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Tripled fixed point of W-compatible mappings in abstract metric spaces

Hassen Aydi[1], Mujahid Abbas[2], Wutiphol Sintunavarat[3*] and Poom Kumam[3*]

*Correspondence:
poom_teun@hotmail.com;
poom.kum@kmutt.ac.th
[3]Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok, 10140, Thailand
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce the concepts of W-compatible mappings for mappings $F: X \times X \times X \to X$ and $g: X \to X$, where (X, d) is an abstract metric space. We establish tripled coincidence and common tripled fixed point theorems in such spaces. The presented theorems generalize and extend several well-known comparable results in literature, in particular the results of Abbas, Ali and Radenović (Appl. Math. Comput. 217:195-202, 2010). We also provide an example to illustrate our obtained results.

MSC: 54H25; 47H10

Keywords: W-compatible mappings; tripled coincidence point; common tripled fixed point; cone metric space

1 Introduction

Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach fixed point theorem. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. There exists vast literature on this topic and this is a very active area of research at present. Banach contraction principle has been generalized in different directions in different spaces by mathematicians over the years, for more details on this and related topics, we refer to [6, 10, 11, 13, 14, 21, 22, 24, 25, 27–30] and references therein.

Fixed point theory in K-metric and K-normed spaces was developed by Perov *et al.* [18], Mukhamadijev and Stetsenko [16], Vandergraft [33] and others. For more details on fixed point theory in K-metric and K-normed spaces, we refer the reader to a fine survey paper by Zabreĭko [32]. The main idea is to use an ordered Banach space instead of the set of real numbers, as the codomain for a metric.

In 2007 Huang and Zhang [12] reintroduced such spaces under the name of cone metric spaces and reintroduced the definition of convergent and Cauchy sequences in terms of interior points of the underlying cone. They also proved some fixed point theorems in such spaces in the same work. Afterwards, many papers about fixed point theory in cone metric spaces appeared (see, for example, [1–4, 7, 17, 19, 23, 26, 31]). In 2011, Abbas *et al.* [1] introduced the concept of w-compatible mappings and obtained a coupled coincidence point and a coupled point of coincidence for mappings satisfying a contractive condition in cone metric spaces. Very recently, Aydi *et al.* [5] introduced the concepts of $\widetilde{w}$-compatible mappings and generalized the results in [1].

The aim of this paper is to introduce the concepts of W-compatible mappings. Based on this notion, a tripled coincidence point and a common tripled fixed point for mappings

$F : X \times X \times X \to X$ and $g : X \to X$ are obtained, where (X, d) is a cone metric space. It is worth mentioning that our results do not rely on the assumption of normality condition of the cone. The presented theorems generalize and extend several well-known comparable results in literature. An example is also given in support of our results.

The following definitions and results will be needed in the sequel.

Definition 1 ([12, 32]) Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, non-empty and $P \neq \{0_E\}$,
(b) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ imply that $ax + by \in P$,
(c) $P \cap (-P) = \{0_E\}$,

where 0_E is the zero vector of E.

Given a cone, define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \ll y$ for $y - x \in \operatorname{Int} P$, where $\operatorname{Int} P$ stands for interior of P. Also we will use $x \prec y$ to indicate that $x \preceq y$ and $x \neq y$.

The cone P in the normed space $(E, \|\cdot\|)$ is called normal whenever there is a number $k \geq 1$ such that for all $x, y \in E$, $0_E \preceq x \preceq y$ implies $\|x\| \leq k\|y\|$. The least positive number satisfying this norm inequality is called a normal constant of P.

Definition 2 Let X be a non-empty set. Suppose that $d : X \times X \to E$ satisfies:

(d1) $0_E \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_E$ if and only if $x = y$,
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric [12] or K-metric [32] on X and (X, d) is called a cone metric space [12] or K-metric space [32].

The concept of a K-metric space is more general than that of a metric space, because each metric space is a K-metric space where $X = R$ and $P = [0, +\infty)$. For other examples of K-metric spaces, we refer to [32], pp.853 and 854.

Definition 3 ([12]) Let X be a K-metric space, $\{x_n\}$ a sequence in X and $x \in X$. For every $c \in E$ with $c \gg 0_E$, we say that $\{x_n\}$ is

(C1) a Cauchy sequence if there is some $k \in \mathbb{N}$ such that, for all $n, m \geq k$, $d(x_n, x_m) \ll c$,
(C2) a convergent sequence if there is some $k \in \mathbb{N}$ such that, for all $n \geq k$, $d(x_n, x) \ll c$. This limit is denoted by $\lim_{n \to +\infty} x_n = x$ or $x_n \to x$ as $n \to +\infty$.

Note that every convergent sequence in a K-metric space X is a Cauchy sequence. A K-metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

2 Main results

For simplicity, we denote from now on $\underbrace{X \times X \cdots X \times X}_{k \text{ terms}}$ by X^k where $k \in \mathbb{N}$ and X is a non-empty set. We start by recalling some definitions.

Definition 4 (Bhashkar and Lakshmikantham [9]) An element $(x, y) \in X^2$ is called a coupled fixed point of the mapping $F : X^2 \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 5 (Lakshmikantham and Ćirić [15]) An element $(x,y) \in X^2$ is called

(i) a coupled coincidence point of mappings $F : X^2 \to X$ and $g : X \to X$ if $gx = F(x,y)$ and $gy = F(y,x)$, and (gx, gy) is called a coupled point of coincidence;

(ii) a common coupled fixed point of mappings $F : X^2 \to X$ and $g : X \to X$ if $x = gx = F(x,y)$ and $y = gy = F(y,x)$.

Note that if g is the identity mapping, then Definition 5 reduces to Definition 4.

Definition 6 (Abbas, Khan and Radenović [1]) The mappings $F : X^2 \to X$ and $g : X \to X$ are called w-compatible if $g(F(x,y)) = F(gx,gy)$ whenever $gx = F(x,y)$ and $gy = F(y,x)$.

In 2010, Samet and Vetro [20] introduced a fixed point of order $N \geq 3$. In particular, for $N = 3$, we have the following definition.

Definition 7 (Samet and Vetro [20]) An element $(x,y,z) \in X^3$ is called a tripled fixed point of a given mapping $F : X^3 \to X$ if $x = F(x,y,z)$, $y = F(y,z,x)$ and $z = F(z,x,y)$.

Note that, Berinde and Borcut [8] defined differently the notion of a tripled fixed point in the case of ordered sets in order to keep true the mixed monotone property. For more details, see [8].

Now, we introduce the following definitions.

Definition 8 An element $(x,y,z) \in X^3$ is called

(i) a tripled coincidence point of mappings $F : X^3 \to X$ and $g : X \to X$ if $gx = F(x,y,z)$, $gy = F(y,x,z)$ and $gz = F(z,x,y)$. In this case (gx, gy, gz) is called a tripled point of coincidence;

(ii) a common tripled fixed point of mappings $F : X^3 \to X$ and $g : X \to X$ if $x = gx = F(x,y,z)$, $y = gy = F(y,z,x)$ and $z = gz = F(z,x,y)$.

Example 1 Let $X = \mathbb{R}$. We define $F : X^3 \to X$ and $g : X \to X$ as follows:

$$F(x,y,z) = \sin(x)\sin(y)\sin(z) \quad \text{and} \quad gx = 1 + x - \frac{\pi}{2}$$

for all $x,y,z \in X$. Then $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ is a tripled coincidence point of F and g, and $(1,1,1)$ is a tripled point of coincidence.

Definition 9 Mappings $F : X^3 \to X$ and $g : X \to X$ are called W-compatible if

$$F(gx,gy,gz) = g\bigl(F(x,y,z)\bigr)$$

whenever $F(x,y,z) = gx$, $F(y,z,x) = gy$ and $F(z,y,x) = gz$.

Example 2 Let $X = \mathbb{R}$. Define $F : X^3 \to X$ and $g : X \to X$ as follows:

$$F(x,y,z) = \frac{x - y + z}{6} \quad \text{and} \quad gx = \frac{x}{3}$$

for all $x,y,z \in X$. One can show that (x,y,z) is a tripled coincidence point of F and g if and only if $x = y = z = 0$. Here $(0,0,0)$ is a common tripled fixed point of F and g. Note that F and g are W-compatible.

Now we prove our first result.

Theorem 1 *Let (X,d) be a K-metric space with a cone P having non-empty interior and $F : X^3 \to X$ and $g : X \to X$ be mappings such that $F(X^3) \subseteq g(X)$. Suppose that for any $x,y,z,u,v,w \in X$, the following condition*

$$\begin{aligned} d\big(F(x,y,z),F(u,v,w)\big) \preceq{}& a_1 d\big(F(x,y,z),gx\big) + a_2 d\big(F(y,z,x),gy\big) + a_3 d\big(F(z,x,y),gz\big) \\ &+ a_4 d\big(F(u,v,w),gu\big) + a_5 d\big(F(v,w,u),gv\big) \\ &+ a_6 d\big(F(w,u,v),gw\big) + a_7 d\big(F(u,v,w),gx\big) \\ &+ a_8 d\big(F(v,w,u),gy\big) + a_9 d\big(F(w,u,v),gz\big) \\ &+ a_{10} d\big(F(x,y,z),gu\big) + a_{11} d\big(F(y,z,x),gv\big) \\ &+ a_{12} d\big(F(z,x,y),gw\big) + a_{13} d(gx,gu) \\ &+ a_{14} d(gy,gv) + a_{15} d(gz,gw) \end{aligned}$$

holds, where a_i, $i = 1,\dots,15$ are nonnegative real numbers such that $\sum_{i=1}^{15} a_i < 1$. Then F and g have a tripled coincidence point provided that $g(X)$ is a complete subspace of X.

Proof Let x_0, y_0 and z_0 be three arbitrary points in X. By given assumption, there exists (x_1,y_1,z_1) such that

$$F(x_0,y_0,z_0) = gx_1, \qquad F(y_0,z_0,x_0) = gy_1 \quad \text{and} \quad F(z_0,x_0,y_0) = gz_1.$$

Continuing this process, we can construct three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$F(x_n,y_n,z_n) = gx_{n+1}, \qquad F(y_n,z_n,x_n) = gy_{n+1} \quad \text{and} \quad F(z_n,x_n,y_n) = gz_{n+1} \quad \forall n \in \mathbb{N}. \tag{1}$$

Now, taking $(x,y,z) = (x_n,y_n,z_n)$ and $(u,v,w) = (x_{n+1},y_{n+1},z_{n+1})$ in the considered contractive condition and using (1), we have

$$\begin{aligned} d(gx_{n+1},gx_{n+2}) ={}& d\big(F(x_n,y_n,z_n),F(x_{n+1},y_{n+1},z_{n+1})\big) \\ \preceq{}& a_1 d\big(F(x_n,y_n,z_n),gx_n\big) + a_2 d\big(F(y_n,z_n,x_n),gy_n\big) + a_3 d\big(F(z_n,x_n,y_n),gz_n\big) \\ &+ a_4 d\big(F(x_{n+1},y_{n+1},z_{n+1}),gx_{n+1}\big) + a_5 d\big(F(y_{n+1},z_{n+1},x_{n+1}),gy_{n+1}\big) \\ &+ a_6 d\big(F(z_{n+1},x_{n+1},y_{n+1}),gz_{n+1}\big) + a_7 d\big(F(x_{n+1},y_{n+1},z_{n+1}),gx_n\big) \\ &+ a_8 d\big(F(y_{n+1},z_{n+1},x_{n+1}),gy_n\big) + a_9 d\big(F(z_{n+1},x_{n+1},y_{n+1}),gz_n\big) \\ &+ a_{10} d\big(F(x_n,y_n,z_n),gx_{n+1}\big) \\ &+ a_{11} d\big(F(y_n,z_n,x_n),gy_{n+1}\big) + a_{12} d\big(F(z_n,x_n,y_n),gz_{n+1}\big) \\ &+ a_{13} d(gx_n,gx_{n+1}) + a_{14} d(gy_n,gy_{n+1}) + a_{15} d(gz_n,gz_{n+1}) \\ ={}& a_1 d(gx_{n+1},gx_n) + a_2 d(gy_{n+1},gy_n) + a_3 d(gz_{n+1},gz_n) + a_4 d(gx_{n+2},gx_{n+1}) \\ &+ a_5 d(gy_{n+2},gy_{n+1}) + a_6 d(gz_{n+2},gz_{n+1}) \end{aligned}$$

$$\begin{aligned}&+ a_7 d(gx_{n+2}, gx_n) + a_8 d(gy_{n+2}, gy_n)\\&+ a_9 d(gz_{n+2}, gz_n) + a_{13} d(gx_n, gx_{n+1})\\&+ a_{14} d(gy_n, gy_{n+1}) + a_{15} d(gz_n, gz_{n+1}).\end{aligned}$$

Then, using the triangular inequality one can write for any $n \in \mathbb{N}$

$$\begin{aligned}(1 - a_4 - a_7) d(gx_{n+2}, gx_{n+1}) \preceq{}& (a_1 + a_7 + a_{13}) d(gx_{n+1}, gx_n) + (a_2 + a_8 + a_{14}) d(gy_n, gy_{n+1})\\&+ (a_3 + a_9 + a_{15}) d(gz_n, gz_{n+1}) + (a_5 + a_8) d(gy_{n+2}, gy_{n+1})\\&+ (a_6 + a_9) d(gz_{n+2}, gz_{n+1}). \qquad (2)\end{aligned}$$

Similarly, following similar arguments to those given above, we obtain

$$\begin{aligned}(1 - a_4 - a_7) d(gy_{n+2}, gy_{n+1}) \preceq{}& (a_1 + a_7 + a_{13}) d(gy_{n+1}, gy_n) + (a_2 + a_8 + a_{14}) d(gz_n, gz_{n+1})\\&+ (a_3 + a_9 + a_{15}) d(gx_n, gx_{n+1}) + (a_5 + a_8) d(gz_{n+2}, gz_{n+1})\\&+ (a_6 + a_9) d(gx_{n+2}, gx_{n+1}) \qquad (3)\end{aligned}$$

and

$$\begin{aligned}(1 - a_4 - a_7) d(gz_{n+2}, gz_{n+1}) \preceq{}& (a_1 + a_7 + a_{13}) d(gz_{n+1}, gz_n) + (a_2 + a_8 + a_{14}) d(gx_n, gx_{n+1})\\&+ (a_3 + a_9 + a_{15}) d(gy_n, gy_{n+1}) + (a_5 + a_8) d(gx_{n+2}, gx_{n+1})\\&+ (a_6 + a_9) d(gy_{n+2}, gy_{n+1}). \qquad (4)\end{aligned}$$

Denote

$$\delta_n = d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n).$$

Adding (2) to (4), we have

$$\begin{aligned}&(1 - a_4 - a_5 - a_6 - a_7 - a_8 - a_9)\delta_{n+1}\\&\quad \preceq (a_1 + a_2 + a_3 + a_7 + a_8 + a_9 + a_{13} + a_{14} + a_{15})\delta_n. \qquad (5)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}d(gx_{n+2}, gx_{n+1}) ={}& d\big(F(x_{n+1}, y_{n+1}, z_{n+1}), F(x_n, y_n, z_n)\big)\\\preceq{}& a_1 d\big(F(x_{n+1}, y_{n+1}, z_{n+1}), gx_{n+1}\big) + a_2 d\big(F(y_{n+1}, z_{n+1}, x_{n+1}), gy_{n+1}\big)\\&+ a_3 d\big(F(z_{n+1}, x_{n+1}, y_{n+1}), gz_{n+1}\big) + a_4 d\big(F(x_n, y_n, z_n), gx_n\big)\\&+ a_5 d\big(F(y_n, z_n, x_n), gy_n\big)\\&+ a_6 d\big(F(z_n, x_n, y_n), gz_n\big) + a_7 d\big(F(x_n, y_n, z_n), gx_{n+1}\big)\\&+ a_8 d\big(F(y_n, z_n, x_n), gy_{n+1}\big) + a_9 d\big(F(z_n, x_n, y_n), gz_{n+1}\big)\\&+ a_{10} d\big(F(x_{n+1}, y_{n+1}, z_{n+1}), gx_n\big)\end{aligned}$$

$$
\begin{aligned}
&+ a_{11}d\big(F(y_{n+1}, z_{n+1}, x_{n+1}), gy_n\big) + a_{12}d\big(F(z_{n+1}, x_{n+1}, y_{n+1}), gz_n\big)\\
&+ a_{13}d(gx_{n+1}, gx_n) + a_{14}d(gy_{n+1}, gy_n) + a_{15}d(gz_{n+1}, gz_n)\\
= {}& a_1 d(gx_{n+2}, gx_{n+1}) + a_2 d(gy_{n+2}, gy_{n+1})\\
&+ a_3 d(gz_{n+2}, gz_{n+1}) + a_4 d(gx_{n+1}, gx_n)\\
&+ a_5 d(gy_{n+1}, gy_n) + a_6 d(gz_{n+1}, gz_n)\\
&+ a_{10} d(gx_{n+2}, gx_n) + a_{11} d(gy_{n+2}, gy_n)\\
&+ a_{12} d(gz_{n+2}, gz_n) + a_{13} d(gx_n, gx_{n+1})\\
&+ a_{14} d(gy_n, gy_{n+1}) + a_{15} d(gz_n, gz_{n+1}).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&(1 - a_1 - a_{10})d(gx_{n+2}, gx_{n+1})\\
&\quad\preceq (a_4 + a_{10} + a_{13})d(gx_{n+1}, gx_n) + (a_5 + a_{11} + a_{14})d(gy_n, gy_{n+1})\\
&\qquad + (a_6 + a_{12} + a_{15})d(gz_n, gz_{n+1}) + (a_2 + a_{11})d(gy_{n+2}, gy_{n+1})\\
&\qquad + (a_3 + a_{12})d(gz_{n+2}, gz_{n+1}).
\end{aligned} \tag{6}
$$

Similarly,

$$
\begin{aligned}
&(1 - a_1 - a_{10})d(gy_{n+2}, gy_{n+1})\\
&\quad\preceq (a_4 + a_{10} + a_{13})d(gy_{n+1}, gy_n) + (a_5 + a_{11} + a_{14})d(gz_n, gz_{n+1})\\
&\qquad + (a_6 + a_{12} + a_{15})d(gx_n, gx_{n+1}) + (a_2 + a_{11})d(gz_{n+2}, gz_{n+1})\\
&\qquad + (a_3 + a_{12})d(gx_{n+2}, gx_{n+1})
\end{aligned} \tag{7}
$$

and

$$
\begin{aligned}
&(1 - a_1 - a_{10})d(gz_{n+2}, gz_{n+1})\\
&\quad\preceq (a_4 + a_{10} + a_{13})d(gz_{n+1}, gz_n) + (a_5 + a_{11} + a_{14})d(gx_n, gx_{n+1})\\
&\qquad + (a_6 + a_{12} + a_{15})d(gy_n, gy_{n+1}) + (a_2 + a_{11})d(gx_{n+2}, gx_{n+1})\\
&\qquad + (a_3 + a_{12})d(gy_{n+2}, gy_{n+1}).
\end{aligned} \tag{8}
$$

Adding (6) to (8), we obtain that

$$
\begin{aligned}
&(1 - a_1 - a_2 - a_3 - a_{10} - a_{11} - a_{12})\delta_{n+1}\\
&\quad\preceq (a_4 + a_5 + a_6 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15})\delta_n.
\end{aligned} \tag{9}
$$

Finally, from (5) and (9), for any $n \in \mathbb{N}$, we have

$$
\left(2 - \sum_{i=1}^{12} a_i\right)\delta_{n+1} \preceq \left(\left(\sum_{i=1}^{15} a_i\right) + a_{13} + a_{14} + a_{15}\right)\delta_n, \tag{10}
$$

that is

$$\delta_{n+1} \preceq \kappa\delta_n \quad \forall n \in \mathbb{N}, \tag{11}$$

where

$$\kappa = \frac{(\sum_{i=1}^{15} a_i) + a_{13} + a_{14} + a_{15}}{2 - \sum_{i=1}^{12} a_i}.$$

Consequently, we have

$$0_E \preceq \delta_n \preceq \kappa\delta_{n-1} \preceq \cdots \preceq \kappa^n \delta_0. \tag{12}$$

If $\delta_0 = 0_E$, we get $d(gx_0, gx_1) + d(gy_0, gy_1) = d(gz_0, gz_1) = 0_E$, that is, $gx_0 = gx_1$, $gy_0 = gy_1$ and $gz_0 = gz_1$. Therefore, from (1) we have

$$F(x_0, y_0, z_0) = gx_1 = gx_0, \qquad F(y_0, z_0, x_0) = gy_1 = gy_0 \quad \text{and} \quad F(z_0, x_0, y_0) = gz_1 = gz_0,$$

that is, (x_0, y_0, z_0) is a tripled coincidence point of F and g. Now, assume that $\delta_0 \succ 0_E$. If $m > n$, we have

$$d(gx_m, gx_n) \preceq d(gx_m, gx_{m-1}) + d(gx_{m-1}, gx_{m-2}) + \cdots + d(gx_{n+1}, gx_n),$$
$$d(gy_m, gy_n) \preceq d(gy_m, gy_{m-1}) + d(gy_{m-1}, gy_{m-2}) + \cdots + d(gy_{n+1}, gy_n),$$

and

$$d(gz_m, gz_n) \preceq d(gz_m, gz_{m-1}) + d(gz_{m-1}, gz_{m-2}) + \cdots + d(gz_{n+1}, gz_n).$$

Adding above inequalities and using (12), we obtain

$$\begin{aligned} d(gx_m, gx_n) + d(gy_m, gy_n) + d(gz_m, gz_n) &\preceq \delta_{m-1} + \delta_{m-2} + \cdots + \delta_n \\ &\preceq \left(\kappa^{m-1} + \kappa^{m-1} + \cdots + \kappa^n\right)\delta_0 \\ &\preceq \frac{\kappa^n}{1-\kappa}\delta_0. \end{aligned}$$

As $0 \leq \sum_{i=1}^{15} a_i < 1$, we have $0 \leq \kappa < 1$. Hence for any $c \in E$ with $c \gg 0_E$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$ we have $\frac{\kappa^n}{1-\kappa}\delta_0 \ll c$. Furthermore, for any $m > n \geq N$, we get

$$d(gx_m, gx_n) + d(gy_m, gy_n) + d(gz_m, gz_n) \ll c.$$

This implies that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in $g(X)$. By completeness of $g(X)$, there exist $x, y, z \in X$ such that

$$gx_n \to gx, \qquad gy_n \to gy \quad \text{and} \quad gz_n \to gz \quad \text{as} \quad n \to +\infty. \tag{13}$$

Now, we prove that $F(x,y,z) = gx$, $F(y,z,x) = gy$ and $F(z,x,y) = gz$. Note that

$$\begin{aligned} d\big(F(x,y,z),gx\big) &\preceq d\big(F(x,y,z),F(x_n,y_n,z_n)\big) + d\big(F(x_n,y_n,z_n),gx\big) \\ &= d\big(F(x,y,z),F(x_n,y_n,z_n)\big) + d(gx_{n+1},gx). \end{aligned} \tag{14}$$

On the other hand, applying given contractive condition, we obtain

$$\begin{aligned} &d\big(F(x,y,z),F(x_n,y_n,z_n)\big) \\ &\quad\preceq a_1 d\big(F(x,y,z),gx\big) + a_2 d\big(F(y,z,x),gy\big) + a_3 d\big(F(z,x,y),gz\big) \\ &\qquad + a_4 d\big(F(x_n,y_n,z_n),gx_n\big) + a_5 d\big(F(y_n,z_n,x_n),gy_n\big) + a_6 d\big(F(z_n,x_n,y_n),gz_n\big) \\ &\qquad + a_7 d\big(F(x_n,y_n,z_n),gx\big) + a_8 d\big(F(y_n,z_n,x_n),gy\big) \\ &\qquad + a_9 d\big(F(z_n,x_n,y_n),gz\big) + a_{10} d\big(F(x,y,z),gx_n\big) \\ &\qquad + a_{11} d\big(F(y,z,x),gy_n\big) + a_{12} d\big(F(z,x,y),gz_n\big) \\ &\qquad + a_{13} d(gx,gx_n) + a_{14} d(gy,gy_n) + a_{15} d(gz,gz_n) \\ &\quad= a_1 d\big(F(x,y,z),gx\big) + a_2 d\big(F(y,z,x),gy\big) + a_3 d\big(F(z,x,y),gz\big) \\ &\qquad + a_4 d(gx_{n+1},gx_n) + a_5 d(gy_{n+1},gy_n) \\ &\qquad + a_6 d(gz_{n+1},gz_n) + a_7 d(gx_{n+1},gx) + a_8 d(gy_{n+1},gy) \\ &\qquad + a_9 d(gz_{n+1},gz) + a_{10} d\big(F(x,y,z),gx_n\big) \\ &\qquad + a_{11} d\big(F(y,z,x),gy_n\big) + a_{12} d\big(F(z,x,y),gz_n\big) \\ &\qquad + a_{13} d(gx,gx_n) + a_{14} d(gy,gy_n) + a_{15} d(gz,gz_n). \end{aligned}$$

Combining above inequality with (14), and using triangular inequality, we have

$$\begin{aligned} d\big(F(x,y,z),gx\big) &\preceq a_1 d\big(F(x,y,z),gx\big) + a_2 d\big(F(y,z,x),gy\big) \\ &\quad + a_3 d\big(F(z,x,y),gz\big) + a_4 d(gx_{n+1},gx_n) \\ &\quad + a_5 d(gy_{n+1},gy_n) + a_6 d(gz_{n+1},gz_n) + a_7 d(gx_{n+1},gx) \\ &\quad + a_8 d(gy_{n+1},gy) + a_9 d(gz_{n+1},gz) \\ &\quad + a_{10} d\big(F(x,y,z),gx\big) + a_{10} d(gx,gx_n) + a_{11} d\big(F(y,z,x),gy\big) \\ &\quad + a_{11} d(gy,gy_n) + a_{12} d\big(F(z,x,y),gz\big) \\ &\quad + a_{12} d(gz,gz_n) + a_{13} d(gx,gx_n) + a_{14} d(gy,gy_n) \\ &\quad + a_{15} d(gz,gz_n) + d(gx_{n+1},gx). \end{aligned}$$

Therefore,

$$\begin{aligned} &(1 - a_1 - a_{10}) d\big(F(x,y,z),gx\big) - (a_2 + a_{11}) d\big(F(y,z,x),gy\big) - (a_3 + a_{12}) d\big(F(z,x,y),gz\big) \\ &\quad\preceq a_4 d(gx_{n+1},gx_n) + a_5 d(gy_{n+1},gy_n) + a_6 d(gz_{n+1},gz_n) + (1 + a_7) d(gx_{n+1},gx) \\ &\qquad + a_8 d(gy_{n+1},gy) + a_9 d(gz_{n+1},gz) + (a_{10} + a_{13}) d(gx,gx_n) \\ &\qquad + (a_{11} + a_{14}) d(gy,gy_n) + (a_{12} + a_{15}) d(gz,gz_n). \end{aligned} \tag{15}$$

Similarly, we obtain

$$\begin{aligned}
&(1-a_1-a_{10})d\bigl(F(y,z,x),gy\bigr)-(a_2+a_{11})d\bigl(F(z,x,y),gz\bigr)-(a_3+a_{12})d\bigl(F(x,y,z),gx\bigr)\\
&\quad\preceq a_4d(gy_{n+1},gy_n)+a_5d(gz_{n+1},gz_n)+a_6d(gx_{n+1},gx_n)+(1+a_7)d(gy_{n+1},gy)\\
&\qquad+a_8d(gz_{n+1},gz)+a_9d(gx_{n+1},gx)+(a_{10}+a_{13})d(gy,gy_n)\\
&\qquad+(a_{11}+a_{14})d(gz,gz_n)+(a_{12}+a_{15})d(gx,gx_n),
\end{aligned}\tag{16}$$

and

$$\begin{aligned}
&(1-a_1-a_{10})d\bigl(F(z,x,y),gz\bigr)-(a_2+a_{11})d\bigl(F(x,y,z),gx\bigr)-(a_3+a_{12})d\bigl(F(y,z,x),gy\bigr)\\
&\quad\preceq a_4d(gz_{n+1},gz_n)+a_5d(gx_{n+1},gx_n)+a_6d(gy_{n+1},gy_n)+(1+a_7)d(gz_{n+1},gz)\\
&\qquad+a_8d(gx_{n+1},gx)+a_9d(gy_{n+1},gy)+(a_{10}+a_{13})d(gz,gz_n)\\
&\qquad+(a_{11}+a_{14})d(gx,gx_n)+(a_{12}+a_{15})d(gy,gy_n).
\end{aligned}\tag{17}$$

Adding (15) and (17), we get

$$\begin{aligned}
&(1-a_1-a_2-a_3-a_{10}-a_{11}-a_{12})\bigl[d\bigl(F(x,y,z),gx\bigr)+d\bigl(F(y,z,x),gy\bigr)+d\bigl(F(z,x,y),gz\bigr)\bigr]\\
&\quad\preceq(a_4+a_5+a_6)\delta_n+(1+a_7+a_8+a_9)\bigl[d(gx_{n+1},gx)+d(gy_{n+1},gy)+d(gz_{n+1},gz)\bigr]\\
&\qquad+(a_{10}+a_{11}+a_{12}+a_{13}+a_{14}+a_{15})\bigl[d(gx,gx_n)+d(gy,gy_n)+d(gz,gz_n)\bigr]\\
&\quad\preceq\delta_n+2\bigl[d(gx_{n+1},gx)+d(gy_{n+1},gy)+d(gz_{n+1},gz)\bigr]\\
&\qquad+\bigl[d(gx,gx_n)+d(gy,gy_n)+d(gz,gz_n)\bigr].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&d\bigl(F(x,y,z),gx\bigr)+d\bigl(F(y,z,x),gy\bigr)+d\bigl(F(z,x,y),gz\bigr)\\
&\quad\preceq\alpha\delta_n+\beta d(gx_{n+1},gx)+\gamma d(gy_{n+1},gy)+\sigma d(gz_{n+1},gz)\\
&\qquad+\theta d(gx,gx_n)+\xi d(gy,gy_n)+\rho d(gz,gz_n),
\end{aligned}$$

where

$$\begin{aligned}
&\alpha=\theta=\xi=\rho=\frac{1}{1-a_1-a_2-a_3-a_{10}-a_{11}-a_{12}},\\
&\beta=\sigma=\gamma=\frac{2}{1-a_1-a_2-a_3-a_{10}-a_{11}-a_{12}}.
\end{aligned}$$

From (12) and (13), for any $c\gg 0_E$ there exists $N\in\mathbb{N}$ such that

$$\begin{aligned}
&\delta_n\preceq\frac{c}{7\alpha},\qquad d(gx_n,gx)\preceq\frac{c}{7\max\{\beta,\theta\}},\\
&d(gy_n,gy)\preceq\frac{c}{7\max\{\gamma,\xi\}},\qquad d(gz_n,gz)\preceq\frac{c}{7\max\{\sigma,\rho\}}
\end{aligned}$$

for all $n \geq N$. Thus, for all $n \geq N$, we have

$$d\big(F(x,y,z),gx\big) + d\big(F(y,z,x),gy\big) + d\big(F(z,x,y),gz\big) \preceq \frac{c}{7} + \frac{c}{7} + \frac{c}{7} + \frac{c}{7} + \frac{c}{7} + \frac{c}{7} + \frac{c}{7} = c.$$

It follows that $d(F(x,y,z),gx) = d(F(y,z,x),gy) = d(F(z,x,y),gz) = 0_E$, that is $F(x,y,z) = gx$, $F(y,z,x) = gy$ and $F(z,x,y) = gz$. □

As consequences of Theorem 1, we give the following corollaries.

Corollary 1 *Let (X,d) be a K-metric space with a cone P having non-empty interior. Let $F : X^3 \to X$ and $g : X \to X$ be mappings such that $F(X^3) \subseteq g(X)$ and for any $x,y,z,u,v,w \in X$, the following condition*

$$\begin{aligned} d\big(F(x,y,z),F(u,v,w)\big) \preceq{} & \alpha_1\big[d\big(F(x,y,z),gx\big) + d\big(F(y,z,x),gy\big) + d\big(F(z,x,y),gz\big)\big] \\ & + \alpha_2\big[d\big(F(u,v,w),gu\big) + d\big(F(v,w,u),gv\big) + d\big(F(w,u,v),gw\big)\big] \\ & + \alpha_3\big[d\big(F(u,v,w),gx\big) + d\big(F(v,w,u),gy\big) + d\big(F(w,u,v),gz\big)\big] \\ & + \alpha_4\big[d\big(F(x,y,z),gu\big) + d\big(F(y,z,x),gv\big) + d\big(F(z,x,y),gw\big)\big] \\ & + \alpha_5\big[d(gx,gu) + d(gy,gv) + d(gz,gw)\big] \end{aligned}$$

holds, where α_i, $i = 1,\ldots,5$ are nonnegative real numbers such that $\sum_{i=1}^{5}\alpha_i < 1/3$. Then F and g have a tripled coincidence point provided that $g(X)$ is a complete subspace of X.

Proof It suffices to take $a_1 = a_2 = a_3 = \alpha_1$, $a_4 = a_5 = a_6 = \alpha_2$, $a_7 = a_8 = a_9 = \alpha_3$, $a_{10} = a_{11} = a_{12} = \alpha_4$ and $a_{13} = a_{14} = a_{15} = \alpha_5$ in Theorem 1 with $\sum_{i=1}^{5}\alpha_i < 1/3$. □

Corollary 2 (Abbas, Khan and Radenović [1]) *Let (X,d) be a K-metric space with a cone P having non-empty interior. Let $\tilde{F} : X^2 \to X$ and $g : X \to X$ be mappings satisfying $\tilde{F}(X^2) \subseteq g(X)$, $(g(X),d)$ is a complete subspace of X and for any $x,y,u,v \in X$,*

$$\begin{aligned} d\big(\tilde{F}(x,y),\tilde{F}(u,v)\big) \preceq{} & a_1 d\big(\tilde{F}(x,y),gx\big) + a_2 d\big(\tilde{F}(u,v),gu\big) + a_3 d\big(\tilde{F}(u,v),gx\big) \\ & + a_4 d\big(\tilde{F}(x,y),gu\big) + a_5 d(gx,gu) + a_6 d(gy,gv), \end{aligned} \tag{18}$$

where a_i, $i = 1,\ldots,6$ are nonnegative real numbers such that $\sum_{i=1}^{6} a_i < 1$. Then $\tilde{F}$ and g have a coupled coincidence point $(x,y) \in X^2$, that is, $\tilde{F}(x,y) = gx$ and $\tilde{F}(y,x) = gy$.

Proof Consider the mappings $F : X^3 \to X$ defined by $F(x,y,z) = \tilde{F}(x,y)$ for all $x,y,z \in X$. Then, the contractive condition (18) implies that, for all $x,y,z,u,v,w \in X$

$$\begin{aligned} d\big(F(x,y,z),F(u,v,w)\big) \preceq{} & a_1 d\big(F(x,y,z),gx\big) + a_2 d\big(F(u,v,w),gu\big) + a_3 d\big(F(x,y,z),gu\big) \\ & + a_4 d\big(F(u,v,w),gx\big) + a_5 d(gx,gu) + a_6 d(gy,gv). \end{aligned}$$

Then F and g satisfy the contractive condition of Theorem 1. Clearly other conditions of Theorem 1 are also satisfied as $\tilde{F}(X^2) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Therefore, from Theorem 1, F and g have a tripled fixed point $(x,y,z) \in X^3$ such that

$F(x,y,z) = gx$, $F(y,z,x) = gy$ and $F(z,x,y) = gz$, that is, $\tilde{F}(x,y) = gx$ and $\tilde{F}(y,x) = gy$. This makes end to the proof. □

Now, we are ready to state and prove the result of a common tripled coupled fixed point.

Theorem 2 *Let $F : X^3 \to X$ and $g : X \to X$ be two mappings which satisfy all the conditions of Theorem 1. If F and g are W-compatible, then F and g have a unique common tripled fixed point. Moreover, a common tripled fixed point of F and g is of the form (u,u,u) for some $u \in X$.*

Proof First, we will show that the tripled point of coincidence is unique. Suppose that (x,y,z) and $(x^*,y^*,z^*) \in X^3$ with

$$\begin{cases} gx = F(x,y,z), \\ gy = F(y,z,x), \\ gz = F(z,x,y), \end{cases} \quad \text{and} \quad \begin{cases} gx^* = F(x^*,y^*,z^*), \\ gy^* = F(y^*,z^*,x^*), \\ gz^* = F(z^*,x^*,y^*). \end{cases}$$

Using the contractive condition in Theorem 1, we obtain

$$\begin{aligned} d(gx,gx^*) &= d(F(x,y,z),F(x^*,y^*,z^*)) \\ &\preceq a_1 d(F(x,y,z),gx) + a_2 d(F(y,z,x),gy) + a_3 d(F(z,x,y),gz) \\ &\quad + a_4 d(F(x^*,y^*,z^*),gx^*) + a_5 d(F(y^*,z^*,x^*),gy^*) + a_6 d(F(z^*,x^*,y^*),gz^*) \\ &\quad + a_7 d(F(x^*,y^*,z^*),gx) + a_8 d(F(y^*,z^*,x^*),gy) \\ &\quad + a_9 d(F(z^*,x^*,y^*),gz) + a_{10} d(F(x,y,z),gx^*) \\ &\quad + a_{11} d(F(y,z,x),gy^*) + a_{12} d(F(z,x,y),gz^*) \\ &\quad + a_{13} d(gx,gx^*) + a_{14} d(gy,gy^*) + a_{15} d(gz,gz^*) \\ &= (a_7 + a_{10} + a_{13}) d(gx^*,gx) + (a_8 + a_{11} + a_{14}) d(gy^*,gy) \\ &\quad + (a_9 + a_{12} + a_{15}) d(gz^*,gz). \end{aligned}$$

Similarly, we have

$$\begin{aligned} d(gy,gy^*) &= d(F(y,z,x),F(y^*,z^*,x^*)) \\ &\preceq (a_7 + a_{10} + a_{13}) d(gy^*,gy) + (a_8 + a_{11} + a_{14}) d(gz^*,gz) \\ &\quad + (a_9 + a_{12} + a_{15}) d(gx^*,gx) \end{aligned}$$

and

$$\begin{aligned} d(gz,gz^*) &= d(F(z,x,y),F(z^*,x^*,y^*)) \\ &\preceq (a_7 + a_{10} + a_{13}) d(gz^*,gz) + (a_8 + a_{11} + a_{14}) d(gx^*,gx) \\ &\quad + (a_9 + a_{12} + a_{15}) d(gy^*,gy). \end{aligned}$$

Adding above three inequalities, we get

$$d(gx,gx^*)+d(gy,gy^*)+d(gz,gz^*)\preceq\left(\sum_{i=7}^{15}a_i\right)\left[d(gx,gx^*)+d(gy,gy^*)+d(gz,gz^*)\right].$$

Since $\sum_{i=7}^{15}a_i<1$, we obtain

$$d(gx,gx^*)+d(gy,gy^*)+d(gz,gz^*)=0_E,$$

which implies that

$$gx=gx^*,\qquad gy=gy^*\quad\text{and}\quad gz=gz^*, \tag{19}$$

which implies the uniqueness of the tripled point of coincidence of F and g, that is, (gx,gy,gz). Note that

$$\begin{aligned}
d(gx,gy^*) &= d\big(F(x,y,z),F(y^*,z^*,x^*)\big)\\
&\preceq a_1d\big(F(x,y,z),gx\big)+a_2d\big(F(y,z,x),gy\big)+a_3d\big(F(z,x,y),gz\big)\\
&\quad+a_4d\big(F(y^*,z^*,x^*),gy^*\big)+a_5d\big(F(z^*,x^*,y^*),gz^*\big)+a_6d\big(F(x^*,y^*,z^*),gx^*\big)\\
&\quad+a_7d\big(F(y^*,z^*,x^*),gx\big)+a_8d\big(F(z^*,x^*,y^*),gy\big)\\
&\quad+a_9d\big(F(x^*,y^*,z^*),gz\big)+a_{10}d\big(F(x,y,z),gy^*\big)\\
&\quad+a_{11}d\big(F(y,z,x),gz^*\big)+a_{12}d\big(F(z,x,y),gx^*\big)\\
&\quad+a_{13}d(gx,gy^*)+a_{14}d(gy,gz^*)+a_{15}d(gz,gx^*)\\
&= (a_7+a_{10}+a_{13})d(gy^*,gx)+(a_8+a_{11}+a_{14})d(gz^*,gy)\\
&\quad+(a_9+a_{12}+a_{15})d(gx^*,gz).
\end{aligned}$$

Similarly,

$$\begin{aligned}
d(gy,gz^*) &\preceq (a_7+a_{10}+a_{13})d(gz^*,gy)+(a_8+a_{11}+a_{14})d(gx^*,gz)\\
&\quad+(a_9+a_{12}+a_{15})d(gy^*,gx),
\end{aligned}$$

and

$$\begin{aligned}
d(gz,gx^*) &\preceq (a_7+a_{10}+a_{13})d(gx^*,gz)+(a_8+a_{11}+a_{14})d(gy^*,gx)\\
&\quad+(a_9+a_{12}+a_{15})d(gz^*,gy).
\end{aligned}$$

Adding above inequalities, we obtain

$$d(gx,gy^*)+d(gy,gz^*)+d(gz,gx^*)\preceq\left(\sum_{i=7}^{15}a_i\right)\big(d(gx,gy^*)+d(gy,gz^*)+d(gz,gx^*)\big).$$

The fact that $\sum_{i=7}^{15} a_i < 1$ yields that

$$gx = gy^*, \qquad gy = gz^* \quad \text{and} \quad gz = gx^*. \tag{20}$$

In view of (19) and (20), one can assert that

$$gx = gy = gz. \tag{21}$$

That is, the unique tripled point of coincidence of F and g is (gx, gy, gz).

Now, let $u = gx$, then we have $u = gx = F(x, y, z) = gy = F(y, z, x) = gz = F(z, x, y)$. Since F and g are W-compatible, we have

$$F(gx, gy, gz) = g\bigl(F(x, y, z)\bigr),$$

which due to (21) gives that

$$F(u, u, u) = gu.$$

Consequently, (u, u, u) is a tripled coincidence point of F and g, and so (gu, gu, gu) is a tripled point of coincidence of F and g, and by its uniqueness, we get $gu = gx$. Thus, we obtain

$$u = gx = gu = F(u, u, u).$$

Hence, (u, u, u) is the unique common tripled fixed point of F and g. This completes the proof. □

Corollary 3 (Abbas, Khan and Radenović [1]) *Let (X, d) be a cone metric space with a cone P having non-empty interior. Let $\tilde{F} : X^2 \to X$ and $g : X \to X$ be mappings satisfying $\tilde{F}(X^2) \subseteq g(X)$, $(g(X), d)$ is a complete subspace of X and for any $x, y, u, v \in X$,*

$$\begin{aligned} d\bigl(\tilde{F}(x, y), \tilde{F}(u, v)\bigr) \preceq{}& a_1 d\bigl(\tilde{F}(x, y), gx\bigr) + a_2 d\bigl(\tilde{F}(u, v), gu\bigr) + a_3 d\bigl(\tilde{F}(u, v), gx\bigr) \\ &+ a_4 d\bigl(\tilde{F}(x, y), gu\bigr) + a_5 d(gx, gu) + a_6 d(gy, gv), \end{aligned}$$

where a_i, $i = 1, \ldots, 6$ are nonnegative real numbers such that $\sum_{i=1}^{6} a_i < 1$. If $\tilde{F}$ and g are w-compatible, then $\tilde{F}$ and g have a unique common coupled fixed point. Moreover, the common fixed point of $\tilde{F}$ and g is of the form (u, u) for some $u \in X$.

Proof Consider the mappings $F : X^3 \to X$ defined by $F(x, y, z) = \tilde{F}(x, y)$ for all $x, y, z \in X$. From the proof of Corollary 2 and the result given by Theorem 2, we have only to show that F and g are W-compatible. Let $(x, y, z) \in X^3$ such that $F(x, y, z) = gx$, $F(y, z, x) = gy$ and $F(z, x, y) = gz$. From the definition of F, we get $\tilde{F}(x, y) = gx$ and $\tilde{F}(y, x) = gy$. Since $\tilde{F}$ and g are w-compatible, this implies that

$$g\bigl(\tilde{F}(x, y)\bigr) = \tilde{F}(gx, gy). \tag{22}$$

Using (22), we have

$$F(gx,gy,gz) = \tilde{F}(gx,gy) = g\big(\tilde{F}(x,y)\big) = g\big(F(x,y,z)\big).$$

Thus, we proved that F and g are W-compatible mappings, and the desired result follows immediately from Theorem 2. □

Remark 1

- Corollary 1 extends Theorem 2.9 of Samet and Vetro [20] to K-metric spaces (corresponding to the case $N = 3$).
- Theorem 2 extends Theorem 2.10 of Samet and Vetro [20] to K-metric spaces (case $N = 3$).
- Theorem 2 extends Theorem 2.11 of Samet and Vetro [20] to K-metric spaces (case $N = 3$).

Similar to Corollaries 2 and 3, by considering $F(x,y,z) = fx$ for all $x,y,z \in X$ where $f : X \to X$, we may state the following consequence of Theorem 2.

Corollary 4 (Olaleru [17]) *Let (X,d) be a K-metric space and $f,g : X \to X$ be mappings such that*

$$\begin{aligned} d(fx,fu) &\preceq a_1 d(fx,gx) + a_2 d(fu,gu) + a_3 d(fu,gx) \\ &\quad + a_4 d(fx,gu) + a_5 d(gu,gx) \end{aligned} \tag{23}$$

for all $x,u \in X$, where $a_i \in [0,1)$, $i = 1,\dots,5$ and $\sum_{i=1}^{5} a_i < 1$. Suppose that f and g are weakly compatible, $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Then the mappings f and g have a unique common fixed point.

Now, we give an example to illustrate our obtained results.

Example 3 Let $X = [0,\infty)$. Take $E = C_R^1[0,1]$ endowed with order induced by $P = \{\phi \in E : \phi(t) \geq 0 \text{ for } t \in [0,1]\}$. The mapping $d : X \times X \to E$ is defined by $d(x,y)(t) = |x-y|e^t$. In this case (X,d) is a complete abstract metric space with a non-normal cone having non-empty interior. Define the mappings $F : X^3 \to X$ and $g : X \to X$ by

$$gx = \frac{x}{3} \quad \text{and} \quad F(x,y,z) = \frac{x-y+z}{12}.$$

We will check that all the hypotheses of Theorem 1 are satisfied. Since, for all $x,y,z,u,v,w \in X$, we have

$$\begin{aligned} &d\big(F(x,y,z),F(u,v,w)\big) \\ &\quad = \big|F(x,y,z) - F(u,v,w)\big|e^t \\ &\quad \leq \frac{1}{12}\big|(x-y+z) - (u-v+w)\big|e^t \\ &\quad \leq \frac{1}{4}\big[d(gx,gu) + d(gy,gv) + d(gz,gw)\big]. \end{aligned}$$

Then, the contractive condition is satisfied with $a_i = 0$ for all $i = 1, \dots, 12$ and $a_{13} = a_{14} = a_{15} = 1/4$. All conditions of Theorem 1 are satisfied. Consequently, F and g have a tripled coincidence point. In this case, (x, y, z) is a tripled coincidence point if and only if $x = y = z = 0$. This implies that F and g are W-compatible. Applying our Theorem 2, we obtain the existence and uniqueness of a common tripled fixed point of F and g. In this example, $(0, 0, 0)$ is the unique common tripled fixed point.

Example 4 Let $X = [0, \infty)$ and $d : X \times X \to E$ be defined as $d(x, y)(t) = |x - y|2^t$. Define the mappings $F : X^3 \to X$ and $g : X \to X$ by

$$F(x,y,z) = \begin{cases} \frac{x}{5} + \frac{y}{5} + \frac{z}{5} & \text{if } x \in [0,1] \text{ and } y,z \in [0,\infty), \\ \frac{x}{6} + \frac{y}{6} + \frac{z}{6} & \text{if } x \in (1,\infty) \text{ and } y,z \in [0,\infty) \end{cases} \quad \text{and}$$

$$g(x) = \begin{cases} 3x & \text{if } x \in [0,1], \\ 2x & \text{if } x \in (1,\infty). \end{cases}$$

Let $a_1 = a_2 = a_7 = \cdots = a_{12} = 0$ and $a_3 = a_6 = \frac{4}{45}$, $a_{13} = a_{14} = a_{15} = \frac{1}{15}$. Now we shall prove that the contractive condition in Theorem 1 holds for all $x, y, z, u, v, w \in X$. By its symmetry and without loss of generality, it suffices to prove it for $x \le y \le z$ and $u \le v \le z$. Define

$$\begin{cases} \text{(I)}: x,y,z \in [0,1], \\ \text{(II)}: x,y \in [0,1], z \in (1,\infty), \\ \text{(III)}: x \in [0,1], y,z \in (1,\infty), \\ \text{(IV)}: x,y,z \in (1,\infty) \end{cases} \quad \text{and} \quad \begin{cases} \text{(i)}: u,v,w \in [0,1], \\ \text{(ii)}: u,v \in [0,1], w \in (1,\infty), \\ \text{(iii)}: u \in [0,1], v,w \in (1,\infty), \\ \text{(iv)}: u,v,w \in (1,\infty). \end{cases}$$

There are 16 possibilities which are (I, i), (I, ii), (I, iii), (I, iv), (II, i), (II, ii), (II, iii), (II, iv), (III, i), (III, ii), (III, iii), (III, iv), (IV, i), (IV, ii), (IV, iii) and (IV, iv).

Case 1. If (I, i) holds, we have

$$\begin{aligned} d\big(F(x,y,z), F(u,v,w)\big) &\le \left[\frac{|x-u|}{5} + \frac{|y-v|}{5} + \frac{|z-w|}{5}\right]2^t \\ &\le a_{13}d(gx,gu) + a_{14}d(gy,gv) + a_{15}d(gz,gw). \end{aligned}$$

Case 2. If (I, ii) holds, we have

$$\begin{aligned} &d\big(F(x,y,z), F(u,v,w)\big) \\ &\quad \le \left[\left|\frac{x}{5} - \frac{u}{5}\right| + \left|\frac{y}{5} - \frac{v}{5}\right| + \left|\frac{z}{5} - \frac{w}{5}\right|\right]2^t \\ &\quad \le a_{13}d(gx,gu) + a_{14}d(gy,gv) + \left[\frac{1}{15}|3z - 2w| + \frac{1}{15}w\right]2^t \\ &\quad \le a_{13}d(gx,gu) + a_{14}d(gy,gv) + a_{15}d(gz,gw) + \frac{2}{45}\left(\frac{11w}{6} - \frac{u}{6} - \frac{v}{6}\right)2^t \\ &\quad \le a_{13}d(gx,gu) + a_{14}d(gy,gv) + a_{15}d(gz,gw) + a_6 d\big(F(w,u,v), gw\big). \end{aligned}$$

Case 3. If (I, iii) holds, we have

$$\begin{aligned}
&d\bigl(F(x,y,z),F(u,v,w)\bigr)\\
&\quad\le\left[\left|\frac{x}{5}-\frac{u}{5}\right|+\left|\frac{y}{5}-\frac{v}{5}\right|+\left|\frac{z}{5}-\frac{w}{5}\right|\right]2^t\\
&\quad\le a_{13}d(gx,gu)+\left[\frac{1}{15}|3y-2v|+\frac{1}{15}w+\frac{1}{15}|3z-2w|+\frac{1}{15}w\right]2^t\\
&\quad\le a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw)+\frac{4}{45}\left(\frac{11w}{6}-\frac{u}{6}-\frac{v}{6}\right)2^t\\
&\quad= a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw)+a_6d\bigl(F(w,u,v),gw\bigr).
\end{aligned}$$

Case 4. If (I, iv) holds, we get

$$\begin{aligned}
&d\bigl(F(x,y,z),F(u,v,w)\bigr)\\
&\quad\le\left[\left|\frac{x}{5}-\frac{u}{6}\right|+\left|\frac{y}{5}-\frac{v}{6}\right|+\left|\frac{z}{5}-\frac{w}{6}\right|\right]2^t\\
&\quad\le\left[\frac{1}{15}|3x-2u|+\frac{1}{30}w+\frac{1}{15}|3y-2v|+\frac{1}{30}w+\frac{1}{15}|3z-2w|+\frac{1}{30}w\right]2^t\\
&\quad\le a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw)+\frac{1}{15}\left(\frac{11w}{6}-\frac{u}{6}-\frac{v}{6}\right)2^t\\
&\quad\le a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw)+a_6d\bigl(F(w,u,v),gw\bigr).
\end{aligned}$$

Case 5. If (II, i) holds, we have

$$\begin{aligned}
&d\bigl(F(x,y,z),F(u,v,w)\bigr)\\
&\quad\le\left[\frac{|x-u|}{5}+\frac{|y-v|}{5}+\frac{|z-w|}{5}\right]2^t\\
&\quad\le a_{13}d(gx,gu)+a_{14}d(gy,gv)+\left[\frac{1}{15}|2z-3w|+\frac{1}{15}z\right]2^t\\
&\quad\le a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw)+\frac{2}{45}\left(\frac{11z}{6}-\frac{x}{6}-\frac{y}{6}\right)2^t\\
&\quad\le a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw)+a_3d\bigl(F(z,x,y),gz\bigr).
\end{aligned}$$

Case 6. If (II, ii) holds, we have

$$\begin{aligned}
&d\bigl(F(x,y,z),F(u,v,w)\bigr)\\
&\quad\le\left[\left|\frac{x}{5}-\frac{u}{5}\right|+\left|\frac{y}{5}-\frac{v}{5}\right|+\left|\frac{z}{5}-\frac{w}{5}\right|\right]2^t\\
&\quad\le a_{13}d(gx,gu)+a_{14}d(gy,gv)+\left[\frac{1}{15}|2z-2w|+\frac{1}{15}w+\frac{1}{15}w\right]2^t\\
&\quad\le a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw)\\
&\qquad+\frac{2}{45}\left(\frac{11w}{6}-\frac{u}{6}-\frac{v}{6}\right)2^t+\frac{2}{45}\left(\frac{11z}{6}-\frac{x}{6}-\frac{y}{6}\right)2^t\\
&\quad\le a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw)+a_6d\bigl(F(w,u,v),gw\bigr)+a_3d\bigl(F(z,x,y),gz\bigr).
\end{aligned}$$

Case 7. If (II, iii) holds, we have

$$\begin{aligned}
&d\bigl(F(x,y,z),F(u,v,w)\bigr)\\
&\quad\le \left[\left|\frac{x}{5}-\frac{u}{5}\right|+\left|\frac{y}{5}-\frac{v}{5}\right|+\left|\frac{z}{5}-\frac{w}{5}\right|\right]2^t\\
&\quad\le a_{13}d(gx,gu)+\left[\frac{1}{15}|3y-2v|+\frac{1}{15}w+\frac{1}{15}|2z-2w|+\frac{1}{15}z+\frac{1}{15}w\right]2^t\\
&\quad\le a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw)\\
&\qquad+\frac{4}{45}\left(\frac{11w}{6}-\frac{u}{6}-\frac{v}{6}\right)2^t+\frac{2}{45}\left(\frac{11z}{6}-\frac{x}{6}-\frac{y}{6}\right)2^t\\
&\quad= a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw)+a_6d\bigl(F(w,u,v),gw\bigr)+a_3d\bigl(F(z,x,y),gz\bigr).
\end{aligned}$$

Case 8. If (II, iv) holds, we get

$$\begin{aligned}
&d\bigl(F(x,y,z),F(u,v,w)\bigr)\\
&\quad\le \left[\left|\frac{x}{5}-\frac{u}{6}\right|+\left|\frac{y}{5}-\frac{v}{6}\right|+\left|\frac{z}{5}-\frac{w}{6}\right|\right]2^t\\
&\quad\le \left[\frac{1}{15}|3x-2u|+\frac{1}{30}w+\frac{1}{15}|3y-2v|+\frac{1}{30}w+\frac{1}{15}|2z-2w|+\frac{1}{15}z+\frac{1}{30}w\right]2^t\\
&\quad\le a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw)\\
&\qquad+\frac{1}{15}\left(\frac{11w}{6}-\frac{u}{6}-\frac{v}{6}\right)2^t+\frac{2}{45}\left(\frac{11z}{6}-\frac{x}{6}-\frac{y}{6}\right)2^t\\
&\quad\le a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw)+a_6d\bigl(F(w,u,v),gw\bigr)+a_3d\bigl(F(z,x,y),gz\bigr).
\end{aligned}$$

Case 9 corresponding to (III, i) is as Case 3.
Case 10 corresponding to (III, ii) is as Case 7.
Case 11. If (III, iii) holds, we have

$$\begin{aligned}
&d\bigl(F(x,y,z),F(u,v,w)\bigr)\\
&\quad\le \left[\left|\frac{x}{5}-\frac{u}{5}\right|+\left|\frac{y}{5}-\frac{v}{5}\right|+\left|\frac{z}{5}-\frac{w}{5}\right|\right]2^t\\
&\quad\le \left[\frac{1}{15}|3x-3u|+\frac{1}{15}|2y-2v|+\frac{1}{15}|2z-2w|+\frac{2}{15}z+\frac{2}{15}w\right]2^t\\
&\quad\le a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw)\\
&\qquad+\frac{4}{45}\left(\frac{11w}{6}-\frac{u}{6}-\frac{v}{6}\right)2^t+\frac{4}{45}\left(\frac{11z}{6}-\frac{x}{6}-\frac{y}{6}\right)2^t\\
&\quad= a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw)+a_6d\bigl(F(w,u,v),gw\bigr)+a_3d\bigl(F(z,x,y),gz\bigr).
\end{aligned}$$

Case 12. If (III, iv) holds, we get

$$\begin{aligned}
&d\bigl(F(x,y,z),F(u,v,w)\bigr)\\
&\quad\le \left[\left|\frac{x}{5}-\frac{u}{6}\right|+\left|\frac{y}{5}-\frac{v}{6}\right|+\left|\frac{z}{5}-\frac{w}{6}\right|\right]2^t
\end{aligned}$$

$$\begin{aligned}
&\le \left[\frac{1}{15}|3x-2u| + \frac{1}{15}|2y-2v| + \frac{1}{15}|2z-2w| + \frac{2}{15}z + \frac{3}{30}w\right]2^t \\
&\le a_{13}d(gx,gu) + a_{14}d(gy,gv) + a_{15}d(gz,gw) \\
&\quad + \frac{1}{15}\left(\frac{11w}{6} - \frac{u}{6} - \frac{v}{6}\right)2^t + \frac{4}{45}\left(\frac{11z}{6} - \frac{x}{6} - \frac{y}{6}\right)2^t \\
&\le a_{13}d(gx,gu) + a_{14}d(gy,gv) + a_{15}d(gz,gw) + a_6 d\big(F(w,u,v),gw\big) + a_3 d\big(F(z,x,y),gz\big).
\end{aligned}$$

Case 13 corresponding to (IV, i) is as Case 4.

Case 14 corresponding to (IV, ii) is as Case 8.

Case 15. If (IV, iii) holds, we have

$$\begin{aligned}
&d\big(F(x,y,z),F(u,v,w)\big) \\
&\quad\le \left[\left|\frac{x}{6} - \frac{u}{6}\right| + \left|\frac{y}{6} - \frac{v}{6}\right| + \left|\frac{z}{6} - \frac{w}{6}\right|\right]2^t \\
&\quad\le \left[\frac{1}{15}|2x-3u| + \frac{1}{15}|2y-2v| + \frac{1}{15}|2z-2w| + \frac{3}{30}z + \frac{3}{30}w\right]2^t \\
&\quad\le a_{13}d(gx,gu) + a_{14}d(gy,gv) + a_{15}d(gz,gw) \\
&\quad\quad + \frac{1}{15}\left(\frac{11w}{6} - \frac{u}{6} - \frac{v}{6}\right)2^t + \frac{1}{15}\left(\frac{11z}{6} - \frac{x}{6} - \frac{y}{6}\right)2^t \\
&\quad\le a_{13}d(gx,gu) + a_{14}d(gy,gv) + a_{15}d(gz,gw) + a_6 d\big(F(w,u,v),gw\big) + a_3 d\big(F(z,x,y),gz\big).
\end{aligned}$$

Case 16. If (IV, iv) holds, we have

$$\begin{aligned}
&d\big(F(x,y,z),F(u,v,w)\big) \\
&\quad\le \left[\left|\frac{x}{6} - \frac{u}{6}\right| + \left|\frac{y}{6} - \frac{v}{6}\right| + \left|\frac{z}{6} - \frac{w}{6}\right|\right]2^t \\
&\quad\le \left[\frac{1}{15}|2x-2u| + \frac{1}{15}|2y-2v| + \frac{1}{15}|2z-2w| + \frac{3}{30}z + \frac{3}{30}w\right]2^t \\
&\quad\le a_{13}d(gx,gu) + a_{14}d(gy,gv) + a_{15}d(gz,gw) \\
&\quad\quad + \frac{1}{15}\left(\frac{11w}{6} - \frac{u}{6} - \frac{v}{6}\right)2^t + \frac{1}{15}\left(\frac{11z}{6} - \frac{x}{6} - \frac{y}{6}\right)2^t \\
&\quad\le a_{13}d(gx,gu) + a_{14}d(gy,gv) + a_{15}d(gz,gw) + a_6 d\big(F(w,u,v),gw\big) + a_3 d\big(F(z,x,y),gz\big).
\end{aligned}$$

All the conditions of Theorem 1 are fulfilled. Moreover, $(0,0,0)$ is a common tripled coincidence point of F and g.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

Author details

[1]Institut Supérieur d'Informatique et des Technologies de Communication de Hammam Sousse, Université de Sousse, Route GP1-4011, H. Sousse, Tunisia. [2]Department of Mathematics, Lahore University of Management Sciences, Lahore, 54792, Pakistan. [3]Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok, 10140, Thailand.

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