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A generalised fixed point theorem of Presic type in cone metric spaces and application to Markov process

Reny George[1,2*], KP Reshma[2] and R Rajagopalan[1]

* Correspondence: renygeorge02@yahoo.com
[1]Department of Mathematics, College of Science, Al-Kharj University, Al-Kharj, Kingdom of Saudi Arabia
Full list of author information is available at the end of the article

Abstract

A generalised common fixed point theorem of Presic type for two mappings $f: X \to X$ and $T: X^k \to X$ in a cone metric space is proved. Our result generalises many well-known results.

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1. Introduction

Considering the convergence of certain sequences, Presic [1] proved the following:

Theorem 1.1. *Let (X, d) be a metric space, k a positive integer, $T: X^k \to X$ be a mapping satisfying the following condition:*

$$d(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \le q_1 \cdot d(x_1, x_2) + q_2 \cdot d(x_2, x_3) + \cdots + q_k \cdot d(x_k, x_{k+1}) \quad (1.1)$$

where $x_1, x_2, \dots, x_{k+1}$ are arbitrary elements in X and $q_1, q_2, \dots, q_k$ are non-negative constants such that $q_1 + q_2 + \cdots + q_k < 1$. Then, there exists some $x \in X$ such that $x = T(x, x, \dots, x)$. Moreover if $x_1, x_2, \dots, x_k$ are arbitrary points in X and for $n \in N$ $x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1})$, then the sequence $<x_n>$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \dots, \lim x_n)$.

Note that for $k = 1$ the above theorem reduces to the well-known Banach Contraction Principle. Ciric and Presic [2] generalising the above theorem proved the following:

Theorem 1.2. *Let (X, d) be a metric space, k a positive integer, $T: X^k \to X$ be a mapping satisfying the following condition:*

$$d(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \le \lambda . max\{d(x_1, x_2), d(x_2, x_3), \dots d(x_k, x_{k+1}) \quad (1.2)$$

where $x_1, x_2, \dots, x_{k+1}$ are arbitrary elements in X and $\lambda \in (0,1)$. Then, there exists some $x \in X$ such that $x = T(x, x, \dots, x)$. Moreover if $x_1, x_2, \dots, x_k$ are arbitrary points in X and for $n \in N x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1})$, then the sequence $<x_n>$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \dots, \lim x_n)$. If in addition T satisfies $D(T(u, u, \dots u), T(v, v, \dots v)) < d(u, v)$, for all $u, v \in X$ then x is the unique point satisfying $x = T(x, x, \dots, x)$.

Huang and Zang [3] generalising the notion of metric space by replacing the set of real numbers by ordered normed spaces, defined a cone metric space and proved some fixed point theorems of contractive mappings defined on these spaces. Rezapour and Hamlbarani [4], omitting the assumption of normality, obtained generalisations of results of [3]. In [5], Di Bari and Vetro obtained results on points of coincidence and common fixed points in non-normal cone metric spaces. Further results on fixed point theorems in such spaces were obtained by several authors, see [5-15].

The purpose of the present paper is to extend and generalise the above Theorems 1.1 and 1.2 for two mappings in non-normal cone metric spaces and by removing the requirement of $D(T(u, u, \dots u), T(v, v, \dots v)) < d(u, v)$, for all $u, v \in X$ for uniqueness of the fixed point, which in turn will extend and generalise the results of [3,4].

2. Preliminaries

Let E be a real Banach space and P a subset of E. Then, P is called a *cone* if

(i) P is closed, non-empty, and satisfies $P \neq \{0\}$,

(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$, i.e. $P \cap (-P) = 0$

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$, and $x \ll y$ if $y - x \in intP$, where $intP$ denote the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $|| x || \leq K || y ||$.

Definition 2.1. [3]*Let X be a non empty set. Suppose that the mapping d: $X \times X \to E$ satisfies:*

(d_1) $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$

(d_2)$d(x, y) = d(y, x)$ *for all* $x, y \in X$

(d_3)$d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$

Then, d is called a *conemetric* on X and (X, d) is called a *conemetricspace*.

Definition 2.2. [3]*Let* (X, d) *be a cone metric space. The sequence* $\{x_n\}$ *in X is said to be:*

(a) A convergent sequence if for every $c \in E$ with $0 \ll c$, there is $n_0 \in N$ such that for all $n \geq n_0$, $d(x_n, x) \ll c$ for some $x \in X$. We denote this by $lim_{n\to\infty} x_n = x$.

(b) A Cauchy sequence if for all $c \in E$ with $0 \ll c$, there is no $\in N$ such that $d(x_m, x_n) \ll c$, for all $m, n \geq n_0$.

(c) A cone metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent in X.

(d) A self-map T on X is said to be *continuous* if $lim_{n\to\infty} x_n = x$ implies that $lim_{n\to\infty} T(x_n) = T(x)$, for every sequence $\{x_n\}$in X.

Definition 2.3. *Let* (X, d) *be a metric space, k a positive integer,* $T: X^k \to X$ *and* $f : X \to X$ *be mappings.*

(a) An element $x \in$ X said to be a *coincidence point* of f and T if and only if $f(x) = T(x, x, \dots, x)$. If $x = f(x) = T(x, x, \dots, x)$, then we say that x; is a *common fixed point* of f and T. If $w = f(x) = T(x, x, \dots, x)$, then w is called a point of coincidence of f and T.

(b) Mappings f and T are said to be *commuting* if and only if $f(T(x, x, \ldots x)) = T(fx, fx, \ldots fx)$ for all $x \in X$.

(c) Mappings f and T are said to be *weakly compatible* if and only if they commute at their coincidence points.

Remark 2.4. *For k = 1, the above definitions reduce to the usual definition of commuting and weakly compatible mappings in a metric space.*

The set of coincidence points of f and T is denoted by $C(f, T)$.

3. Main results

Consider a function $\varphi: E^k \to E$ such that

(a) φ is an increasing function, i.e $x_1 < y_1, x_2 < y_2, \ldots, x_k < y_k$ implies $\varphi(x_1, x_2, \ldots, x_k) < \varphi(y_1, y_2, \ldots, y_k)$.

(b) $\varphi(t, t, t, \ldots) \leq t$, for all $t \in X$

(c) φ is continuous in all variables.

Now, we present our main results as follows:

Theorem 3.1. *Let (X, d) be a cone metric space with solid cone P contained in a real Banach space E. For any positive integer k, let $T: X^k \to X$ and $f: X \to X$ be mappings satisfying the following conditions:*

$$T(X^k) \subseteq f(X) \tag{3.1}$$

$$d(T(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})) \leq \lambda\phi(d(fx_1, fx_2), d(fx_2, fx_3), \ldots, (fx_k, fx_{k+1})) \tag{3.2}$$

where $x_1, x_2, \ldots, x_{k+1}$ are arbitrary elements in X and $\lambda \in (0, \frac{1}{k})$*and*

$$f(X) \text{ is complete} \tag{3.3}$$

there exist elements $x_1, x_2, \ldots, x_k$ in X such that

$$R = max\left(\frac{d(fx_1, fx_2)}{\theta}, \frac{d(fx_2, fx_3)}{\theta^2}, \ldots, \frac{d(fx_k, T(x_1, x_2, \ldots, x_k))}{\theta^k}\right) \text{ exist in } E \tag{3.4}$$

where $\theta = \lambda^{\frac{1}{k}}$. Then, f and T have a coincidence point, i.e. $C(f, T) \neq \emptyset$.

Proof. By (3.1) and (3.4) we define sequence $\langle y_n \rangle$in $f(X)$ as $y_n = fx_n$ for $n = 1, 2, \ldots, k$ and $y_{n+k} = f(x_{n+k}) = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$, $n = 1, 2, \ldots$ Let $\alpha_n = d(y_n, y_{n+1})$. By the method of mathematical induction, we will now prove that

$$\alpha_n \leq R.\theta^n \tag{3.5}$$

for all n. Clearly by the definition of R, (3.5) is true for $n = 1, 2, \ldots, k$. Let the k inequalities $\alpha_n \leq R\theta^n, \alpha_{n+1} \leq R\theta^{n+1}, \ldots, \alpha_{n+k-1} \leq R\theta^{n+k-1}$ be the induction hypothesis. Then, we have

$$\begin{aligned}
&\alpha_{n+k} = d(y_{n+k}, y_{n+k+1}) \\
&= d(T(x_n, x_{n+1}, \ldots, x_{n+k-1}), T(x_{n+1}, x_{n+2}, \ldots, x_{n+k})) \\
&\leq \lambda\phi(d(fx_n, fx_{n+1}), d(fx_{n+1}, fx_{n+2}), \ldots, d(fx_{n+k-1}, fx_{n+k})) \\
&= \lambda\phi(\alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+k-1}) \\
&\leq \lambda\phi(R\theta^n, R\theta^{n+1}, \ldots, R.\theta^{n+k-1}) \\
&\leq \lambda\phi(R\theta^n, R\theta^n, \ldots, R\theta^n) \leq \lambda R\theta^n = R.\theta^{n+k}.
\end{aligned}$$

Thus inductive proof of (3.5) is complete. Now for $n, p \in N$, we have

$$\begin{aligned} d(y_n, y_{n+p}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p-1}, y_{n+p}), \\ &\leq R\theta^n + R\theta^{n+1} + \cdots + R\theta^{n+p-1} \\ &\leq R\theta^n(1 + \theta + \theta^2 + \cdots) \\ &= \frac{R\theta^n}{1-\theta} \end{aligned}$$

Let $0 \ll c$ be given. Choose $\delta > 0$ such that $c + N_\delta(0) \subseteq P$ where $N_\delta(0) = \{y \in E; \|y\| < \delta\}$. Also choose a natural number N_1 such that $\frac{R\theta^n}{1-\theta} \in N_\delta(0)$, for all $n > N_1$. Then, $\frac{R\theta^n}{1-\theta} \ll c$ for all $n \geq N_1$. Thus, $d(y_n, y_{n+p}) \leq \frac{R\theta^n}{1-\theta} \ll c$ for all $n \geq N_1$. Hence, sequence $< y_n >$ is a Cauchy sequence in $f(X)$, and since f(X) is complete, there exists $v, u \in X$ such that $lim_{n\to\infty} y_n = v = f(u)$. Choose a natural number N_2 such that $d(y_n, y_{n+1}) \ll \frac{c}{\lambda(k+1)}$ and $d(x, y_{n+1}) \ll \frac{c}{k+1}$ for all $n \geq N_2$.

Then for all $n \geq N_2$

$$\begin{aligned} & d(fu, T(u, u, \ldots u)) \leq d(fu, y_{n+k}) + d(y_{n+k}, T(u, u, \ldots u)) \\ &= d(fu, y_{n+k}) + d(T(x_n, x_{n+1}, \ldots x_{n+k-1}), T(u, u, \ldots u)) \\ &\leq d(fu, y_{n+k}) + d(T(u, u, \ldots u), T(u, u, \ldots x_n)) + d(T(u, u, \ldots x_n), T(u, u, \ldots x_n, x_{n+1})) \\ &+ \cdots d(T(u, x_n, \ldots x_{n+k-2}), T(x_n, x_{n+1} \ldots x_{n+k-1}) \\ &\leq d(fu, y_{n+k}) + \lambda\phi\{d(fu, fu), d(fu, fu), \ldots, d(fu, fx_n)\} \\ &+\lambda\phi\{d(fu, fu), d(fu, fu), \ldots, d(fu, fx_n), d(fx_n, fx_{n+1})\} + \cdots \\ &+\lambda\phi\{d(fu, fx_n), d(fx_n, fx_{n+1}), \ldots d(fx_{n+k-2}, fx_{n+k-1})\}. \\ &= d(fu, y_{n+k}) + \lambda\phi(0, 0, \ldots, d(fu, fx_n)) \\ &+\lambda\phi(0, 0, \ldots, d(fu, fx_n), d(fx_n, fx_{n+1})) + \cdots \\ &+\lambda\phi(d(fu, fx_n), d(fx_n, fx_{n+1}), \ldots d(fx_{n+k-2}, fx_{n+k-1})). \\ &\ll \frac{c}{k+1} + \lambda\phi\left(\frac{c}{\lambda(k+1)}, \frac{c}{\lambda(k+1)}, \ldots, \frac{c}{\lambda(k+1)}\right) + \lambda\phi\left(\frac{c}{\lambda(k+1)}, \frac{c}{\lambda(k+1)}, \ldots, \frac{c}{\lambda(k+1)}\right) \\ &+ \cdots + \lambda\phi\left(\frac{c}{\lambda(k+1)}, \frac{c}{\lambda(k+1)}, \ldots, \frac{c}{\lambda(k+1)}\right) \\ &\ll \frac{c}{k+1} + \lambda\frac{c}{\lambda(k+1)} \cdots + \lambda\frac{c}{\lambda(k+1)} = c. \end{aligned}$$

Thus, $d(fu, T(u, u, \ldots u)) \ll \frac{c}{m}$ for all $m \geq 1$.

So, $\frac{c}{m} - d(fu, T(u, u, \ldots u)) \in P$ for all $m \geq 1$. Since $\frac{c}{m} \to 0$ as $m \to \infty$ and P is closed, $-d(fu, T(u, u, \ldots u)) \in P$, but $P \cap (-P) = /0/$. Therefore, $d(fu, T(u, u, \ldots u)) = 0$. Thus, $fu = T(u, u, u, \ldots, u)$, i.e. $C(f, T) \neq \emptyset$. □

Theorem 3.2. *Let* (X, d) *be a cone metric space with solid cone* P *contained in a real Banach space* E. *For any positive integer* k, *let* $T: X^k \to X$ *and* $f: X \to X$ *be mappings satisfying* (3.1), (3.2), (3.3) *and let there exist elements* $x_1, x_2, \ldots x_k$ *in* X *satisfying* (3.4). *If* f *and* T *are weakly compatible, then* f *and* T *have a unique common fixed point. Moreover if* $x_1, x_2, \ldots, x_k$ *are arbitrary points in* X *and for* $n \in N$, $y_{n+k} = f(x_{n+k}) = T(x_n, x_{n+1}, \ldots x_{n+k-1})$, $n = 1, 2, \ldots$, *then the sequence* $< y_n >$ *is convergent and* $\lim y_n = f(\lim y_n) = T(\lim y_n, \lim y_n, \ldots, \lim y_n)$.

Proof. As proved in Theorem 3.1, there exists $v, u \in X$ such that $lim_{n\to\infty} y_n = v = f(u) = T(u, u, u \ldots u)$. Also since f and T are weakly compatible $f(T(u, u, \ldots u) = T(fu, fu, fu \ldots fu)$. By (3.2) we have,

$$
\begin{aligned}
&d(ffu, fu) = d(fT(u, u, \ldots u), T(u, u, \ldots u)) = d(T(fu, fu, fu, \ldots fu), T(u, u, \ldots u)) \\
&\leq d(T(fu, fu, fu, \ldots fu), T(fu, fu, \ldots fu, u)) + d(T(fu, fu, \ldots fu, u), \\
&T(fu, fu, \ldots, u, u)) + \cdots + d(T(fu, u, \ldots u, u), T(u, u, \ldots u)) \\
&\leq \lambda\phi(d(ffu, ffu), \ldots d(ffu, ffu), d(ffu, fu)) + \lambda\phi(d(ffu, ffu), \ldots d(ffu, fu), \\
&d(fu, fu)) + \cdots \lambda\phi(d(ffu, fu), \ldots d(fu, fu), d(fu, fu)) \\
&= \lambda\phi(0, 0, 0, \ldots d(ffu, fu)) + \lambda\phi(0, 0 \ldots 0, d(ffu, fu), 0) + \cdots . \lambda\phi(d(ffu, fu), 0, 0 \ldots 0) = k\lambda d(ffu, fu).
\end{aligned}
$$

Repeating this process n times we get, $d(ffu, fu) < k^n \lambda^n d(ffu, fu)$. So $k^n \lambda^n d(ffu, fu) - d(ffu, fu) \in P$ for all $n \geq 1$. Since $k^n \lambda^n \to 0$ as $n \to \infty$ and P is closed, $-d(ffu, fu) \in P$, but $P \cap (-P) = \{0\}$. Therefore, $d(ffu, fu) = 0$ and so $ffu = fu$. Hence, we have, $fu = ffu = f(T(u, u, \ldots u)) = T(fu, fu, fu \ldots fu)$, i.e. fu is a common fixed point of f and T, and $lim\ y_n = f(lim\ y_n) = T(lim\ y_n, lim\ y_n, \ldots lim\ y_n)$. Now suppose x, y be two fixed points of f and T. Then,

$$
\begin{aligned}
&d(x, y) = d(T(x, x, x \ldots x), T(y, y, y \ldots y)) \\
&\leq d(T(x, x, \ldots x), T(x, x, \ldots x, y)) + d(T(x, x, \ldots x, y), T(x, x, x \ldots x, y, y)) \\
&+ \cdots + d(T(x, y, y, \ldots y), T(y, y, \ldots y)) \\
&\leq \lambda\phi\{d(fx, fx), d(fx, fx), \ldots, d(fx, fy)\} \\
&+\lambda\phi\{d(fx, fx), d(fx, fx), \ldots d(fx, fy), d(fy, fy)\} \\
&+ \cdots + \lambda\phi\{d(fx, fy), d(fy, fy), \ldots d(fy, fy)\}. \\
&= \lambda\phi(0, 0, \ldots, d(fx, fy)) + \lambda\phi(0, 0, \ldots d(fx, fy), 0) + \cdots + \lambda\phi(d(fx, fy), 0, 0, 0, \ldots 0)). \\
&= k\lambda d(fx, fy) = k\lambda d(x, y).
\end{aligned}
$$

Repeating this process n times we get as above, $d(x, y) \leq k^n \lambda^n d(x, y)$ and so as $n \to \infty d(x, y) = 0$, which implies $x = y$. Hence, the common fixed point is unique. □

Remark 3.3. *Theorem 3.2 is a proper extension and generalisation of Theorems 1.1 and 1.2.*

Remark 3.4. *If we take $k = 1$ in Theorem, 3.2, we get the extended and generalised versions of the result of* [3]*and* [4].

Example 3.5. *Let $E = R^2$, $P = \{(x, y) \in E \backslash x, y \geq 0\}$, $X = [0, 2]$ and d: $X \times X \to E$ such that $d(x, y) = (|x - y|, |x - y|)$. Then, d is a cone metric on X. Let T: $X^2 \to X$ and f: $X \to X$ be defined as follows:*

$$
\begin{aligned}
&T(x, y) = \tfrac{(x^2+y^2)}{4} + \tfrac{1}{2}\ if (x, y) \in [0, 1] \times [0, 1] \\
&T(x, y) = \tfrac{(x+y)}{4} + \tfrac{1}{2}\ if (x, y) \in [1, 2] \times [1, 2] \\
&T(x, y) = \tfrac{(x^2+y)}{4} + \tfrac{1}{2}\ if (x, y) \in [0, 1] \times [1, 2] \\
&T(x, y) = \tfrac{(x+y^2)}{4} + \tfrac{1}{2}\ if (x, y) \in [1, 2] \times [0, 1] \\
&f(x) = x^2\ if\ x \in [0, 1] \\
&f(x) = x\ if\ x \in [1, 2]
\end{aligned}
$$

T and f satisfies condition (*3.2*) *as follows:*

Case 1. x, y, $z \in$ [0, 1]

$$
\begin{aligned}
&d(T(x, y), T(y, z)) = (|T(x, y) - T(y, z)|, |T(x, y) - T(y, z)|) \\
&= (|\tfrac{x^2-z^2}{4}|, |\tfrac{x^2-z^2}{4}|) \\
&\leq (|\tfrac{x^2-z^2}{4}| + |\tfrac{y^2-z^2}{4}|, |\tfrac{x^2-y^2}{4}| + |\tfrac{y^2-z^2}{4}|) \\
&\leq \tfrac{1}{2}.max\{d(fx, fy), d(fy, fz)\}
\end{aligned}
$$

Case 2. $x, y \in [0, 1]$ *and* $z \in [1,2]$

$$d(T(x,y),T(y,z)) = (|\tfrac{x^2+y^2}{4} - \tfrac{y^2+z}{4}|, |\tfrac{x^2+y^2}{4} - \tfrac{y^2+z}{4}|)$$
$$\leq (|\tfrac{x^2-y^2}{4}| + |\tfrac{y^2-z}{4}|, |\tfrac{x^2-y^2}{4}| + |\tfrac{y^2-z}{4}|)$$
$$\leq \tfrac{1}{2}.max\{(fx,fy), d(fy,fz)\}$$

Case 3. $x \in [0, 1]$ *and* $y; z \in [1,2]$

$$d(T(x,y),T(y,z)) = (|\tfrac{x^2+y}{4} - \tfrac{y+z}{4}|, |\tfrac{x^2+y}{4} - \tfrac{y+z}{4}|)$$
$$= (|\tfrac{x^2-z}{4}|, |\tfrac{x^2-z}{4}|)$$
$$\leq (|\tfrac{x^2-y}{4}1| + |\tfrac{y-z}{4}|, |\tfrac{x^2-y}{4}| + |\tfrac{y-z}{4}|)$$
$$\leq \tfrac{1}{./2}.max\{d(fx,fy), d(fy,fz)\}$$

Case 4. $x, y, z \in [1,2]$

$$d(T(x,y),T(y,z)) = (|\tfrac{x+y}{4} - \tfrac{y+z}{4}|, |\tfrac{x+y}{4} - \tfrac{y+z}{4}|)$$
$$\leq (|\tfrac{x-y}{4}| + |\tfrac{y-z}{4}|, |\tfrac{x-y}{4}| + |\tfrac{y-z}{4}|)$$
$$\leq \tfrac{1}{2}. \max\{(fx,fy), d(fy,fz)\}.$$

Similarly in all other cases $d(T(x,y),T(y,z)) \leq \frac{1}{2}.max\{(fx,fy), d(fy,fz)\}$. *Thus, f and T satisfy condition (3.2) with* $\varphi(x_1, x_2) = max\{x_1, x_2\}$. *We see that* $C(f, T) = 1$, *f and T commute at 1. Finally, 1 is the unique common fixed point of f and T.*

4. An application to markov process

Let $\Delta_{n-1} = \{x \in R_+^n : \Sigma_{i=1}^n x_i = 1\}$ denote the n - 1 dimensional unit simplex. Note that any $x \in \Delta_{n-1}$ may be regarded as a probability over the n possible states. A random process in which one of the n states is realised in each period t = 1, 2, ... with the probability conditioned on the current realised state is called Markov Process. Let a_{ij} denote the conditional probability that state i is reached in succeeding period starting in state j. Then, given the prior probability vector x^t in period t, the posterior probability in period $t + 1$ is given by $x_i^{t+1} = \sum a_{ij}x_j^t$ for each i = 1, 2, ... To express this in matrix notation, we let x^t denote a column vector. Then, $x^{t+1} = Ax^t$. Observe that the properties of conditional probability require each $a_{ij} \geq 0$ and $\sum_{i=1}^n a_{ij} = 1$ for each j. If for any period t, $x^{t+1} = x^t$ then x^t is a stationary distribution of the Markov Process. Thus, the problem of finding a stationary distribution is equivalent to the fixed point problem $Ax^t = x^t$.

For each i, let $\varepsilon_i = min_j a_{ij}$ and define $\varepsilon = \sum_{i=1}^n \varepsilon_i$.

Theorem 4.1. *Under the assumption* $a_{i,j} > 0$, *a unique stationary distribution exist for the Markov process.*

Proof. Let d: $\Delta_{n-1} \times \Delta_{n-1} \to R^2$ be given by $d(x,y) = (\sum_{i=1}^n |x_i - y_i|, \alpha \sum_{i=1}^n |x_i - y_i|)$ for all $x, y \in \Delta_{n-1}$ and some $\alpha \geq 0$.

Clearly $d(x, y) \geq (0, 0)$ for all $x, y \in \Delta_{n-1}$ and $d(x,y) = (0,0) \Rightarrow (\sum_{i=1}^n |x_i - y_i|, \alpha \sum_{i=1}^n |x_i - y_i|) = (0,0) \Rightarrow |x_i - y_i| = 0$ for all $i \Rightarrow x = y$. Also $x = y \Rightarrow x_i = y_i$ for all $i \Rightarrow |x_i - y_i| = 0 \Rightarrow \sum_{i=1}^n |x_i - y_i| = 0 \Rightarrow d(x,y) = (0,0)$

$$
\begin{aligned}
d(x,y) &= \left(\sum_{i=1}^{n} |x_i - y_i|, \alpha \sum_{i=1}^{n} |x_i - y_i|\right) \\
&= \left(\sum_{i=1}^{n} |y_i - x_i|, \alpha \sum_{i=1}^{n} |y_i - x_i|\right) = d(y,x) \\
d(x,y) &= \left(\sum_{i=1}^{n} |x_i - y_i|, \alpha \sum_{i=1}^{n} |x_i - y_i|\right) \\
&= \left(\sum_{i=1}^{n} |(x_i - z_i) + (z_i - y_i)|, \alpha \sum_{i=1}^{n} |(x_i - z_i)| + |(z_i - y_i)|\right) \\
&\leq \left(\sum_{i=1}^{n} |(x_i - z_i)| + |(z_i - y_i)|, \alpha \sum_{i=1}^{n} |(x_i - z_i)| + |(z_i - y_i)|\right) \\
&= \left(\sum_{i=1}^{n} |(x_i - z_i)|, \alpha \sum_{i=1}^{n} |(x_i - z_i)|\right) + \left(\sum_{i=1}^{n} |(z_i - y_i)|, \alpha \sum_{i=1}^{n} |(z_i - y_i)|\right) \\
&= d(x,z) + d(z,x).
\end{aligned}
$$

So Δ_{n-1} is a cone metric space. For $x \in \Delta_{n-1}$, let $y = Ax$. Then each $y_i = \sum_{j=1}^{n} a_{ij} x_j \geq 0$. Further more, since each $\sum_{i=1}^{n} a_{ij} = 1$, we have $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} x_j \sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} x_j = 1$, so $y \in \Delta_{n-1}$. Thus, we see that $A: \Delta_{n-1} \to \Delta_{n-1}$. We will show that A is a contraction. Let A_i denote the ith row of A. Then for any $x, y \in \Delta_{n-1}$, we have

$$
\begin{aligned}
d(Ax, Ay) &= \left(\sum_{i=1}^{n} |(Ax)_i - (Ay)_i|, \alpha \sum_{i=1}^{n} |(Ax)_i - (Ay)_i|\right) \\
&= \left(\sum_{i=1}^{n} \left|\sum_{j=1}^{n} a_{ij} x_j - a_{ij} y_j\right|, \alpha \sum_{i=1}^{n} \left|\sum_{j=1}^{n} a_{ij} x_j - a_{ij} y_j\right|\right) \\
&= \left(\sum_{i=1}^{n} \left|\sum_{j=1}^{n} (a_{ij} - \varepsilon_i)(x_j - y_j) + \varepsilon_i (x_j - y_j)\right|,\right. \\
&\quad \left. \alpha \sum_{i=1}^{n} \left|\sum_{j=1}^{n} (a_{ij} - \varepsilon_i)(x_j - y_j) + \varepsilon_i (x_j - y_j)\right|\right) \\
&\leq \left(\sum_{i=1}^{n} \left(\left|\sum_{j=1}^{n} (a_{ij} - \varepsilon_i)(x_j - y_j)\right| + \varepsilon_i \left|\sum_{j=1}^{n} (x_j - y_j)\right|\right),\right. \\
&\quad \left. \alpha \left(\sum_{i=1}^{n} \left(\left|\sum_{j=1}^{n} (a_{ij} - \varepsilon_i)(x_j - y_j)\right| + \varepsilon_i \left|\sum_{j=1}^{n} (x_j - y_j)\right|\right)\right)\right) \\
&\leq \left(\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - \varepsilon_i)|x_j - y_j|, \alpha \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - \varepsilon_i)|x_j - y_j|\right) \\
&\quad \left(\mathit{Since} \sum_{j=1}^{n} (x_j - y_j) = 0\right) \\
&= \left(\sum_{j=1}^{n} |x_j - y_j| \sum_{i=1}^{n} (a_{ij} - \varepsilon_i),\right. \\
&\quad \left. \alpha \sum_{j=1}^{n} |x_j - y_j| \sum_{i=1}^{n} (a_{ij} - \varepsilon_i)\right) \\
&= \left(\sum_{j=1}^{n} |x_j - y_j|(1 - \varepsilon), \alpha \sum_{j=1}^{n} |x_j - y_j|(1 - \varepsilon)\right) \\
&\quad \left(\mathit{Since} \sum_{i=1}^{n} a_{ij} = 1 \text{ and } \sum_{i=1}^{n} \varepsilon_i = \varepsilon\right) \\
&= d(x,y)(1 - \varepsilon)
\end{aligned}
$$

which establishes that A is a contraction mapping. Thus, Theorem 3.2 with $k = 1$ and f as identity mapping ensures a unique stationary distribution for the Markov Process. Moreover for any $x^0 \in \Delta_{n-1}$, the sequence $\langle A^n x^0 \rangle$ converges to the unique stationary distribution. □

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Author details
[1]Department of Mathematics, College of Science, Al-Kharj University, Al-Kharj, Kingdom of Saudi Arabia [2]Department of Mathematics and Computer Science, St. Thomas College, Ruabandha Bhilai, Durg, Chhattisgarh State, 490006, India

Authors' contributions
RG gave the idea of this work. All authors worked on the proofs and examples. KPR and RR drafted the manuscript. RG read the manuscript and made necessary corrections. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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