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Coupled best proximity point theorem in metric Spaces

Wutiphol Sintunavarat and Poom Kumam*

* Correspondence: poom.kum@kmutt.ac.th
Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangmod, Bangkok 10140, Thailand

Abstract

In this article the concept of coupled best proximity point and cyclic contraction pair are introduced and then we study the existence and convergence of these points in metric spaces. We also establish new results on the existence and convergence in a uniformly convex Banach spaces. Furthermore, we give new results of coupled fixed points in metric spaces and give some illustrative examples. An open problems are also given at the end for further investigation.

1 Introduction

The Banach contraction principle [1] states that if (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping (i.e., $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$, where α is a non-negative number such that $\alpha < 1$), then T has a unique fixed point. This principle has been generalized in many ways over the years [2-15].

One of the most interesting is the study of the extension of Banach contraction principle to the case of non-self mappings. In fact, given nonempty closed subsets A and B of a complete metric space (X, d), a contraction non-self-mapping $T : A \to B$ does not necessarily has a fixed point.

Eventually, it is quite natural to find an element x such that $d(x, Tx)$ is minimum for a given problem which implies that x and Tx are in close proximity to each other.

A point x in A for which $d(x, Tx) = d(A, B)$ is call a best proximity point of T. Whenever a non-self-mapping T has no fixed point, a best proximity point represent an optimal approximate solution to the equation $Tx = x$. Since a best proximity point reduces to a fixed point if the underlying mapping is assumed to be self-mappings, the best proximity point theorems are natural generalizations of the Banach contraction principle.

In 1969, Fan [16] introduced and established a classical best approximation theorem, that is, if A is a nonempty compact convex subset of a Hausdorff locally convex topological vector space B and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Afterward, many authors have derived extensions of Fan's Theorem and the best approximation theorem in many directions such as Prolla [17], Reich [18], Sehgal and Singh [19,20], Wlodarczyk and Plebaniak [21-24], Vetrivel et al. [25], Eldred and Veeramani [26], Mongkolkeha and Kumam [27] and Sadiq Basha and Veeramani [28-31].

On the other hand, Bhaskar and Lakshmikantham [32] introduced the notions of a mixed monotone mapping and proved some coupled fixed point theorems for mappings satisfying the mixed monotone property. They have observation that their theorem can be used to investigate a large class of problems and discussed the existence and uniqueness of a solution for a periodic boundary value problem. For several improvements and generalizations see in [33-36] and reference therein.

The purpose of this article is to first introduce the notion of coupled best proximity point and cyclic contraction pair. We also establish the existence and convergence theorem of coupled best proximity points in metric spaces. Moreover, we apply this results in uniformly convex Banach space. We also study some results on the existence and convergence of coupled fixed point in metric spaces and give illustrative examples of our theorems. An open problem are also given at the end for further investigations.

2 Preliminaries

In this section, we give some basic definitions and concepts related to the main results of this article. Throughout this article we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. For nonempty subsets A and B of a metric space (X, d), we let

$$d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\}$$

stands for the distance between A and B.

A Banach space X is said to be

(1) *strictly convex* if the following implication holds for all $x, y \in X$:

$$\|x\| = \|y\| = 1 \text{ and } x \neq y \Rightarrow \left\|\frac{x+y}{2}\right\| < 1.$$

(2) *uniformly convex* if for each ε with $0 < \varepsilon \leq 2$, there exists $\delta > 0$ such that the following implication holds for all $x, y \in X$:

$$||x|| \leq 1, ||y|| \leq 1 \text{ and } ||x - y|| \geq \varepsilon \Rightarrow \left\|\frac{x+y}{2}\right\| < 1 - \delta.$$

It easily to see that a uniformly convex Banach space X is strictly convex but the converse is not true.

Definition 2.1. [37] Let A and B be nonempty subsets of a metric space (X, d). The ordered pair (A, B) satisfies the *property UC* if the following holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $d(x_n, y_n) \to d(A, B)$ and $d(z_n, y_n) \to d(A, B)$, then $d(x_n, z_n) \to 0$.

Example 2.2. [37] *The following are examples of a pair of nonempty subsets* (A, B) *satisfying the property UC.*

(1) Every pair of nonempty subsets A, B *of a metric space* (X, d) *such that* $d(A, B) = 0$.

(2) Every pair of nonempty subsets A, B *of a uniformly convex Banach space* X *such that* A *is convex.*

(3) Every pair of nonempty subsets A, B *of a strictly convex Banach space which* A *is convex and relatively compact and the closure of* B *is weakly compact.*

Definition 2.3. Let A and B be nonempty subsets of a metric space (X, d) and $T : A \to B$ be a mapping. A point $x \in A$ is said to be a *best proximity point* of T if it satisfies the condition that

$$d(x, Tx) = d(A, B).$$

It can be observed that a best proximity point reduces to a fixed point if the underlying mapping is a self-mapping.

Definition 2.4. [32] Let A be a nonempty subset of a metric space X and $F : A \text{ X } A \to A$. A point $(x, x') \in A \times A$ is called a *coupled fixed point* of F if

$$x = F(x, x') \text{ and } x' = F(x', x).$$

3 Coupled best proximity point theorem

In this section, we study the existence and convergence of coupled best proximity points for cyclic contraction pairs. We begin by introducing the notion of property UC* and a coupled best proximity point.

Definition 3.1. Let A and B be nonempty subsets of a metric space (X, d). The ordered pair (A, B) satisfies the *property UC** if (A, B) has property UC and the following condition holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B satisfying:

(1) $d(z_n, y_n) \to d(A, B)$.

(2) For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_m, y_n) \le d(A, B) + \varepsilon$$

for all $m > n \ge N$,

then, for every $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that

$$d(x_m, z_n) \le d(A, B) + \varepsilon$$

for all $m > n \ge N_1$.

Example 3.2. *The following are examples of a pair of nonempty subsets* (A, B) *satisfying the property UC**.

(1) Every pair of nonempty subsets A, B *of a metric space* (X, d) *such that* $d(A, B) = 0$.

(2) Every pair of nonempty closed subsets A, B *of a uniformly convex Banach space* X *such that* A *is convex* [[38], Lemma 3.7].

Definition 3.3. Let A and B be nonempty subsets of a metric space X and $F : A \times A \to B$. A point $(x, x') \in A \times A$ is called a *coupled best proximity point* of F if

$$d(x, F(x, x')) = d(x', F(x', x)) = d(A, B).$$

It is easy to see that if $A = B$ in Definition 3.3, then a coupled best proximity point reduces to a coupled fixed point.

Next, we introduce the notion of a cyclic contraction for a pair of two binary mappings.

Definition 3.4. Let A and B be nonempty subsets of a metric space X, $F : A \times A \to B$ and $G : B \times B \to A$. The ordered pair (F, G) is said to be a *cyclic contraction* if there exists a non-negative number $\alpha < 1$ such that

$$d(F(x,\ x'),\ G(y,\ y')) \leq \frac{\alpha}{2}[d(x,\ y) + d(x',\ y')] + (1-\alpha)d(A,\ B)$$

for all $(x, x') \in A \times A$ and $(y, y') \in B \times B$.

Note that if (F, G) is a cyclic contraction, then (G, F) is also a cyclic contraction.

Example 3.5. *Let* $X = \mathbb{R}$ *with the usual metric* $d(x, y) = |x - y|$ *and let* $A = [2,4]$ *and* $B = [-4, -2]$. *It easy to see that* $d(A, B) = 4$. *Define* $F : A \times A \to B$ *and* $G : B \times B \to A$ *by*

$$F(x,\ x') = \frac{-x - x' - 4}{4}$$

and

$$G(x,\ x') = \frac{-x - x' + 4}{4}.$$

For arbitrary $(x, x') \in A \times A$ *and* $(y, y') \in B \times B$ *and fixed* $\alpha = \frac{1}{2}$, *we get*

$$\begin{aligned} d(F(x,\ x'),\ G(y,\ y')) &= \left| \frac{-x - x' - 4}{4} - \frac{-y - y' + 4}{4} \right| \\ &\leq \frac{|x - y| + |x' - y'|}{4} + 2 \\ &= \frac{\alpha}{2}[d(x,\ y) + d(x',\ y')] + (1-\alpha)d(A,\ B). \end{aligned}$$

This implies that (F, G) *is a cyclic contraction with* $\alpha = \frac{1}{2}$.

Example 3.6. *Let* $X = \mathbb{R}^2$ *with the metric* $d((x, y), (x', y')) = \max\{|x - x'|, |y - y'|\}$ *and let* $A = \{(x, 0): 0 \leq x \leq 1\}$ *and* $B = \{(x, 1): 0 \leq x \leq 1\}$. *It easy to prove that* $d(A, B) = 1$. *Define* $F : A \times A \to B$ *and* $G : B \times B \to A$ *by*

$$F((x,\ 0),\ (x',\ 0)) = \left(\frac{x + x'}{2},\ 1\right)$$

and

$$G((x,\ 1),\ (x',\ 1)) = (\frac{x + x'}{2},\ 0).$$

We obtain that

$$d(F((x,\ 0),\ (x',\ 0)),\ G((y,\ 1),\ (y',\ 1))) = d((\frac{x + x'}{2},\ 1),\ (\frac{y + y'}{2},\ 0)) = 1$$

Also for all $\alpha > 0$, *we get*

$$\begin{aligned} &\frac{\alpha}{2}[d((x,\ 0),(y,\ 1)) + d((x',\ 0),\ (y',\ 1))] + (1-\alpha)d(A,\ B) \\ &\qquad = \frac{\alpha}{2}[\max\{|x - y|,\ 1\} + \max\{|x' - y'|,\ 1\}] + (1-\alpha)d(A,\ B) \\ &\qquad = \frac{\alpha}{2} \times 2 + (1-\alpha) \\ &\qquad = 1. \end{aligned}$$

This implies that (F, G) *is cyclic contraction.*

The following lemma plays an important role in our main results.

Lemma 3.7. *Let* A *and* B *be nonempty subsets of a metric space* X, $F : A \times A \to B$, $G : B \times B \to A$ *and* (F, G) *be a cyclic contraction. If* $(x_0, x'_0) \in A \times A$ *and we define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}),\ x'_{2n+1} = F(x'_{2n}, x_{2n})$$

and

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}),\ x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$, *then* $d(x_{2n}, x_{2n+1}) \to d(A, B)$, $d(x_{2n+1}, x_{2n+2}) \to d(A, B)$, $d(x'_{2n}, x'_{2n+1}) \to d(A, B)$ *and* $d(x'_{2n+1}, x'_{2n+2}) \to d(A, B)$.

Proof. For each $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
d(x_{2n}, x_{2n+1}) &= d(x_{2n}, F(x_{2n}, x'_{2n})) \\
&= d(G(x_{2n-1}, x'_{2n-1}), F(G(x_{2n-1}, x'_{2n-1}), G(x'_{2n-1}, x_{2n-1}))) \\
&\le \frac{\alpha}{2}[d(x_{2n-1}, G(x_{2n-1}, x'_{2n-1})) + d(x'_{2n-1}, G(x'_{2n-1}, x_{2n-1}))] + (1-\alpha)d(A, B) \\
&= \frac{\alpha}{2}\big[d(F(x_{2n-2}, x'_{2n-2}), G(F(x_{2n-2}, x'_{2n-2}), F(x'_{2n-2}, x_{2n-2}))) \\
&\quad + d(F(x'_{2n-2}, x_{2n-2}), G(F(x'_{2n-2}, x_{2n-2}), F(x_{2n-2}, x'_{2n-2})))\big] + (1-\alpha)d(A, B) \\
&\le \frac{\alpha}{2}\Big[\frac{\alpha}{2}[d(x_{2n-2}, F(x_{2n-2}, x'_{2n-2})) + d(x'_{2n-2}, F(x'_{2n-2}, x_{2n-2})) + (1-\alpha)d(A, B)] \\
&\quad + \frac{\alpha}{2}[d(x'_{2n-2}, F(x'_{2n-2}, x_{2n-2})) + d(x_{2n-2}, F(x_{2n-2}, x'_{2n-2})) + (1-\alpha)d(A, B)\Big] \\
&\quad + (1-\alpha)d(A, B) \\
&= \frac{\alpha^2}{2}[d(x_{2n-2}, F(x_{2n-2}, x'_{2n-2})) + d(x'_{2n-2}, F(x'_{2n-2}, x_{2n-2}))] + (1-\alpha^2)d(A, B).
\end{aligned}$$

By induction, we see that

$$d(x_{2n}, x_{2n+1}) \le \frac{\alpha^{2n}}{2}[d(x_0, F(x_0, x'_0)) + d(x'_0, F(x'_0, x_0))] + (1-\alpha^{2n})d(A, B).$$

Taking $n \to \infty$, we obtain

$$d(x_{2n}, x_{2n+1}) \to d(A, B). \tag{3.1}$$

For each $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &= d(x_{2n+1}, G(x_{2n+1}, x'_{2n+1})) \\
&= d(F(x_{2n}, x'_{2n}), G(F(x_{2n}, x'_{2n}), F(x'_{2n}, x_{2n}))) \\
&\le \frac{\alpha}{2}[d(x_{2n}, F(x_{2n}, x'_{2n})) + d(x'_{2n}, F(x'_{2n}, x_{2n}))] + (1-\alpha)d(A, B) \\
&= \frac{\alpha}{2}\big[d(G(x_{2n-1}, x'_{2n-1}), F(G(x_{2n-1}, x'_{2n-1}), G(x'_{2n-1}, x_{2n-1}))) \\
&\quad + d(G(x'_{2n-1}, x_{2n-1}), F(G(x'_{2n-1}, x_{2n-1}), G(x_{2n-1}, x'_{2n-1})))\big] + (1-\alpha)d(A, B) \\
&\le \frac{\alpha}{2}\Big[\frac{\alpha}{2}[d(x_{2n-1}, G(x_{2n-1}, x'_{2n-1})) + d(x'_{2n-1}, G(x'_{2n-1}, x_{2n-1})) + (1-\alpha)d(A, B)] \\
&\quad + \frac{\alpha}{2}[d(x'_{2n-1}, G(x'_{2n-1}, x_{2n-1})) + d(x_{2n-1}, G(x_{2n-1}, x'_{2n-1})) + (1-\alpha)d(A, B)\Big] \\
&\quad + (1-\alpha)d(A, B) \\
&= \frac{\alpha^2}{2}[d(x_{2n-1}, G(x_{2n-1}, x'_{2n-2})) + d(x'_{2n-1}, G(x'_{2n-1}, x_{2n-1}))] + (1-\alpha^2)d(A, B).
\end{aligned}$$

By induction, we see that

$$d(x_{2n+1},\ x_{2n+2}) \leq \frac{\alpha^{2n}}{2}[d(x_1,\ G(x_1,\ x_1')) + d(x_1',\ G(x_1',\ x_1))] + (1-\alpha^{2n})d(A,\ B).$$

Setting $n \to \infty$, we obtain

$$d(x_{2n+1},\ x_{2n+2}) \to d(A,\ B). \tag{3.2}$$

By similar argument, we also have $d(x_{2n}',\ x_{2n+1}') \to d(A,\ B)$ and $d(x_{2n+1}',\ x_{2n+2}') \to d(A,\ B)$ for all $n \in \mathbb{N} \cup \{0\}$. □

Lemma 3.8. *Let A and B be nonempty subsets of a metric space X such that* (*A*, *B*) *and* (*B*, *A*) *have a property UC,* $F : A \times A \to B$, $G : B \times B \to A$ *and let the ordered pair* (*F*, *G*) *is a cyclic contraction. If* $(x_0, x_0') \in A \times A$ *and define*

$$x_{2n+1} = F(x_{2n}, x_{2n}'), x_{2n+1}' = F(x_{2n}', x_{2n})$$

and

$$x_{2n+2} = G(x_{2n+1}, x_{2n+1}'), x_{2n+2}' = G(x_{2n+1}', x_{2n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$, *then for* $\varepsilon > 0$, *there exists a positive integer* N_0 *such that for all* $m > n \geq N_0$,

$$\frac{1}{2}[d(x_{2m}', x_{2n+1}') + d(x_{2m},\ x_{2n+1})] < d(A,B) + \varepsilon. \tag{3.3}$$

Proof. By Lemma 3.7, we have $d(x_{2n}, x_{2n+1}) \to d(A, B)$ and $d(x_{2n+1}, x_{2n+2}) \to d(A, B)$. Since (*A*, *B*) has a property UC, we get $d(x_{2n}, x_{2n+2}) \to 0$. A similar argument shows that $d(x_{2n}', x_{2n+2}') \to 0$. As (*B*, *A*) has a property UC, we also have $d(x_{2n+1}, x_{2n+3}) \to 0$ and $d(x_{2n+1}', x_{2n+3}') \to 0$. Suppose that (3.3) does not hold. Then there exists $\varepsilon' > 0$ such that for all $k \in \mathbb{N}$, there is $m_k > n_k \geq k$ satisfying

$$\frac{1}{2}[d(x_{2m_k}', x_{2n_k+1}') + d(x_{2m_k},\ x_{2n_k+1})] \geq d(A,B) + \varepsilon'.$$

and

$$\frac{1}{2}[d(x_{2m_k-2}', x_{2n_k+1}') + d(x_{2m_k-2},\ x_{2n_k+1})] < d(A,B) + \varepsilon'.$$

Therefore, we get

$$\begin{aligned} d(A,B) + \varepsilon' &\leq \frac{1}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k},\ x_{2n_k+1})] \\ &\leq \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k-2}) + d(x'_{2m_k-2}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2m_k-2}) + d(x_{2m_k-2}, x_{2n_k+1})] \\ &< \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k-2}) + d(x_{2m_k}, x_{2m_k-2})] + d(A,B) + \varepsilon'. \end{aligned}$$

Letting $k \to \infty$, we obtain to see that

$$\frac{1}{2}[d(x_{2m_k}', x_{2n_k+1}') + d(x_{2m_k}, x_{2n_k+1})] \to d(A,B) + \varepsilon'. \tag{3.4}$$

By using the triangle inequality we get

$$
\begin{aligned}
&\tfrac{1}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k}, \ x_{2n_k+1})] \\
&\le \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(x'_{2m_k+2}, x'_{2n_k+3}) + d(x'_{2n_k+3}, x'_{2n_k+1}) \\
&\quad + d(x_{2m_k-2}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1})\Big] \\
&= \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(G(x'_{2m_k+1}, x_{2m_k+1}), F(x'_{2n_k+2}, x_{2n_k+2})) + d(x'_{2n_k+3}, x'_{2n_k+1}) \\
&+ d(x_{2m_k}, x_{2m_k+2}) + d(G(x_{2m_k+1}, x_{2m_k+1}), F(x_{2n_k+2}, x'_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&\le \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + \frac{\alpha}{2}[d(x'_{2m_k+1}, x'_{2n_k+2}) + d(x_{2m_k+1}, x_{2n_k+2})] \\
&\quad + (1-\alpha)d(A, B) + d(x'_{2n_k+3}, x'_{2n_k+1}) \\
&\quad + d(x_{2m_k-2}, x_{2m_k+2}) + \frac{\alpha}{2}[d(x_{2m_k+1}, x_{2n_k+2}) + d(x'_{2m_k+1}, x'_{2n_k+2})] \\
&\quad + (1-\alpha)d(A, B) + d(x_{2n_k+3}, x_{2n_k+1})\Big] \\
&= \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(x'_{2n_k+3}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&\quad + \frac{\alpha}{2}[d(x_{2m_k+1}, x_{2n_k+2}) + d(x'_{2m_k+1}, x'_{2n_k+2})] + (1-\alpha)d(A, B) \\
&= \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(x'_{2n_k+3}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&\quad + \frac{\alpha}{2}[d(F(x_{2m_k}, x'_{2m_k}), G(x_{2n_k+1}, x'_{2n_k+1})) + d(F(x'_{2m_k}, x_{2m_k}), G(x'_{2n_k+1}, x_{2n_k+1}))] \\
&\quad + (1-\alpha)d(A, B) \\
&\le \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(x'_{2n_k+3}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&\quad + \frac{\alpha}{2}\Big[\frac{\alpha}{2}\ [d(x_{2m_k}, x_{2n_k+1}) + d(x'_{2m_k+1}, x'_{2n_k+1}) + (1-\alpha)d(A, B)] \\
&+ \frac{\alpha}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k+1}, x_{2n_k+1}) + (1-\alpha)d(A, B)]\Big] \\
&\quad + (1-\alpha)d(A, B) \\
&= \frac{1}{2}[d(x'_{2m_k}, x'_{2m_k+2}) + d(x'_{2n_k+3}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})] \\
&\quad + \frac{\alpha^2}{2}[d(x_{2m_k}, x_{2n_k+1}) + d(x'_{2m_k}, x'_{2n_k+1})] + (1-\alpha^2)d(A, B).
\end{aligned}
$$

Taking $k \to \infty$, we get

$$d(A, B) + \varepsilon' \le \alpha^2[d(A, B) + \varepsilon'] + (1-\alpha^2)d(A, B) = d(A, B) + \alpha^2\varepsilon'$$

which contradicts. Therefore, we can conclude that (3.3) holds. □

Lemma 3.9. *Let A and B be nonempty subsets of a metric space X, (A, B) and (B, A) satisfy the property UC*. Let $F : A \times A \to B$, $G : B \times B \to A$ and (F, G) be a cyclic contraction. If $(x_0, x'_0) \in A \times A$ and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

and

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$, then $\{x_{2n}\}$, $\{x'_{2n}\}$, $\{x_{2n+1}\}$ and $\{x'_{2n+1}\}$ are Cauchy sequences.

Proof. By Lemma 3.7, we have $d(x_{2n}, x_{2n+1}) \to d(A, B)$ and $d(x_{2n+1}, x_{2n+2}) \to d(A, B)$. Since (A, B) has a property UC*, we get $d(x_{2n}, x_{2n+2}) \to 0$. As (B, A) has a property UC*, we also have $d(x_{2n+1}, x_{2n+3}) \to 0$.

We now show that for every $\varepsilon > 0$ there exists N such that

$$d(x_{2m}, x_{2n+1}) \leq d(A, B) + \varepsilon \tag{3.5}$$

for all $m > n \geq N$.

Suppose (3.5) not, then there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$ there exists $m_k > n_k \geq k$ such that

$$d(x_{2m_k}, x_{2n_k+1}) > d(A, B) + \varepsilon. \tag{3.6}$$

Now we have

$$\begin{aligned} d(A, B) + \varepsilon &< d(x_{2m_k}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k+1}) \\ &\leq d(A, B) + \varepsilon + d(x_{2n_k-1}, x_{2n_k+1}) \end{aligned}$$

Taking $k \to \infty$, we have $d(x_{2m_k}, x_{2n_k+1}) \to d(A, B) + \varepsilon$.

By Lemma 3.8, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{2}[d(x'_{2m_k}, x'_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1})] < d(A, B) + \varepsilon \tag{3.7}$$

for all $m > n \geq N$. By using the triangle inequality we get

$$\begin{aligned} & d(x_{2m_k}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &= d(x_{2m_k}, x_{2m_k+2}) + d(G(x_{2m_k+1}, x'_{2m_k} + 1), F(x_{2n_k+2}, x'_{2n_k} + 2)) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2m_k+2}) + \frac{\alpha}{2}[d(x_{2m_k+1}, x_{2n_k+2}) + d(x'_{2m_k} + 1, x'_{2n_k} + 2)] + (1-\alpha)d(A, B) \\ &\quad + d(x_{2n_k+3}, x_{2n_k+1}) \\ &= \frac{\alpha}{2}[d(F(x_{2m_k}, x'_{2m_k}), G(x_{2n_k+1}, x'_{2n_k} + 1)) + d(F(x'_{2m_k}, x_{2m_k}), G(x'_{2n_k} + 1, x_{2n_k+1}))] \\ &\quad + (1-\alpha)d(A, B) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &\leq \frac{\alpha}{2}\Big[\frac{\alpha}{2}\,[d(x_{2m_k}, x_{2n_k+1}) + d(x'_{2m_k}, x'_{2n_k} + 1) + (1-\alpha)d(A, B)] \\ &\quad + \frac{\alpha}{2}[d(x'_{2m_k}, x'_{2n_k} + 1) + d(x_{2m_k}, x_{2n_k+1}) + (1-\alpha)d(A, B)]\Big] \\ &\quad + (1-\alpha)d(A, B) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &= \alpha^2\frac{1}{2}[d(x_{2m_k}, x_{2n_k+1}) + d(x'_{2m_k}, x'_{2n_k} + 1)] + (1-\alpha^2)d(A, B) \\ &\quad + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &< \alpha^2(d(A, B) + \varepsilon) + (1-\alpha^2)d(A, B) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\ &= d(A, B) + \alpha^2\varepsilon + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \end{aligned}$$

Taking $k \to \infty$, we get

$$d(A, B) + \varepsilon \leq d(A, B) + \alpha^2\varepsilon$$

which contradicts. Therefore, condition (3.5) holds. Since (3.5) holds and $d(x_{2n}, x_{2n+1}) \to d(A, B)$, by using property UC* of (A, B), we have $\{x_{2n}\}$ is a Cauchy sequence. In similar way, we can prove that $\{x'_{2n}\}$, $\{x_{2n+1}\}$ and $\{x'_{2n+1}\}$ are Cauchy sequences. □

Here we state the main results of this article on the existence and convergence of coupled best proximity points for cyclic contraction pairs on nonempty subsets of metric spaces satisfying the property UC*.

Theorem 3.10. *Let A and B be nonempty closed subsets of a complete metric space X such that (A, B) and (B, A) satisfy the property UC*. Let $F : A \times A \to B$, $G : B \times B \to A$ and (F, G) be a cyclic contraction. Let $(x_0, x'_0) \in A \times A$ and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

and

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$. Then F has a coupled best proximity point $(p, q) \in A \times A$ and G has a coupled best proximity point $(p', q') \in B \times B$ such that

$$d(p, p') + d(q, q') = 2d(A, B).$$

Moreover, we have $x_{2n} \to p$, $x'_{2n} \to q$, $x_{2n+1} \to p'$ and $x'_{2n+1} \to q'$.

Proof. By Lemma 3.7, we get $d(x_{2n}, x_{2n+1}) \to d(A, B)$. Using Lemma 3.9, we have $\{x_{2n}\}$ and $\{x'_{2n}\}$ are Cauchy sequences. Thus, there exists $p, q \in A$ such that $x_{2n} \to p$ and $x'_{2n} \to q$. We obtain that

$$d(A, B) \le d(p, x_{2n-1}) \le d(p, x_{2n}) + d(x_{2n}, x_{2n-1}). \tag{3.8}$$

Letting $n \to \infty$ in (3.8), we have $d(p, x_{2n-1}) \to d(A, B)$. By a similar argument we also have $d(q, x'_{2n-1}) \to d(A, B)$. It follows that

$$\begin{aligned} d(x_{2n}, F(p, q)) &= d(G(x_{2n-1}, x'_{2n-1}), F(p, q)) \\ &\le \frac{\alpha}{2}[d(x_{2n-1}, p) + d(x'_{2n-1}, q)] + (1 - \alpha)d(A, B). \end{aligned}$$

Taking $n \to \infty$, we get $d(p, F(p, q)) = d(A, B)$. Similarly, we can prove that $d(q, F(q, p)) = d(A, B)$. Therefore, we have (p, q) is a coupled best proximity point of F.

In similar way, we can prove that there exists $p', q' \in B$ such that $x_{2n+1} \to p'$ and $x'_{2n+1} \to q'$. Moreover, we also have $d(p', G(p', q')) = d(A, B)$ and $d(q', G(q', p')) = d(A, B)$ and so (p', q') is a coupled best proximity point of G.

Finally, we show that $d(p, p') + d(q, q') = 2d(A, B)$. For $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(G(x_{2n-1}, x'_{2n-1}), F(x_{2n}, x'_{2n})) \\ &\le \frac{\alpha}{2}[d(x_{2n-1}, x_{2n}) + d(x'_{2n-1}, x'_{2n})] + (1 - \alpha)d(A, B). \end{aligned}$$

Letting $n \to \infty$, we have

$$d(p, p') \le \frac{\alpha}{2}[d(p, p') + d(q, q')] + (1 - \alpha)d(A, B). \tag{3.9}$$

For $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} d(x'_{2n}, x'_{2n+1}) &= d(G(x'_{2n-1}, x_{2n-1}), F(x'_{2n}, x_{2n})) \\ &\le \frac{\alpha}{2}[d(x'_{2n-1}, x'_{2n}) + d(x_{2n-1}, x_{2n})] + (1 - \alpha)d(A, B). \end{aligned}$$

Letting $n \to \infty$, we have

$$d(q, q') \le \frac{\alpha}{2}[d(q, q') + d(p, p')] + (1 - \alpha)d(A, B). \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$d(p,p') + d(q,q') \leq \alpha[d(p,p') + d(q,q')] + 2(1-\alpha)d(A,B).$$

which implies that

$$d(p,p') + d(q,q') \leq 2d(A,B). \tag{3.11}$$

Since $d(A, B) \leq d(p, p')$ and $d(A, B) \leq d(q, q')$, we have

$$2d(A,B) \leq d(p,p') + d(q,q'). \tag{3.12}$$

From (3.11) and (3.12), we get

$$d(p,p') + d(q,q') = 2d(A,B).$$

This complete the proof. □

Note that every pair of nonempty closed subsets A, B of a uniformly convex Banach space X such that A is convex satisfies the property UC*. Therefore, we obtain the following corollary.

Corollary 3.11. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space X, $F : A \times A \to B$, $G : B \times B \to A$ and (F, G) be a cyclic contraction. Let $(x_0, x'_0) \in A \times A$ and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

and

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$. Then F has a coupled best proximity point $(p, q) \in A \times A$ and G has a coupled best proximity point $(p', q') \in B \times B$ such that

$$d(p,p') + d(q,q') = 2d(A,B).$$

Moreover, we have $x_{2n} \to p$, $x'_{2n} \to q$, $x_{2n+1} \to p'$ and $x'_{2n+1} \to q'$.

Next, we give some illustrative example of Corollary 3.11.

Example 3.12. *Consider uniformly convex Banach space $X = \mathbb{R}$ with the usual norm. Let $A = [1,2]$ and $B = [-2, -1]$. Thus $d(A, B) = 2$. Define $F : A \times A \to B$ and $G : B \times B \to A$ by*

$$F(x,x') = \frac{-x-x'-2}{4}$$

and

$$G(x,x') = \frac{-x-x'+2}{4}.$$

For arbitrary $(x, x') \in A \times A$ and $(y, y') \in B \times B$ and fixed $\alpha = \frac{1}{2}$, we get

$$\begin{aligned} d(F(x,x'), G(y,y')) &= \left| \frac{-x-x'-2}{4} - \frac{-y-y'+2}{4} \right| \\ &\leq \frac{|x-y| + |x'-y'|}{4} + 1 \\ &= \frac{\alpha}{2}[d(x,y) + d(x',y')] + 1(1-\alpha)d(A,B). \end{aligned}$$

This implies that (F, G) *is a cyclic contraction with* $\alpha = \frac{1}{2}$. *Since A and B are convex, we have* (A, B) *and* (B, A) *satisfy the property UC*. Therefore, all hypothesis of Corollary 3.11 hold. So F has a coupled best proximity point and G has a coupled best proximity point. We note that a point* $(1, 1) \in A \times A$ *is a unique coupled best proximity point of F and a point* $(-1, -1) \in B \times B$ *is a unique coupled best proximity point of G. Furthermore, we get*

$$d(1,-1) + d(1,-1) = 4 = 2d(A,B).$$

Theorem 3.13. *Let A and B be nonempty compact subsets of a metric space X,* $F : A \times A \to B$, $G : B \times B \to A$ *and* (F, G) *be a cyclic contraction pair. If* $(x_0, x'_0) \in A \times A$ *and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

and

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$, *then F has a coupled best proximity point* $(p, q) \in A \times A$ *and G has a coupled best proximity point* $(p', q') \in B \times B$ *such that*

$$d(p,p') + d(q,q') = 2d(A,B).$$

Proof. Since $x_0, x'_0 \in A$ and

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

and

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$, we have $x_{2n}, x'_{2n} \in A$ and $x_{2n+1}, x'_{2n+1} \in B$ for all $n \in \mathbb{N} \cup \{0\}$. As A is compact, the sequence $\{x_{2n}\}$ and $\{x'_{2n}\}$ have convergent subsequences $\{x_{2n_k}\}$ and $\{x'_{2n_k}\}$, respectively, such that

$$x_{2n_k} \to p \in A \text{ and } x'_{2n_k} \to q \in A.$$

Now, we have

$$d(A,B) \le d(p, x_{2n_k-1}) \le d(p, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}). \tag{3.13}$$

By Lemma 3.7, we have $d(x_{2n_k}, x_{2n_k-1}) \to d(A,B)$. Taking $k \to \infty$ in (3.13), we get $d(p, x_{2n_k-1}) \to d(A,B)$. By a similar argument we observe that $d(q, x_{2n_k-1}) \to d(A,B)$. Note that

$$\begin{aligned} d(A,B) &\le d((x_{2n_k}, F(p,q)) \\ &= d(G(x_{2n_k-1}, x'_{2n_k-1}), F(p,q)) \\ &\le \frac{\alpha}{2}[d(x_{2n_k-1}, p) + d(x'_{2n_k-1}, q)] + (1-\alpha)d(A,B). \end{aligned}$$

Taking $k \to \infty$, we get $d(p, F(p, q)) = d(A, B)$. Similarly, we can prove that $d(q, F(q, p)) = d(A, B)$. Thus F has a coupled best proximity $(p, q) \in A \times A$. In similar way, since B is compact, we can also prove that G has a coupled best proximity point in $(p', q') \in B \times B$.

For $d(p, p') + d(q, q') = 2d(A, B)$ similar to the final step of the proof of Theorem 3.10. This complete the proof. □

4 Coupled fixed point theorem

In this section, we give the new coupled fixed point theorem for a cyclic contraction pair.

Theorem 4.1. *Let A and B be nonempty closed subsets of a complete metric space X, $F : A \times A \to B$, $G : B \times B \to A$ and (F, G) be a cyclic contraction. Let* $(x_0, x'_0) \in A \times A$ *and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

and

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$. If $d(A, B) = 0$, then F and G have a unique common coupled fixed point $(p, q) \in A \cap B \times A \cap B$. Moreover, we have $x_{2n} \to p$, $x'_{2n} \to q$, $x_{2n+1} \to p$ and $x'_{2n+1} \to q$.

Proof. Since $d(A, B) = 0$, we get (A, B) and (B, A) have the property UC*. Therefore, by Theorem 3.10 claim that F has a coupled best proximity point $(p, q) \in A \times A$ that is

$$d(p, F(p, q)) = d(q, F(q, p)) = d(A, B) \tag{4.1}$$

and G has a coupled best proximity point $(p', q') \in B \times B$ that is

$$d(p', G(p', q')) = d(q', G(q', p')) = d(A, B). \tag{4.2}$$

Moreover, we have

$$d(p, p') + d(q, q') = 2d(A, B). \tag{4.3}$$

From (4.1) and $d(A, B) = 0$, we conclude that

$$p = F(p, q) \text{ and } q = F(q, p)$$

that is (p, q) is a coupled fixed point of F . It follows from (4.2) and $d(A, B) = 0$, we get

$$p' = G(p', q') \text{ and } q' = G(q', p')$$

that is (p', q') is a coupled fixed point of G. Using (4.3) and the fact that $d(A, B) = 0$, we have

$$d(p, p') + d(q, q') = 0$$

which implies that $p = p'$ and $q = q'$. Therefore, we conclude that $(p, q) \in A \cap B \times A \cap B$ *i*s a common coupled fixed point of F and G.

Finally, we show the uniqueness of common coupled fixed point of F and G. Let $(\hat{p}, \hat{q})$ be another common coupled fixed point of F and G. So $\hat{p} = G(\hat{p}, \hat{q})$ and $\hat{q} = G(\hat{q}, \hat{p})$. Now, we obtain that

$$d(p, \hat{p}) = d(F(p, q), G(\hat{p}, \hat{q})) \leq \frac{\alpha}{2}[d(p, \hat{p}) + d(q, \hat{q})] \tag{4.4}$$

and also

$$d(q,\hat{q}) = d(F(q,p), G(\hat{q},\hat{p})) \leq \frac{\alpha}{2}[d(q,\hat{q}) + d(p,\hat{p})]. \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$d(p,\hat{p}) + d(q,\hat{q}) \leq \alpha[d(p,\hat{p}) + d(q,\hat{q})],$$

which implies that $d(\hat{p},\hat{q}) + d(q,\hat{q}) = 0$ and so $d(p,\hat{p}) = 0$ and $d(q,\hat{q}) = 0$. Therefore, (p, q) is a unique common coupled fixed point in $A \cap B \times A \cap B$. □

Example 4.2. *Consider* $X = \mathbb{R}$ *with the usual metric,* $A = [-1, 0]$ *and* $B = [0,1]$ *. Define* $F : A \times A \to B$ *by* $F(x,y) = -\frac{x+y}{4}$ *and* $G(x,y) = -\frac{x+y}{8}$*. Then* $d(A, B) = 0$ *and* (F, G) *is a cyclic contraction with* $\alpha = \frac{1}{2}$*. Indeed, for arbitrary* $(x, x') \in A \times A$ *and* $(y, y') \in B \times B$*, we have*

$$\begin{aligned} d(F(x,x'), G(y,y')) &= \left| -\frac{x+x'}{4} + \frac{y+y'}{4} \right| \\ &\leq \left| -\frac{x+x'}{4} + \frac{2y+2y'}{8} \right| \\ &= \frac{1}{4}(|x-y| + |x'-y'|) \\ &= \frac{\alpha}{2}[d(x,y) + d(x',y')] + (1-\alpha)d(A,B). \end{aligned}$$

Therefore, all hypothesis of Theorem 4.1 *hold. So F and G have a unique common coupled fixed point and this point is* $(0, 0) \in A \cap B \times A \cap B$.

If we take $A = B$ in Theorem 4.1, then we get the following results.

Corollary 4.3. *Let A be nonempty closed subsets of a complete metric space X,* $F : A \times A \to A$ *and* $G : A \times A \to A$ *and let the order pair* (F, G) *is a cyclic contraction. Let* $(x_0, x'_0) \in A \times A$ *and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}), x'_{2n+1} = F(x'_{2n}, x_{2n})$$

and

$$x_{2n+2} = G(x_{2n+1}, x'_{2n+1}), x'_{2n+2} = G(x'_{2n+1}, x_{2n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$*. Then F and G have a unique common coupled fixed point* $(p, q) \in A \times A$*. Moreover, we have* $x_{2n} \to p$, $x'_{2n} \to q$, $x_{2n+1} \to p$ *and* $x'_{2n+1} \to q$

We take $F = G$ in Corollary 4.3, then we get the following results.

Corollary 4.4. *Let A be nonempty closed subsets of a complete metric space X,* $F : A \times A \to A$ *and*

$$d(F(x,x'), F(y,y')) \leq \frac{\alpha}{2}[d(x,y) + d(x',y')] \tag{4.6}$$

for all $(x, x'), (y, y') \in A \times A$*. Then F has a unique coupled fixed point* $(p, q) \in A \times A$.

Example 4.5. *Consider* $X = \mathbb{R}$ *with the usual metric and* $A = [0, \frac{1}{2}]$*. Define* $F : A \times A \to A$ *by*

$$F(x, y) = \begin{cases} \dfrac{x^2 - y^2}{4}; & x \geq y \\ 0; & x < y. \end{cases}$$

We show that F satisfies (4.6) *with* $\alpha = \frac{1}{2}$. *Let* (x, x'), $(y, y') \in A \times A$.

Case 1: *If* $x < x'$ *and* $y < y'$, *then*

$$d(F(x, x'), F(y, y')) = 0 \leq \frac{1}{4}[|x - y| + |x' - y'|] = \frac{\alpha}{2}[d(x, y) + d(x', y')].$$

Case 2: *If* $x < x'$ *and* $y \geq y'$, *then*

$$\begin{aligned} d(F(x, x'), F(y, y')) &= \left|0 - \frac{y^2 - y'^2}{4}\right| \\ &\leq \frac{1}{4}[|y - y'|\ |y + y'|] \\ &\leq \frac{1}{4}|y - y'| \\ &= \frac{1}{4}(y - y') \\ &< \frac{1}{4}[(y - y') + (x' - x)] \\ &\leq \frac{1}{4}[|x - y| + |x' - y'|] \\ &= \frac{\alpha}{2}[d(x, y) + d(x', y')]. \end{aligned}$$

Case 3: *If* $x \geq x'$ *and* $y < y'$. *In this case we can prove by a similar argument as in case* 2.

Case 4: *If* $x \geq x'$ *and* $y \geq y'$, *then*

$$\begin{aligned} d(F(x, x'), F(y, y')) &= \left|\frac{x^2 - x'^2}{4} - \frac{y^2 - y'^2}{4}\right| \\ &\leq \frac{1}{4}[|x - x'|\ |x + x'| + |y - y'|\ |y + y'|] \\ &\leq \frac{1}{4}[|x - x'| + |y - y'|] \\ &= \frac{\alpha}{2}[d(x, y) + d(x', y')]. \end{aligned}$$

Thus condition (4.6) *holds with* $\alpha = \frac{1}{2}$. *Therefore, by Corollary* 4.4 *F has the unique coupled fixed point in A that is a point* (0, 0).

Open problems:

- In Theorem 3.10, can be replaced the property UC* by a more general condition ?
- In Theorem 3.10, can be drop the property UC* ?
- Can be extend the result in this article to another spaces ?

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Authors' contributions

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Competing interests

The authors declare that they have no competing interests.

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