WEAK AND STRONG CONVERGENCE THEOREMS FOR NONEXPANSIVE SEMIGROUPS IN BANACH SPACES

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We introduce an implicit iterative process for a nonexpansive semigroup and then we prove a weak convergence theorem for the nonexpansive semigroup in a uniformly convex Banach space which satisfies Opial's condition. Further, we discuss the strong convergence of the implicit iterative process.

1. Introduction

Let C be a closed convex subset of a Hilbert space and let T be a nonexpansive mapping from C into itself. For each $t \in (0,1)$, the contraction mapping T_t of C into itself defined by

$$T_t x = tu + (1-t)Tx \tag{1.1}$$

for every $x \in C$, has a unique fixed point x_t, where u is an element of C. Browder [4] proved that $\{x_t\}$ converges strongly to a fixed point of T as $t \to 0$ in a Hilbert space. Motivated by Browder's theorem [4], Takahahi and Ueda [20] proved the strong convergence of the following iterative process in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm (see also [14]):

$$x_k = \frac{1}{k}x + \left(1 - \frac{1}{k}\right)Tx_k \tag{1.2}$$

for every $k = 1,2,3,\ldots$, where $x \in C$. On the other hand, Xu and Ori [21] studied the following implicit iterative process for finite nonexpansive mappings $T_1, T_2, \ldots, T_r$ in a Hilbert space: $x_0 = x \in C$ and

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n \tag{1.3}$$

for every $n = 1,2,\ldots$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $T_n = T_{n+r}$. And they proved a weak convergence of the iterative process defined by (1.3) in a Hilbert space. Sun et al. [17] studied the iterations defined by (1.3) and proved the strong convergence of the iterations in a uniformly convex Banach space, requiring one mapping T_i in the family to be semi compact.

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In this paper, using the idea of [17, 21], we introduce an implicit iterative process for a nonexpansive semigroup and then prove a weak convergence theorem for the nonexpansive semigroup in a uniformly convex Banach space which satisfies Opial's condition. Further, we discuss the strong convergence of the implicit iterative process (see also [1, 2, 7, 12, 13]).

2. Preliminaries and notations

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{Z}^+$ the set of all positive integers and the set of all nonnegative integers, respectively. Let E be a real Banach space. We denote by B_r the set $\{x \in E : \|x\| \le r\}$. A Banach space E is said to be *strictly convex* if $\|x+y\|/2 < 1$ for each $x, y \in B_1$ with $x \neq y$, and it is said to be *uniformly convex* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x+y\|/2 \le 1-\delta$ for each $x, y \in B_1$ with $\|x-y\| \ge \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex (see [19]). Let C be a closed subset of a Banach space and let T be a mapping from C into itself. We denote by $F(T)$ and $F_\varepsilon(T)$ for $\varepsilon > 0$, the sets $\{x \in C : x = Tx\}$ and $\{x \in C : \|x - Tx\| \le \varepsilon\}$, respectively.

A mapping T of C into itself is said to be *compact* if T is continuous and maps bounded sets into relatively compact sets. A mapping T of C into itself is said to be *demicompact at* $\xi \in C$ if for any bounded sequence $\{y_n\}$ in C such that $y_n - Ty_n \to \xi$ as $n \to \infty$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $y \in C$ such that $y_{n_k} \to y$ as $k \to \infty$ and $y - Ty = \xi$. In particular, a continuous mapping T is *demicompact at* 0 if for any bounded sequence $\{y_n\}$ in C such that $y_n - Ty_n \to 0$ as $n \to \infty$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $y \in C$ such that $y_{n_k} \to y$ as $k \to \infty$. T is also said to be *semicompact* if T is continuous and *demicompact at* 0 (e.g., see [21]). T is said to be *demicompact on* C if T is demicompact for each $y \in C$. If T is compact on C, then T is demicompact on C. For examples of demicompact mappings, see [1, 2, 12, 13]. We also denote by I the identity mapping. A mapping T of C into itself is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for every $x, y \in C$. We denote by $N(C)$ the set of all nonexpansive mappings from C into itself. We know from [5] that if C is a nonempty closed convex subset of a strictly convex Banach space, then $F(T)$ is convex for each $T \in N(C)$ with $F(T) \neq \emptyset$. The following are crucial to prove our results (see [5, 6]).

Proposition 2.1 (Browder). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space and let T be a nonexpansive mapping from C into itself. Let $\{x_n\}$ be a sequence in C such that it converges weakly to an element x of C and $\{x_n - Tx_n\}$ converges strongly to* 0. *Then x is a fixed point of T.*

Proposition 2.2 (Bruck). *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any nonexpansive mapping T of C into itself with $F(T) \neq \emptyset$,*

$$\overline{\text{co}}F_\delta(T) \subset F_\varepsilon(T). \tag{2.1}$$

Let E^* be the dual space of a Banach space E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. We say that a Banach space E satisfies *Opial's condition* [11] if for each sequence $\{x_n\}$ in E which converges weakly to x,

$$\varliminf_{n\to\infty} ||x_n - x|| < \varliminf_{n\to\infty} ||x_n - y|| \tag{2.2}$$

for each $y \in E$ with $y \neq x$. Since if the duality mapping $x \mapsto \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$ from E into E^* is single-valued and weakly sequentially continuous, then E satisfies Opial's condition. Each Hilbert space and the sequence spaces ℓ^p with $1 < p < \infty$ satisfy Opial's condition (see [8, 11]). Though an L^p-space with $p \neq 2$ does not usually satisfy Opial's condition, each separable Banach space can be equivalently renormed so that it satisfies Opial's condition (see [11, 22]).

Let S be a semigroup. Let $B(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm. For $s \in S$ and $f \in B(S)$, we define an element $l_s f$ in $B(S)$ by $(l_s f)(t) = f(st)$ for each $t \in S$. Let X be a subspace of $B(S)$ containing 1. An element μ in X^* is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be l_s-invariant, that is, $l_s(X) \subset X$ for each $s \in S$. A mean μ on X is said to be *left invariant* if $\mu(l_s f) = \mu(f)$ for each $s \in S$ and $f \in X$. A sequence $\{\mu_n\}$ of means on X is said to be *strongly left regular* if $\|\mu_n - l_s^* \mu_n\| \to 0$ for each $s \in S$, where l_s^* is the adjoint operator of l_s. In the case when S is commutative, a strongly left regular sequence is said to be *strongly regular* [9, 10]. Let E be a Banach space, let X be a subspace of $B(S)$ containing 1 and let μ be a mean on X. Let f be a mapping from S into E such that $\{f(t) : t \in S\}$ is contained in a weakly compact convex subset of E and the mapping $t \mapsto \langle f(t), x^* \rangle$ is in X for each $x^* \in E^*$. We know from [9, 18] that there exists a unique element $x_0 \in E$ such that $\langle x_0, x^* \rangle = \mu_t \langle f(t), x^* \rangle$ for all $x^* \in E^*$. Following [9], we denote such x_0 by $\int f(t)\, d\mu(t)$. Let C be a nonempty closed convex subset of a Banach space E. A family $\mathcal{S} = \{T(t) : t \in S\}$ is said to be a *nonexpansive semigroup* on C if it satisfies the following:

(1) for each $t \in S$, $T(t)$ is a nonexpansive mapping from C into itself;

(2) $T(ts) = T(t)T(s)$ for each $t, s \in S$.

We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, that is, $\bigcap_{t \in S} F(T(t))$. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that for each $x \in C$, $\{T(t)x : t \in S\}$ is contained in a weakly compact convex subset of C. Let X be a subspace of $B(S)$ with $1 \in X$ such that the mapping $t \mapsto \langle T(t)x, x^* \rangle$ is in X for each $x \in C$ and $x^* \in E^*$, and let μ be a mean on X. Following [15], we also write $T_\mu x$ instead of $\int T(t)x\, d\mu(t)$ for $x \in C$. We remark that T_μ is nonexpansive on C and $T_\mu x = x$ for each $x \in F(\mathcal{S})$; for more details, see [19].

We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors converges strongly to x. Similarly, we write $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$) will symbolize weak convergence. For any element z and any set A, we denote the distance between z and A by $d(z, A) = \inf\{\|z - y\| : y \in A\}$.

3. Weak convergence theorem

Throughout the rest of this paper, we assume that S is a semigroup. Let C be a nonempty weakly compact convex subset of a Banach space E and let $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup of C. We consider the following iterative procedure (see [21]): $x_0 = x \in C$ and

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{\mu_n} x_n \tag{3.1}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0,1)$.

Lemma 3.1. *Let C be a nonempty weakly compact convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let X be a subspace of $B(S)$ with $1 \in X$ such that the function $t \mapsto \langle T(t)x, x^* \rangle$ is in X for each $x \in C$ and $x^* \in E^*$. Let $\{\mu_n\}$ be a sequence of means on S and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$ for every $n \in \mathbb{N}$. Let $x \in C$ and let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{\mu_n} x_n \tag{3.2}$$

for every $n \in \mathbb{N}$. Then, $\|x_{n+1} - w\| \le \|x_n - w\|$ and $\lim_{n\to\infty} \|x_n - w\|$ exists for each $w \in F(\mathcal{S})$.

Proof. Let $w \in F(\mathcal{S})$. By the definition of $\{x_n\}$, we obtain that

$$\begin{aligned} ||x_n - w|| &= ||\alpha_n (x_{n-1} - w) + (1-\alpha_n)(T_{\mu_n} x_n - w)|| \\ &\le \alpha_n ||x_{n-1} - w|| + (1-\alpha_n) ||T_{\mu_n} x_n - w|| \\ &\le \alpha_n ||x_{n-1} - w|| + (1-\alpha_n) ||x_n - w|| \end{aligned} \tag{3.3}$$

and hence

$$\alpha_n ||x_n - w|| \le \alpha_n ||x_{n-1} - w||. \tag{3.4}$$

It follows from $\alpha_n \neq 0$ that $\{\|x_n - w\|\}$ is a nonincreasing sequence. Hence, it follows that $\lim_{n\to\infty} \|x_n - w\|$ exists. □

The following lemma was proved by Shioji and Takahashi [16] (see also [3, 9]).

Lemma 3.2 (Shioji and Takahashi). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let X be a subspace of $B(S)$ with $1 \in X$ such that it is l_s-invariant for each $s \in S$, and the function $t \mapsto \langle T(t)x, x^* \rangle$ is in X for each $x \in C$ and $x^* \in E^*$. Let $\{\mu_n\}$ be a sequence of means on S which is strongly left regular. For each $r > 0$ and $t \in S$,*

$$\lim_{n\to\infty} \sup_{y \in C\cap B_r} ||T_{\mu_n} y - T(t) T_{\mu_n} y|| = 0. \tag{3.5}$$

The following lemma is crucial in the proofs of the main theorems.

Lemma 3.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let X be a subspace of $B(S)$ with $1 \in X$ such that it is l_s-invariant for each $s \in S$, and the function $t \mapsto \langle T(t)x, x^* \rangle$ is in X for each $x \in C$ and $x^* \in E^*$. Let $\{\mu_n\}$ be a sequence of means on S*

which is strongly left regular and let $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 < \alpha_n < 1$ *for every* $n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$*. Let* $x \in C$ *and let* $\{x_n\}$ *be the sequence defined by* $x_0 = x$ *and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{\mu_n} x_n \tag{3.6}$$

for every $n \in \mathbb{N}$*. Then, for each* $t \in S$*,*

$$\lim_{n\to\infty} ||x_n - T(t)x_n|| = 0. \tag{3.7}$$

Proof. For $x \in C$ and $w \in F(\mathcal{S})$, put $r = \|x - w\|$ and set $D = \{u \in E : \|u - w\| \le r\} \cap C$. Then, D is a nonempty bounded closed convex subset of C which is $T(s)$-invariant for each $s \in S$ and contains $x_0 = x$. So, without loss of generality, we may assume that C is bounded. Fix $\varepsilon > 0$, $t \in S$ and set $M_0 = \sup\{\|z\| : z \in C\}$. Then, from Proposition 2.2, there exists $\delta > 0$ such that

$$\overline{\text{co}} F_\delta(T(t)) \subset F_\varepsilon(T(t)). \tag{3.8}$$

From Lemma 3.2 there exists $l \in \mathbb{N}$ such that

$$||T_{\mu_i} y - T(t) T_{\mu_i} y|| < \delta \tag{3.9}$$

for every $i \ge l$ and $y \in C$. We have, for each $k \in \mathbb{N}$,

$$\begin{aligned}
x_{l+k} &= \alpha_{l+k} x_{l+k-1} + (1-\alpha_{l+k}) T_{\mu_{l+k}} x_{l+k} \\
&= \alpha_{l+k}\{\alpha_{l+k-1} x_{l+k-2} + (1-\alpha_{l+k-1}) T_{\mu_{l+k-1}} x_{l+k-1}\} + (1-\alpha_{l+k}) T_{\mu_{l+k}} x_{l+k} \\
&\ \ \vdots \\
&= \left(\prod_{i=l}^{l+k} \alpha_i\right) x_{l-1} + \sum_{j=l}^{l+k-1} \left\{ \left(\prod_{i=j+1}^{l+k} \alpha_i\right)(1-\alpha_j) T_{\mu_j} x_j \right\} + (1-\alpha_{l+k}) T_{\mu_{l+k}} x_{l+k}.
\end{aligned} \tag{3.10}$$

Put

$$y_k = \frac{1}{1 - \prod_{i=l}^{l+k} \alpha_i} \left\{ \sum_{j=l}^{l+k-1} \left\{ \left(\prod_{i=j+1}^{l+k} \alpha_i\right)(1-\alpha_j) T_{\mu_j} x_j \right\} + (1-\alpha_{l+k}) T_{\mu_{l+k}} x_{l+k} \right\}. \tag{3.11}$$

From

$$\sum_{j=l}^{l+k-1} \left\{ \left(\prod_{i=j+1}^{l+k} \alpha_i\right)(1-\alpha_j) \right\} + (1-\alpha_{l+k}) = 1 - \prod_{i=l}^{l+k} \alpha_i, \tag{3.12}$$

we obtain $y_k \in \mathrm{co}(\{T_{\mu_i}x_i\}_{i=l}^{i=l+k})$ and

$$x_{l+k} = \left(\prod_{i=l}^{l+k}\alpha_i\right)x_{l-1} + \left(1-\prod_{i=l}^{l+k}\alpha_i\right)y_k. \tag{3.13}$$

From (3.9), we know that for every $k \in \mathbb{N}$, $T_{\mu_i}x_i \in F_\delta(T(t))$ for $i = l, l+1, \ldots, l+k$. So, it follows from (3.8) that $y_k \in \mathrm{co}F_\delta(T(t)) \subset F_\varepsilon(T(t))$ for every $k \in \mathbb{N}$. We know from Abel-Dini theorem that $\sum_{i=l}^{\infty}(1-\alpha_i) = \infty$ implies $\prod_{i=l}^{\infty}\alpha_i = 0$. Then, there exists $m \in \mathbb{N}$ such that $\prod_{i=l}^{l+k}\alpha_i < \varepsilon/(2M_0)$ for every $k \geq m$. From (3.13), we obtain

$$||x_{l+k} - y_k|| = \left(\prod_{i=l}^{l+k}\alpha_i\right)||x_{l-1} - y_k|| < \frac{\varepsilon}{2M_0}\cdot 2M_0 = \varepsilon \tag{3.14}$$

for every $k \geq m$. Hence,

$$\begin{aligned} ||T(t)x_{l+k} - x_{l+k}|| &\leq ||T(t)x_{l+k} - T(t)y_k|| + ||T(t)y_k - y_k|| + ||y_k - x_{l+k}|| \\ &\leq 2||x_{l+k} - y_k|| + ||T(t)y_k - y_k|| \leq 2\varepsilon + \varepsilon = 3\varepsilon \end{aligned} \tag{3.15}$$

for every $k \geq m$. Since $\varepsilon > 0$ is arbitrary, we get $\lim_{n\to\infty}\|T(t)x_n - x_n\| = 0$ for each $t \in S$. □

Now, we prove a weak convergence theorem for a nonexpansive semigroup in a Banach space.

Theorem 3.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let X be a subspace of $B(S)$ with $1 \in X$ such that it is l_s-invariant for each $s \in S$, and the function $t \mapsto \langle T(t)x, x^*\rangle$ is in X for each $x \in C$ and $x^* \in E^*$. Let $\{\mu_n\}$ be a sequence of means on S which is strongly left regular and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Let $x \in C$ and let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_{\mu_n}x_n \tag{3.16}$$

for every $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to an element of $F(\mathcal{S})$.

Proof. Since E is reflexive and $\{x_n\}$ is bounded, $\{x_n\}$ must contain a subsequence of $\{x_n\}$ which converges weakly to a point in C. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ which converge weakly to y and z, respectively. From Lemma 3.3 and Proposition 2.1, we know $y, z \in F(\mathcal{S})$. We will show $y = z$. Suppose $y \neq z$. Then from Lemma 3.1 and Opial's condition, we have

$$\begin{aligned} \lim_{n\to\infty}||x_n - y|| &= \lim_{i\to\infty}||x_{n_i} - y|| < \lim_{i\to\infty}||x_{n_i} - z|| \\ &= \lim_{n\to\infty}||x_n - z|| = \lim_{j\to\infty}||x_{n_j} - z|| \\ &< \lim_{j\to\infty}||x_{n_j} - y|| = \lim_{j\to\infty}||x_n - y||. \end{aligned} \tag{3.17}$$

This is a contradiction. Hence $\{x_n\}$ converges weakly to an element of $F(\mathcal{S})$. □

4. Strong convergence theorems

In this section, we discuss the strong convergence of the iterates defined by (3.1). Now, we can prove a strong convergence theorem for a nonexpansive semigroup in a Banach space (see also [2]).

THEOREM 4.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let X be a subspace of $B(S)$ with $1 \in X$ such that it is l_s-invariant for each $s \in S$, and the function $t \mapsto \langle T(t)x, x^* \rangle$ is in X for each $x \in C$ and $x^* \in E^*$. Let $\{\mu_n\}$ be a sequence of means on S which is strongly left regular and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Let $x \in C$ and let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{\mu_n} x_n \tag{4.1}$$

for every $n \in \mathbb{N}$. If there exists some $T(s) \in \mathcal{S}$ which is semicompact, then $\{x_n\}$ converges strongly to an element of $F(\mathcal{S})$.

Proof. Since the nonexpansive mapping $T(s)$ is semicompact, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $y \in C$ such that $x_{n_j} \to y$ as $j \to \infty$. By Lemma 3.3, we have that

$$0 = \lim_{j\to\infty} \|x_{n_j} - T(t)x_{n_j}\| = \|y - T(t)y\| \tag{4.2}$$

for each $t \in S$ and hence $y \in F(\mathcal{S})$. Then, it follows from Lemma 3.1 that

$$\lim_{n\to\infty} \|x_n - y\| = \lim_{j\to\infty} \|x_{n_j} - y\| = 0. \tag{4.3}$$

Therefore, $\{x_n\}$ converges strongly to an element of $F(\mathcal{S})$. □

Next, we give a necessary and sufficient condition for the strong convergence of the iterates.

THEOREM 4.2. *Let C be a nonempty weakly compact convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let X be a subspace of $B(S)$ with $1 \in X$ such that the function $t \mapsto \langle T(t)x, x^* \rangle$ is in X for each $x \in C$ and $x^* \in E^*$. Let $\{\mu_n\}$ be a sequence of means on S and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$ for every $n \in \mathbb{N}$. Let $x \in C$ and let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{\mu_n} x_n \tag{4.4}$$

for every $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$ if and only if $\underline{\lim}_{n\to\infty} d(x_n, F(\mathcal{S})) = 0$.

Proof. The necessity is obvious. So, we will prove the sufficiency. Assume

$$\underline{\lim}_{n\to\infty} d(x_n, F(\mathcal{S})) = 0. \tag{4.5}$$

By Lemma 3.1, we have

$$\|x_{n+1} - w\| \le \|x_n - w\| \tag{4.6}$$

for each $w \in F(\mathcal{S})$. Taking the infimum over $w \in F(\mathcal{S})$,

$$d(x_{n+1}, F(\mathcal{S})) \le d(x_n, F(\mathcal{S})) \tag{4.7}$$

and hence the sequence $\{d(x_n, F(\mathcal{S}))\}$ is nonincreasing. So, from $\underline{\lim}_{n\to\infty} d(x_n, F(\mathcal{S})) = 0$, we obtain that

$$\lim_{n\to\infty} d(x_n, F(\mathcal{S})) = 0. \tag{4.8}$$

We will show that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$. There exists a positive integer N such that for each $n \ge N$, $d(x_n, F(\mathcal{S})) < \varepsilon/2$. For any $l, k \ge N$ and $w \in F(\mathcal{S})$, we obtain

$$\|x_l - w\| \le \|x_N - w\|, \qquad \|x_k - w\| \le \|x_N - w\| \tag{4.9}$$

by Lemma 3.1. So, we obtain $\|x_l - x_k\| \le \|x_l - w\| + \|w - x_k\| \le 2\|x_N - w\|$ and hence

$$\|x_l - x_k\| \le 2\inf\{\|x_N - y\| : y \in F(\mathcal{S})\} = 2d(x_N, F(\mathcal{S})) < \varepsilon \tag{4.10}$$

for every $l, k \ge N$. This implies that $\{x_n\}$ is a Cauchy sequence. Since C is a closed subset of E, $\{x_n\}$ converges strongly to $z_0 \in C$. Further, since $F(\mathcal{S})$ is a closed subset of C, (4.8) implies that $z_0 \in F(\mathcal{S})$. Thus, we have that $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$. □

Theorem 4.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let X be a subspace of $B(S)$ with $1 \in X$ such that it is l_s-invariant for each $s \in S$, and the function $t \mapsto \langle T(t)x, x^*\rangle$ is in X for each $x \in C$ and $x^* \in E^*$. Let $\{\mu_n\}$ be a sequence of means on S which is strongly left regular and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$. Assume that there exist $s \in S$ and $k > 0$ such that*

$$\|(I - T(s))z\| \ge kd(z, F(\mathcal{S})) \tag{4.11}$$

for every $z \in C$*. Let* $x \in C$ *and let* $\{x_n\}$ *be the sequence defined by* $x_0 = x$ *and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{\mu_n} x_n \tag{4.12}$$

for every $n \in \mathbb{N}$*. Then,* $\{x_n\}$ *converges strongly to an element of* $F(\mathcal{S})$*.*

Proof. From Lemma 3.3, we obtain that $\|(I - T(s))x_n\| \to 0$ as $n \to 0$. Then, it follows from (4.11) that

$$\lim_{n\to\infty} kd(x_n, F(\mathcal{S})) = 0 \tag{4.13}$$

for some $k > 0$. Therefore, we can conclude that $\{x_n\}$ converges strongly to an element of $F(\mathcal{S})$ by Theorem 4.2. □

5. Deduced theorems from main results

Throughout this section, we assume that C is a nonempty closed convex subset of a uniformly convex Banach space E, x is an element of C, and $\{\alpha_n\}$ is a sequence of real numbers such that $0 < \alpha_n < 1$ for each $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. As direct consequences of Theorems 3.4 and 4.1, we can show some convergence theorems.

Theorem 5.1. *Let* T *be a nonexpansive mapping from* C *into itself such that* $F(T) \neq \emptyset$*. Let* $\{x_n\}$ *be the sequence defined by* $x_0 = x$ *and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)\frac{1}{n+1}\sum_{i=0}^{n} T^i x_n \tag{5.1}$$

for every $n \in \mathbb{N}$*. If* E *satisfies Opial's condition, then* $\{x_n\}$ *converges weakly to a fixed point of* T*, and if* T *is semicompact, then* $\{x_n\}$ *converges strongly to a fixed point of* T*.*

Theorem 5.2. *Let* T *be as in Theorem 5.1. Let* $\{s_n\}$ *be a sequence of positive real numbers with* $s_n \uparrow 1$*. Let* $\{x_n\}$ *be the sequence defined by* $x_0 = x$ *and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)(1-s_n)\sum_{i=0}^{\infty} {s_n}^i T^i x_n \tag{5.2}$$

for every $n \in \mathbb{N}$*. If* E *satisfies Opial's condition, then* $\{x_n\}$ *converges weakly to a fixed point of* T*, and if* T *is semicompact, then* $\{x_n\}$ *converges strongly to a fixed point of* T*.*

Theorem 5.3. *Let* T *be as in Theorem 5.1. Let* $\{q_{n,m} : n, m \in \mathbb{Z}^+\}$ *be a sequence of real numbers such that* $q_{n,m} \geq 0$, $\sum_{m=0}^{\infty} q_{n,m} = 1$ *for every* $n \in \mathbb{Z}^+$ *and* $\lim_{n\to\infty}\sum_{m=0}^{\infty}|q_{n,m+1} - q_{n,m}| = 0$*. Let* $\{x_n\}$ *be the sequence defined by* $x_0 = x$ *and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)\sum_{m=0}^{\infty} q_{n,m} T^m x_n \tag{5.3}$$

for every $n \in \mathbb{N}$*. If* E *satisfies Opial's condition, then* $\{x_n\}$ *converges weakly to a fixed point of* T*, and if* T *is semicompact, then* $\{x_n\}$ *converges strongly to a fixed point of* T*.*

Theorem 5.4. *Let T and U be commutative nonexpansive mappings from C into itself such that $F(T) \cap F(U) \neq \varnothing$. Let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)\frac{1}{(n+1)^2}\sum_{i,j=0}^{n} T^i U^j x_n \tag{5.4}$$

for every $n \in \mathbb{N}$. If E satisfies Opial's condition, then $\{x_n\}$ converges weakly to a common fixed point of T and U, and if either T or U is semicompact, then $\{x_n\}$ converges strongly to a common fixed point of T and U.

Let C be a closed convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in [0,\infty)\}$ be a family of nonexpansive mappings of C into itself. Then, $\mathcal{S}$ is called a one-parameter nonexpansive semigroup on C if it satisfies the following conditions: $T(0) = I$, $T(t+s) = T(t)T(s)$ for all $t,s \in [0,\infty)$ and $T(t)x$ is continuous in $t \in [0,\infty)$ for each $x \in C$.

Theorem 5.5. *Let $\mathcal{S} = \{T(t) : t \in [0,\infty)\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \varnothing$. Let $\{s_n\}$ be a sequence of positive real numbers with $s_n \to \infty$. Let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)\frac{1}{s_n}\int_0^{s_n} T(t)x_n\,dt \tag{5.5}$$

for every $n \in \mathbb{N}$. If E satisfies Opial's condition, then $\{x_n\}$ converges weakly to a common fixed point of $\mathcal{S}$, and if there exists some $T(s) \in \mathcal{S}$ which is semicompact, then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$.

Theorem 5.6. *Let $\mathcal{S}$ be as in Theorem 5.5. Let $\{r_n\}$ be a sequence of positive real numbers with $r_n \to 0$. Let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) r_n \int_0^{\infty} e^{-r_n t} T(t)x_n\,dt \tag{5.6}$$

for every $n \in \mathbb{N}$. If E satisfies Opial's condition, then $\{x_n\}$ converges weakly to a common fixed point of $\mathcal{S}$, and if there exists some $T(s) \in \mathcal{S}$ which is semicompact, then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$.

Theorem 5.7. *Let $\mathcal{S}$ be as in Theorem 5.5. Let $\{q_n\}$ be a sequence of continuous functions from $[0,\infty)$ into $[0,\infty)$ such that $\int_0^\infty q_n(t)dt = 1$ for every $n \in \mathbb{N}$, $\lim_{n\to\infty} q_n(t) = 0$ for $t \geq 0$ and $\lim_{n\to\infty}\int_0^\infty |q_n(t+s) - q_n(t)|dt = 0$ for all $s \geq 0$. Let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)\int_0^{\infty} q_n(t)T(t)x_n\,dt \tag{5.7}$$

for every $n \in \mathbb{N}$. If E satisfies Opial's condition, then $\{x_n\}$ converges weakly to a common fixed point of $\mathcal{S}$, and if there exists some $T(s) \in \mathcal{S}$ which is semicompact, then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$.

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