real numbers, rational numbers or integers. Following the terminology adopted in [20] we say that a mapping $P : S \times E \times \mathcal{B} \to [0,1]$, called a *transition function*, defines a *Markov process* $P(t,x,A)$ on the *phase space* $(E,\mathcal{B})$ if the following conditions hold:

(i) $P(t,\cdot,A)$ is a $\mathcal{B}$-measurable function on E for all fixed $t \in S$ and $A \in \mathcal{B}$.

(ii) $P(t,x,\cdot) \in \mathcal{M}$ for all fixed $t \in S$ and $x \in E$.

(iii) $P(t+s,x,A) = \int_E P(s,y,A)P(t,x,dy)$ for all fixed $t,s \in S$, $x \in E$ and $A \in \mathcal{B}$.

We say that $p \in \mathcal{M}$ is an *invariant measure* of $P(t,x,A)$ if

$$p(A) = \int_E P(t,x,A)p(dx) \quad \forall t \in S, A \in \mathcal{B}. \tag{4.2}$$

Conditions (i) and (ii) ensure that the equation

$$G_t p(A) = \int_E P(t,x,A)p(dx), \quad A \in \mathcal{B}, \tag{4.3}$$

defines for each $t \in S$ a mapping $G_t : \mathcal{M} \to \mathcal{M}$. Thus $p \in \mathcal{M}$ is an invariant measure of $P(t,x,A)$ if and only if p is a common fixed point of the operators G_t, $t \in S$. As an easy consequence of the *Chapman-Kolmogorov equation* (iii) and the definition (4.3) of G_t we obtain the following result.

Lemma 4.2. *The operators G_t, $t \in S$ commute, that is, $G_tG_s = G_sG_t$ for all $t,s \in S$.*

Results of Section 3 and Subsection 4.1 will now be applied to derive existence results for extremal invariant measures of a Markov process $P(t,x,A)$.

Theorem 4.3. *Assume there exists a $t_0 \in S$ such that*
(P0) *$x \leq y$ in E implies $P(t_0,x,\cdot) \preceq P(t_0,y,\cdot)$ in $\mathcal{M}$;*
(P1) *$P(t,x,\cdot) \preceq P(t_0,x,\cdot)$ for all $t \in S$ and $x \in E$;*
(P2) *there exists a closed subset B of E whose monotone sequences converge such that $P(t_0,x,B) = 1$ for each $x \in E$.*
If B in (P2) has a lower bound in E, then $P(t,x,A)$ has the least invariant measure, and it is increasing with respect to P.

Proof. It suffices to show that the family of the operators $G_t : \mathcal{M} \to \mathcal{M}$, $t \in S$, defined by (4.3) satisfies the hypotheses of Theorem 3.2. The hypothesis (G0) holds by Lemma 4.2, (P1) implies that the hypothesis (G1) is valid, and G_{t_0} is increasing by (P0) (cf. [13]). The hypothesis (P2) implies that $G_{t_0}p(B) = 1$ for each $p \in \mathcal{M}$, whence $G_{t_0}[\mathcal{M}]$ is relatively well-order complete by Lemma 4.1. If B has a lower bound $a \in E$, then relation

$$\underline{p}(A) = \begin{cases} 1 & \text{if } a \in A, \\ 0 & \text{otherwise}, \end{cases} \quad A \in \mathcal{B}, \tag{4.4}$$

defines a lower bound of $G_{t_0}[\mathcal{M}]$. Thus also the hypothesis (Ga) holds, whence the family $\{G_t\}_{t \in S}$ has by Theorem 3.2 the least common fixed point p_*. This result, (4.2) and (4.3) imply that p_* is the least invariant measure of $P(t,x,A)$. Moreover, it follows from Theorem 3.2 that p_* is increasing with respect to G_{t_0}, and hence also to P by (4.3). □

The next result is the dual to that of Theorem 4.3, and it is a consequence of Theorem 3.2(b).

Proposition 4.4. *If the hypothesis (P1) is replaced in Theorem 4.3 by*
(P3) *$P(t_0,x,\cdot) \preceq P(t,x,\cdot)$ for all $t \in S$ and $x \in E$,*

and if B in (P2) has an upper bound in E, then $P(t,x,A)$ has the greatest invariant measure, and it is increasing with respect to P.

The next result is an application of Theorem 3.1 and Lemma 4.1,

THEOREM 4.5. *Assume there exists a $t_0 \in S$ such that the hypotheses (P0) and (P2) hold, and that the set $\mathcal{P} = \{p \in \mathcal{M} \mid p(B) = 1\}$, where B is given by (P2), has an up-map or a down-map. Then $P(t,x,A)$ has*

(a) *a minimal invariant measure if (P1) holds;*
(b) *a maximal invariant measure if (P3) holds.*

Proof. (a) The hypothesis (P0) implies that G_{t_0} is increasing. Since B in (P2) is closed, and since $G_{t_0}[\mathcal{M}] \subseteq \mathcal{P}$ by (P2), it follows from Lemma 4.1 that $G_{t_0}[\mathcal{M}]$ is relatively well-order complete, and that $\mathcal{P}$ contains both $G_{t_0}[\mathcal{M}]_+$ and $G_{t_0}[\mathcal{M}]_-$. Thus an up-map of $\mathcal{P}$ is also an up-map of $G_{t_0}[\mathcal{M}]_+$, and a down-map of $\mathcal{P}$ is a down-map of $G_{t_0}[\mathcal{M}]_-$. The hypothesis (P1) implies that (G1) is valid for $G = G_{t_0}$, which thus satisfies all the hypotheses of Theorem 3.1(a). Consequently, the family $\{G_t\}_{t\in S}$ has a minimal common fixed point p_-. In view of (4.2) and (4.3) p_- is a minimal invariant measure of $P(t,x,A)$.

(b) The given hypotheses and (P3) imply that the hypotheses of Theorem 3.1(b) hold, which implies the assertion of (b). □

4.3. Special cases. In this subsection we consider the special case when the transition function $P : E \times \mathcal{B} \to [0,1]$ which defines a Markov process $P(x,A)$ on $(E,\mathcal{B})$ is independent on the parameter t, and satisfies the following hypotheses.

(a) $P(\cdot,A)$ is a $\mathcal{B}$-measurable function on E for all fixed $A \in \mathcal{B}$.
(b) $P(x,\cdot) \in \mathcal{M}$ for all fixed $x \in E$.

In this case (4.2) is reduced to the form

$$p(A) = \int_E P(x,A)p(dx), \quad A \in \mathcal{B}. \tag{4.5}$$

Thus $p \in \mathcal{M}$ is a invariant measure of $P(x,A)$ if and only if p is a fixed point of $G : \mathcal{M} \to \mathcal{M}$, defined by

$$Gp(A) = \int_E P(x,A)p(dx), \quad A \in \mathcal{B}. \tag{4.6}$$

The hypotheses (P1) and (P2) are reduced to the following form.

(Pa) $x \le y$ in E implies $P(x,\cdot) \preceq P(y,\cdot)$ in $\mathcal{M}$.
(Pb) There exists a closed set B in E whose monotone sequences converge in E such that $P(x,B) = 1$ for each $x \in E$.

As a special case of Theorem 4.3 and Proposition 4.4 we obtain the following.

PROPOSITION 4.6. *Let the hypotheses (Pa) and (Pb) hold.*

(a) *If B in (Pb) has a lower bound in E, then $P(x,A)$ has the least invariant measure, and it is increasing with respect to P.*

(b) *If B in (Pb) has an upper bound in E, then $P(x,A)$ has the greatest invariant measure, and it is increasing with respect to P.*

As a consequence of Proposition 4.6 we obtain the following result.

Corollary 4.7. *Let H be a p.o. Polish space whose order-bounded and monotone sequences converge, and assume that a closed subset E of H contains an order interval $[a,b]$ of H. If the transition function P of a Markov process $P(x,A)$ on $(E,\mathcal{B})$ satisfies the hypotheses (a), (b) and (Pa), and if $P(x,[a,b]) = 1$ for each $x \in E$, then $P(x,A)$ has least and greatest invariant measures, and they are increasing with respect to P.*

Proof. $[a,b]$ is a closed subset of E whose monotone sequences converge by assumption, and $[a,b]$ has least and greatest elements, so that the assertions follow from Proposition 4.6 when $B = [a,b]$. □

The next result is a consequence of Theorem 2.4 and Propositions 2.5 and 2.6.

Proposition 4.8. *Let the hypotheses (Pa) and (Pb) hold, let B is as in (Pb), and denote $\mathcal{P} = \{p \in \mathcal{M} \mid p(B) = 1\}$.*

(a) *If $\mathcal{P}$ has an up-map, then $P(x,A)$ has minimal and maximal invariant measures, and the greatest invariant measure p^* in $(\overline{p}]$, where $\overline{p} = \min\{p \mid p = f^c(\int_E P(x,\cdot)p(dx))\}$.*

(b) *If $\mathcal{P}$ has a down-map, then $P(x,A)$ has minimal and maximal invariant measures, and the least invariant measure p_* in $[\underline{p})$, where $\underline{p} = \max\{p \mid p = f_c(\int_E P(x,\cdot)p(dx))\}$.*

(c) *p_*, p^*, $\underline{p}$ and $\overline{p}$ are increasing with respect to P.*

Proof. (a) As in the proof of Theorem 4.5 it can be shown that G, defined by (4.6), satisfies the hypotheses of Theorem 2.4, whence the assertions follow from Theorem 2.4 and Proposition 2.6.

The assertions of (b) are similar consequences of Propositions 2.5. and 2.6, and the assertions of (c) follow from Theorem 2.4(b) and from Proposition 2.5(b). □

Remarks 4.9. In [13] the existence of an invariant measure of a Markov process $P(x,A)$ with property (Pa) is proved in the case when E is compact and has the least element. This result follows, for example, from Proposition 4.6(a) when $B = E$, since in a compact partially ordered Polish space all monotone sequences converge. This convergence property holds also when E (or B) is a closed and order-bounded subset of an ordered separable Banach space H whose order cone K is regular. E (or B) may be also norm-bounded if K is fully regular. This holds, for instance, if H is weakly sequentially complete and K is normal (cf. [7, Theorem 2.4.5]). In particular, if H is infinite-dimensional, and if its subset E contains an open set, then E is not compact, not even locally compact, as assumed in [20] in the proof of the existence of an invariant measure. As for examples of such ordered Banach spaces H see, for example, [7, Section 2], and [11, Section 5.8].

In [12] a number of existence results for invariant measures are derived for a Markov process whose transition function has properties (a) and (b).

As remarked in [15, page 901], $\mathcal{M}$ is not in general a lattice under $\preceq$ even if E is a lattice. The existence of an up-map or a down-map is assumed in Theorem 4.5 and in

Proposition 4.8 to overcome this difficulty. This situation is demonstrated in the following example.

4.4. An example. Assume that an Euclidean m-space $\mathbb{R}^m$ is ordered coordinatewise. Given $\beta, R \in (0,\infty)$ and $(a_1,\dots,a_m) \in \mathbb{R}^m$, denote

$$B = \left\{(x_1,\dots,x_m) \in \mathbb{R}^m \mid |x_1 - a_1|^\beta + \cdots + |x_m - a_m|^\beta \le R^\beta\right\}. \tag{4.7}$$

Let $\mathcal{M}$ be the set of all probability measures on a closed subset E of $\mathbb{R}^m$ which contains B. Assume that $P(x,A)$ is a Markov process on $(E,\mathcal{B})$ which satisfies the hypothesis (Pa), and for which $P(x,B) = 1$ for each $x \in E$. The set B is closed and each monotone sequence of B converges, whence the hypothesis (Pb) holds. Defining

$$c(A) = \begin{cases} 1 & \text{if } (a_1,\dots,a_m) \in A, \\ 0 & \text{otherwise,} \end{cases} \qquad A \in \mathcal{B}, \tag{4.8}$$

we obtain an element c of $\mathcal{M}$. For each $p \in \mathcal{M}$, denote by F_p the (cumulative) distribution function of p, and let $f^c(p)$ and $f_c(p)$ be the probability measures on B whose distribution functions are

$$\begin{aligned} F_{f^c(p)}(x_1,\dots,x_m) &= \begin{cases} F_p(x_1,\dots,x_m) & \text{if } x_i \ge a_i,\ i = 1,\dots,m, \\ 0 & \text{otherwise,} \end{cases} \\ F_{f_c(p)}(x_1,\dots,x_m) &= \begin{cases} 1 & \text{if } x_i \ge a_i,\ i = 1,\dots,m, \\ F_p(x_1,\dots,x_m) & \text{otherwise.} \end{cases} \end{aligned} \tag{4.9}$$

Routine calculations show that f^c is an up-map and f_c is a down-map of $\mathcal{P} = \{p \in \mathcal{M} \mid p(B) = 1\}$. Thus the hypotheses of Proposition 4.8 hold.

According to the conclusions of Proposition 4.8 the Markov process $P(x,A)$ has minimal and maximal invariant measures p_m and p^m, the greatest invariant measure p^* in the order interval $(\overline{p}]$ of $\mathcal{M}$, where $\overline{p} = \min\{p \in \mathcal{M} \mid p(\cdot) = f^c(\int_E P(x,\cdot)p(dx))\}$, and the least invariant measure in the order interval $[\underline{p})$ of $\mathcal{M}$, where $\underline{p} = \max\{p \in \mathcal{M} \mid p(\cdot) = f_c(\int_E P(x,\cdot)p(dx))\}$.

Consider next the case when the mapping $p \mapsto \int_E P(x,\cdot)p(dx)$ has a finite number of values. In view of Remarks 2.10 the definition (4.6) of G and the above choices of f_c and f^c we can determine $\overline{p}$, $\underline{p}$, p^*, p_*, p_m and p^m in the following manner.

(1) $\overline{p}$ is the last element of the finite sequence of iterations

$$p_0 = c, \qquad p_{n+1}(\cdot) = f^c\left(\int_E P(x,\cdot)p_n(dx)\right), \quad \text{as long as } p_n \prec p_{n+1}. \tag{4.10}$$

(2) $\underline{p}$ is the last element of the finite sequence of iterations

$$p_0 = c, \qquad p_{n+1}(\cdot) = f_c\left(\int_E P(x,\cdot)p_n(dx)\right), \quad \text{as long as } p_{n+1} \prec p_n. \tag{4.11}$$

(3) p^* is the last element of the finite sequence of iterations

$$q_0 = \overline{p}, \qquad q_{n+1}(\cdot) = \int_E P(x,\cdot)q_n(dx), \quad \text{as long as } q_{n+1} \prec q_n. \tag{4.12}$$

(4) p_* is the last element of the finite sequence of iterations

$$q_0 = \underline{p}, \qquad q_{n+1}(\cdot) = \int_E P(x,\cdot)q_n(dx), \quad \text{as long as } q_n \prec q_{n+1}. \tag{4.13}$$

(5) p_m is the last element of the finite sequence formed by $q_0 = \overline{p}$, and q_{n+1} is a strict lower bound of $\int_E P(x,\cdot)q_n(dx)$ as long as such a lower bound exists.

(6) p^m is the last element of the finite sequence formed by $q_0 = \underline{p}$, and $q_{n+1}$is a strict upper bound of $\int_E P(x,\cdot)q_n(dx)$ as long as such an upper bound exists.

Remarks 4.10. Notice that B, defined by (4.7), is not convex when $\beta \in (0,1)$. In fact, B can be any closed subset of $\mathbb{R}^m$ which has an order center $a = (a_1,\dots,a_m) \in B$, that is, $\sup\{a,x\}$ and $\inf\{a,x\}$ exist and belong to B for each $x \in B$.

The sequences (q_n) in (5) and (6), as well their last elements, minimal and maximal invariant measures of $P(x,A)$, may not be uniquely determined.

The above example can easily be extended to the case when the transition function depends also on the parameter $t \in S$.

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