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PPF dependent fixed point theorems for an α_c-admissible non-self mapping in the Razumikhin class

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Abstract

In this paper, we introduce the concept of α_c-admissible non-self mappings and prove the existence and convergence of the past-present-future (briefly, PPF) dependent fixed point theorems for such mappings in the Razumikhin class. We use these results to prove the PPF dependent fixed point of Bernfeld *et al.* (Appl. Anal. 6:271-280, 1977) and also apply our results to PPF dependent coincidence point theorems.

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1 Introduction

The applications of fixed point theory are very important and useful in diverse disciplines of mathematics. The theory can be applied to solve many problem in real world, for example: equilibrium problems, variational inequalities and optimization problems. A very powerful tool in fixed point theory is the Banach fixed point theorem or Banach's contraction principle for a single-valued mapping. It is no surprise that there is a great number of generalizations of this principle. Several mathematicians have gone in several directions modifying Banach's contractive condition, changing the space or extending a single-valued mapping to a multivalued mapping (see [1–10]).

One of the most interesting results is the extension of Banach's contraction principle in case of non-self mappings. In 1997, Bernfeld *et al.* [11] introduced the concept of fixed point for mappings that have different domains and ranges, the so called past-present-future (briefly, PPF) dependent fixed point or the fixed point with PPF dependence. Furthermore, they gave the notion of Banach-type contraction for a non-self mapping and also proved the existence of PPF dependent fixed point theorems in the Razumikhin class for Banach-type contraction mappings. These results are useful for proving the solutions of nonlinear functional differential and integral equations which may depend upon the past history, present data and future consideration. Several PPF dependence fixed point theorems have been proved by many researchers (see [12–15]).

On the other hand, Samet *et al.* [16] were first to introduce the concept of α-admissible self-mappings and they proved the existence of fixed point results using contractive conditions involving an α-admissible mapping in complete metric spaces. They also gave some examples and applications to ordinary differential equations of the obtained results. Subsequently, there are a number of results proved for contraction mappings via the concept of α-admissible mapping in metric spaces and other spaces (see [17–19] and references therein).

To the best of our knowledge, there has been no discussion so far concerning the PPF dependent fixed point theorems via α-admissible mappings. In this paper, we introduce the concept of α_c-admissible non-self mappings and establish the existence and convergence of PPF dependent fixed point theorems for contraction mappings involving α_c-admissible non-self mappings in the Razumikhin class. Furthermore, we apply our results to the existence of PPF dependent fixed point theorems in [11] and also apply to PPF dependent coincidence point theorems.

2 Preliminaries

Throughout this paper, E denotes a Banach space with the norm $\|\cdot\|_E$, I denotes a closed interval $[a,b]$ in $\mathbb{R}$, and $E_0 = C(I,E)$ denotes the set of all continuous E-valued functions on I equipped with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|\phi\|_{E_0} = \sup_{t\in I}\big\|\phi(t)\big\|_E$$

for $\phi \in E_0$.

For a fixed element $c \in I$, the Razumikhin or minimal class of functions in E_0 is defined by

$$\mathcal{R}_c = \big\{\phi \in E_0 : \|\phi\|_{E_0} = \big\|\phi(c)\big\|_E\big\}.$$

It is easy to see that the constant function is one of the mapping in $\mathcal{R}_c$. The class $\mathcal{R}_c$ is said to be algebraically closed with respect to difference if $\phi - \xi \in \mathcal{R}_c$ whenever $\phi, \xi \in \mathcal{R}_c$. Also, we say that the class $\mathcal{R}_c$ is topologically closed if it is closed with respect to the topology on E_0 generated by the norm $\|\cdot\|_{E_0}$.

Definition 2.1 (Bernfeld *et al.* [11]) A point $\phi \in E_0$ is said to be *a PPF dependent fixed point* or *a fixed point with PPF dependence* of the non-self mapping $T : E_0 \to E$ if $T\phi = \phi(c)$ for some $c \in I$.

Definition 2.2 (Bernfeld *et al.* [11]) The mapping $T : E_0 \to E$ is called a Banach-type contraction if there exists a real number $k \in [0,1)$ such that

$$\|T\phi - T\xi\|_E \le k\|\phi - \xi\|_{E_0} \tag{2.1}$$

for all $\phi, \xi \in E_0$.

Definition 2.3 (Samet *et al.* [16]) Let X be a nonempty set, $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. We say that T is an α-admissible mapping if it satisfies the following condition:

$$\text{for } x, y \in X \quad \text{for which } \alpha(x,y) \ge 1 \quad \Longrightarrow \quad \alpha(Tx,Ty) \ge 1.$$

Example 2.4 Let $X = [1, \infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Tx = x^2$ for all $x \in X$ and

$$\alpha(x,y) = \begin{cases} 2 & \text{if } x \geq y, \\ 0 & \text{otherwise.} \end{cases}$$

Then T is α-admissible.

Example 2.5 Let $X = [1, \infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Tx = \log x$ for all $x \in X$ and

$$\alpha(x,y) = \begin{cases} e^{x-y} & \text{if } x \geq y, \\ 0 & \text{otherwise.} \end{cases}$$

Then T is α-admissible.

Remark 2.6 In the setting of Examples 2.4 and 2.5, every nondecreasing self-mapping T is ß-admissible.

Example 2.7 Let $X = \mathbb{R}$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$Tx = \begin{cases} \ln x & \text{if } x > 1, \\ \frac{x}{2} & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Then T is α-admissible.

3 PPF dependent fixed point theorems for α_c-admissible mappings

First of all, we introduce the concept of α_c-admissible non-self mappings.

Definition 3.1 Let $c \in I$ and $T : E_0 \to E$, $\alpha : E \times E \to [0, \infty)$. We say that T is an α_c-admissible mapping if for $\phi, \xi \in E_0$,

$$\alpha\bigl(\phi(c), \xi(c)\bigr) \geq 1 \quad \text{implies} \quad \alpha(T\phi, T\xi) \geq 1.$$

Example 3.2 Let $E = \mathbb{R}$ be real Banach spaces with usual norms and $I = [0, 1]$. Define $T : E_0 \to E$ and $\alpha : E \times E \to [0, \infty)$ by $T\phi = \phi(1)$ for all $\phi \in E_0$ and

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \geq y, \\ 0 & \text{otherwise.} \end{cases}$$

Then T is α_1-admissible.

Next, we prove the following result for a PPF dependent fixed point.

Theorem 3.3 *Let $T : E_0 \to E$, $\alpha : E \times E \to [0,\infty)$ be two mappings satisfying the following conditions*:

(a) *There exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference.*
(b) *T is α_c-admissible.*
(c) *For all $\phi, \xi \in E_0$,*

$$\alpha\big(\phi(c), T\phi\big)\alpha\big(\xi(c), T\xi\big)\|T\phi - T\xi\|_E \leq k\|\phi - \xi\|_{E_0},$$

where $k \in [0,1)$.
(d) *If $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to \infty$ and $\alpha(\phi_n(c), T\phi_n) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(\phi(c), T\phi) \geq 1$.*

If there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$, then T has a unique PPF dependent fixed point ϕ^* in $\mathcal{R}_c$ such that $\alpha(\phi^*(c), T\phi^*) \geq 1$.

Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$, if a sequence $\{\phi_n\}$ of iterates of T in $\mathcal{R}_c$ is defined by

$$T\phi_{n-1} = \phi_n(c) \tag{3.1}$$

for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathcal{R}_c$.

Proof Let ϕ_0 be a point in $\mathcal{R}_c \subseteq E_0$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$. Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathcal{R}_c$ such that

$$x_1 = \phi_1(c).$$

Since $\phi_1 \in \mathcal{R}_c \subseteq E_0$ and by hypothesis, we get $T\phi_1 \in E$. This implies that there exists $x_2 \in E$ such that $T\phi_1 = x_2$. Thus, we can choose $\phi_2 \in \mathcal{R}_c$ such that

$$x_2 = \phi_2(c).$$

By continuing this process, by induction, we can construct the sequence $\{\phi_n\}$ in $\mathcal{R}_c \subseteq E_0$ such that

$$T\phi_{n-1} = \phi_n(c)$$

for all $n \in \mathbb{N}$.

It follows from the fact that $\mathcal{R}_c$ is algebraically closed with respect to difference that

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \big\|\phi_{n-1}(c) - \phi_n(c)\big\|_E$$

for all $n \in \mathbb{N}$.

Since T is α_c-admissible and $\alpha(\phi_0(c), \phi_1(c)) = \alpha(\phi_0(c), T\phi_0) \geq 1$, we deduce that

$$\alpha\big(\phi_1(c), T\phi_1\big) = \alpha(T\phi_0, T\phi_1) \geq 1.$$

By continuing this process, we get $\alpha(\phi_{n-1}(c), T\phi_{n-1}) \geq 1$ for all $n \in \mathbb{N}$.

Next, we show that $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}\|\phi_n - \phi_{n+1}\|_{E_0} &= \big\|\phi_n(c) - \phi_{n+1}(c)\big\|_E \\ &= \|T\phi_{n-1} - T\phi_n\|_E \\ &\leq \alpha\big(\phi_{n-1}(c), T\phi_{n-1}\big)\alpha\big(\phi_n(c), T\phi_n\big)\|T\phi_{n-1} - T\phi_n\|_E \\ &\leq k\|\phi_{n-1} - \phi_n\|_{E_0}.\end{aligned}$$

By repeating the above relation, we get

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq k^n\|\phi_0 - \phi_1\|_{E_0}$$

for all $n \in \mathbb{N}$.

For $m, n \in \mathbb{N}$ with $m > n$, we obtain that

$$\begin{aligned}\|\phi_n - \phi_m\|_{E_0} &\leq \|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_{n+1} - \phi_{n+2}\|_{E_0} + \cdots + \|\phi_{m-1} - \phi_m\|_{E_0} \\ &\leq \big(k^n + k^{n+1} + \cdots + k^{m-1}\big)\|\phi_0 - \phi_1\|_{E_0} \leq \frac{k^n}{1-k}\|\phi_0 - \phi_1\|_{E_0}.\end{aligned}$$

This implies that the sequence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c \subseteq E_0$. By the completeness of E_0, we get that $\{\phi_n\}$ converges to a limit point $\phi^* \in E_0$, that is, $\lim_{n\to\infty}\phi_n = \phi^*$. Since $\mathcal{R}_c$ is topologically closed, we have $\phi^* \in \mathcal{R}_c$.

Now we prove that ϕ^* is a PPF dependent fixed point of T. By (d), we have $\alpha(\phi^*(c), T\phi^*) \geq 1$. From assumption (c), we get

$$\begin{aligned}\big\|T\phi^* - \phi^*(c)\big\|_E &\leq \big\|T\phi^* - \phi_n(c)\big\|_E + \big\|\phi_n(c) - \phi^*(c)\big\|_E \\ &= \big\|T\phi^* - T\phi_{n-1}\big\|_E + \big\|\phi_n - \phi^*\big\|_{E_0} \\ &\leq \alpha\big(\phi^*(c), T\phi^*\big)\alpha\big(\phi_{n-1}(c), T\phi_{n-1}\big)\big\|T\phi^* - T\phi_{n-1}\big\|_E + \big\|\phi_n - \phi^*\big\|_{E_0} \\ &\leq k\big\|\phi^* - \phi_{n-1}\big\|_{E_0} + \big\|\phi_n - \phi^*\big\|_{E_0}\end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the above inequality, we have

$$\big\|T\phi^* - \phi^*(c)\big\|_E = 0$$

and so

$$T\phi^* = \phi^*(c).$$

This implies that ϕ^* is a PPF dependent fixed point of T in $\mathcal{R}_c$.

Finally, we prove the uniqueness of a PPF dependent fixed point of T in $\mathcal{R}_c$. Let ϕ^* and ξ^* be two PPF dependent fixed points of T in $\mathcal{R}_c$ such that $\alpha(\phi^*(c), T\phi^*) \geq 1$ and $\alpha(\xi^*(c), T\xi^*) \geq 1$. Now we obtain that

$$\begin{aligned}\big\|\phi^* - \xi^*\big\|_{E_0} &= \big\|\phi^*(c) - \xi^*(c)\big\|_E \\ &= \big\|T\phi^* - T\xi^*\big\|_E\end{aligned}$$

$$\begin{aligned}&\leq \alpha\big(\phi^*(c), T\phi^*\big)\alpha\big(\xi^*(c), T\xi^*\big)\big\|T\phi^* - T\xi^*\big\|_E\\&\leq k\big\|\phi^* - \xi^*\big\|_{E_0}.\end{aligned}$$

Since $0 \leq k < 1$, we get $\|\phi^* - \xi^*\|_{E_0} = 0$ and then $\phi^* = \xi^*$. Therefore, T has a unique PPF dependent fixed point in $\mathcal{R}_c$. This completes the proof. □

Theorem 3.4 *Let $T : E_0 \to E$, $\alpha : E \times E \to [0,\infty)$ be two mappings satisfying the following conditions*:

(a) *There exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference.*

(b) *T is α_c-admissible.*

(c) *For all $\phi, \xi \in E_0$,*

$$\big(\|T\phi - T\xi\|_E + \epsilon\big)^{\alpha(\phi(c),T\phi)\alpha(\xi(c),T\xi)} \leq k\|\phi - \xi\|_{E_0} + \epsilon,$$

where $k \in [0,1)$ and $\epsilon \geq 1$.

(d) *If $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to \infty$ and $\alpha(\phi_n(c), T\phi_n) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(\phi(c), T\phi) \geq 1$.*

If there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$, then T has a unique PPF dependent fixed point ϕ^* in $\mathcal{R}_c$ such that $\alpha(\phi^*(c), T\phi^*) \geq 1$.

Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$, if a sequence $\{\phi_n\}$ of iterates of T in $\mathcal{R}_c$ is defined by

$$T\phi_{n-1} = \phi_n(c) \tag{3.2}$$

for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathcal{R}_c$.

Proof Let ϕ_0 be a point in $\mathcal{R}_c \subseteq E_0$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$. Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Now, we choose $\phi_1 \in \mathcal{R}_c$ such that

$$x_1 = \phi_1(c).$$

From the fact that $T\phi_1 \in E$, we obtain that there exists $x_2 \in E$ such that $T\phi_1 = x_2$. Thus, we can choose $\phi_2 \in \mathcal{R}_c$ such that

$$x_2 = \phi_2(c).$$

By continuing this process, we can construct the sequence $\{\phi_n\}$ in $\mathcal{R}_c \subseteq E_0$ such that

$$T\phi_{n-1} = \phi_n(c)$$

for all $n \in \mathbb{N}$.

By algebraic closedness with respect to difference of $\mathcal{R}_c$, we get

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \big\|\phi_{n-1}(c) - \phi_n(c)\big\|_E$$

for all $n \in \mathbb{N}$.

Since T is α_c-admissible and $\alpha(\phi_0(c), \phi_1(c)) = \alpha(\phi_0(c), T\phi_0) \geq 1$, we have

$$\alpha\big(\phi_1(c), T\phi_1\big) = \alpha(T\phi_0, T\phi_1) \geq 1.$$

By repeating this process and by induction, we get

$$\alpha\big(\phi_{n-1}(c), T\phi_{n-1}\big) \geq 1 \tag{3.3}$$

for all $n \in \mathbb{N}$.

Next, we show that $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|\phi_n - \phi_{n+1}\|_{E_0} + \epsilon &= \big\|\phi_n(c) - \phi_{n+1}(c)\big\|_E + \epsilon \\
&= \|T\phi_{n-1} - T\phi_n\|_E + \epsilon \\
&\leq \big(\|T\phi_{n-1} - T\phi_n\|_E + \epsilon\big)^{\alpha(\phi_{n-1}(c), T\phi_{n-1})\alpha(\phi_n(c), T\phi_n)} \\
&\leq k\|\phi_{n-1} - \phi_n\|_{E_0} + \epsilon.
\end{aligned}$$

This implies that

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq k\|\phi_{n-1} - \phi_n\|_{E_0}$$

for all $n \in \mathbb{N}$. Repeated application of the above relation yields

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq k^n\|\phi_0 - \phi_1\|_{E_0}$$

for all $n \in \mathbb{N}$.

For $m, n \in \mathbb{N}$ with $m > n$, we obtain that

$$\begin{aligned}
\|\phi_n - \phi_m\|_{E_0} &\leq \|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_{n+1} - \phi_{n+2}\|_{E_0} + \cdots + \|\phi_{m-1} - \phi_m\|_{E_0} \\
&\leq \big(k^n + k^{n+1} + \cdots + k^{m-1}\big)\|\phi_0 - \phi_1\|_{E_0} \\
&\leq \frac{k^n}{1-k}\|\phi_0 - \phi_1\|_{E_0}.
\end{aligned}$$

This implies that the sequence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c \subseteq E_0$. Since $\mathcal{R}_c$ is topologically closed and E_0 is complete, we get $\{\phi_n\}$ converges to a limit point $\phi^* \in \mathcal{R}_c$, that is, $\lim_{n\to\infty} \phi_n = \phi^*$.

Now we show that ϕ^* is a PPF dependent fixed point of T. By (3.3) and assumption (d), we get $\alpha(\phi^*(c), T\phi^*) \geq 1$. From assumption (c), we get

$$\begin{aligned}
\big\|T\phi^* - \phi^*(c)\big\|_E + \epsilon &\leq \big\|T\phi^* - \phi_n(c)\big\|_E + \big\|\phi_n(c) - \phi^*(c)\big\|_E + \epsilon \\
&= \big\|T\phi^* - T\phi_{n-1}\big\|_E + \big\|\phi_n - \phi^*\big\|_{E_0} + \epsilon \\
&\leq \big(\big\|T\phi^* - T\phi_{n-1}\big\|_E + \epsilon\big)^{\alpha(\phi^*(c), T\phi^*)\alpha(\phi_{n-1}(c), T\phi_{n-1})} + \big\|\phi_n - \phi^*\big\|_{E_0} \\
&\leq \big(k\big\|\phi^* - \phi_{n-1}\big\|_{E_0} + \epsilon\big) + \big\|\phi_n - \phi^*\big\|_{E_0}
\end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the above inequality, we have

$$\|T\phi^* - \phi^*(c)\|_E = 0.$$

This implies that $T\phi^* = \phi^*(c)$ and so ϕ^* is a PPF dependent fixed point of T in $\mathcal{R}_c$.

Finally, we prove the uniqueness of a PPF dependent fixed point of T in $\mathcal{R}_c$. Let ϕ^* and ξ^* be two PPF dependent fixed points of T in $\mathcal{R}_c$ such that $\alpha(\phi^*(c), T\phi^*) \geq 1$ and $\alpha(\xi^*(c), T\xi^*) \geq 1$. By assumption (c), we have

$$\begin{aligned}
\|\phi^* - \xi^*\|_{E_0} + \epsilon &= \|\phi^*(c) - \xi^*(c)\|_E + \epsilon \\
&= \|T\phi^* - T\xi^*\|_E + \epsilon \\
&\leq \left(\|T\phi^* - T\xi^*\|_E + \epsilon\right)^{\alpha(\phi^*(c),T\phi^*)\alpha(\xi^*(c),T\xi^*)} \\
&\leq k\|\phi^* - \xi^*\|_{E_0} + \epsilon \\
&\leq k\|\phi^* - \xi^*\|_{E_0} + \epsilon
\end{aligned}$$

and so $\|\phi^* - \xi^*\|_{E_0} \leq k\|\phi^* - \xi^*\|_{E_0}$. Since $0 < k < 1$, we have $\|\phi^* - \xi^*\|_{E_0} = 0$ and hence $\phi^* = \xi^*$. Therefore, T has a unique PPF dependent fixed point in $\mathcal{R}_c$. This completes the proof. □

Theorem 3.5 *Let $T : E_0 \to E$, $\alpha : E \times E \to [0, \infty)$ be two mappings satisfying the following conditions*:

(a) *There exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference.*

(b) *T is α_c-admissible.*

(c) *For all $\phi, \xi \in E_0$,*

$$\left(\alpha\left(\phi(c), T\phi\right)\alpha\left(\xi(c), T\xi\right) - 1 + \epsilon'\right)^{\|T\phi - T\xi\|_E} \leq \epsilon^{k\|\phi-\xi\|_{E_0}},$$

where $k \in [0, 1)$ and $1 < \epsilon \leq \epsilon'$.

(d) *If $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to \infty$ and $\alpha(\phi_n(c), T\phi_n) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(\phi(c), T\phi) \geq 1$.*

If there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$, then T has a unique PPF dependent fixed point ϕ^* in $\mathcal{R}_c$ such that $\alpha(\phi^*(c), T\phi^*) \geq 1$.

Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$, if a sequence $\{\phi_n\}$ of iterates of T in $\mathcal{R}_c$ is defined by

$$T\phi_{n-1} = \phi_n(c) \tag{3.4}$$

for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathcal{R}_c$.

Proof For fixed ϕ_0 in $\mathcal{R}_c \subseteq E_0$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$. Here we construct the sequence $\{\phi_n\}$ in $\mathcal{R}_c$.

Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathcal{R}_c$ such that

$$x_1 = \phi_1(c).$$

Since $T\phi_1 \in E$, we can find $x_2 \in E$ such that $T\phi_1 = x_2$. By the same argument, we can choose $\phi_2 \in \mathcal{R}_c$ such that

$$x_2 = \phi_2(c).$$

By induction, we produce the sequence $\{\phi_n\}$ in $\mathcal{R}_c \subseteq E_0$ such that

$$T\phi_{n-1} = \phi_n(c)$$

for all $n \in \mathbb{N}$.

We also obtain that

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \bigl\|\phi_{n-1}(c) - \phi_n(c)\bigr\|_E$$

for all $n \in \mathbb{N}$ since $\mathcal{R}_c$ is algebraically closed with respect to difference.

Since T is α_c-admissible and $\alpha(\phi_0(c), \phi_1(c)) = \alpha(\phi_0(c), T\phi_0) \geq 1$, we have

$$\alpha\bigl(\phi_1(c), T\phi_1\bigr) = \alpha(T\phi_0, T\phi_1) \geq 1.$$

By continuing this process, we get $\alpha(\phi_{n-1}(c), T\phi_{n-1}) \geq 1$ for all $n \in \mathbb{N}$.

Next, we show that $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} \epsilon^{\|\phi_n - \phi_{n+1}\|_{E_0}} &= \epsilon^{\|\phi_n(c) - \phi_{n+1}(c)\|_E} \\ &= \epsilon^{\|T\phi_{n-1} - T\phi_n\|_E} \\ &\leq \bigl(\alpha\bigl(\phi_{n-1}(c), T\phi_{n-1}\bigr)\alpha\bigl(\phi_n(c), T\phi_n\bigr) - 1 + \epsilon'\bigr)^{\|T\phi_{n-1} - T\phi_n\|_E} \\ &\leq \epsilon^{k\|\phi_{n-1} - \phi_n\|_{E_0}}. \end{aligned}$$

Since $\epsilon > 1$, we have

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq k\|\phi_{n-1} - \phi_n\|_{E_0}$$

for all $n \in \mathbb{N}$. By repeating this inequality, we have

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq k^n\|\phi_0 - \phi_1\|_{E_0}$$

for all $n \in \mathbb{N}$.

For $m, n \in \mathbb{N}$ with $m > n$, we obtain that

$$\begin{aligned} \|\phi_n - \phi_m\|_{E_0} &\leq \|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_{n+1} - \phi_{n+2}\|_{E_0} + \cdots + \|\phi_{m-1} - \phi_m\|_{E_0} \\ &\leq \bigl(k^n + k^{n+1} + \cdots + k^{m-1}\bigr)\|\phi_0 - \phi_1\|_{E_0} \\ &\leq \frac{k^n}{1-k}\|\phi_0 - \phi_1\|_{E_0}. \end{aligned}$$

This implies that the sequence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c \subseteq E_0$.

Since $\mathcal{R}_c$ is topologically closed, by the completeness of E_0, we get $\{\phi_n\}$ converges to a limit point $\phi^* \in \mathcal{R}_c$, that is, $\lim_{n\to\infty} \phi_n = \phi^*$.

Now we prove that ϕ^* is a PPF dependent fixed point of T. Since $\alpha(\phi_{n-1}(c), T\phi_{n-1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \phi_n = \phi^*$, by using condition (d), we have $\alpha(\phi^*(c), T\phi^*) \geq 1$. From condition (c), we get

$$\begin{aligned}
\epsilon^{\|T\phi^*-\phi^*(c)\|_E} &\leq \epsilon^{\|T\phi^*-\phi_n(c)\|_E+\|\phi_n(c)-\phi^*(c)\|_E} \\
&= \epsilon^{\|T\phi^*-T\phi_{n-1}\|_E+\|\phi_n-\phi^*\|_{E_0}} \\
&= \epsilon^{\|T\phi^*-T\phi_{n-1}\|_E}\epsilon^{\|\phi_n-\phi^*\|_{E_0}} \\
&\leq \left(\alpha\left(\phi^*(c), T\phi^*\right)\alpha\left(\phi_{n-1}(c), T\phi_{n-1}\right) - 1 + \epsilon'\right)^{\|T\phi^*-T\phi_{n-1}\|_E}\epsilon^{\|\phi_n-\phi^*\|_{E_0}} \\
&\leq \epsilon^{k\|\phi^*-\phi_{n-1}\|_{E_0}}\epsilon^{\|\phi_n-\phi^*\|_{E_0}} \\
&\leq \epsilon^{k\|\phi^*-\phi_{n-1}\|_{E_0}+\|\phi_n-\phi^*\|_{E_0}}
\end{aligned}$$

for all $n \in \mathbb{N}$.

Since the exponential function is a real continuous function, we can take the limit as $n \to \infty$ in the above inequality, and so

$$\left\|T\phi^* - \phi^*(c)\right\|_E = 0.$$

This implies that $T\phi^* = \phi^*(c)$ and hence ϕ^* is a PPF dependent fixed point of T in $\mathcal{R}_c$.

Finally, we prove the uniqueness of PPF dependent fixed point of T in $\mathcal{R}_c$. Let ϕ^* and ξ^* be two PPF dependent fixed points of T in $\mathcal{R}_c$ such that $\alpha(\phi^*(c), T\phi^*) \geq 1$ and $\alpha(\xi^*(c), T\xi^*) \geq 1$. Now we obtain that

$$\begin{aligned}
\epsilon^{\|\phi^*-\xi^*\|_{E_0}} &= \epsilon^{\|\phi^*(c)-\xi^*(c)\|_E} \\
&= \epsilon^{\|T\phi^*-T\xi^*\|_E} \\
&\leq \left(\alpha\left(\phi^*(c), T\phi^*\right)\alpha\left(\xi^*(c), T\xi^*\right) - 1 + \epsilon'\right)^{\|T\phi^*-T\xi^*\|_E} \\
&\leq \epsilon^{k\|\phi^*-\xi^*\|_{E_0}}
\end{aligned}$$

and then $\|\phi^* - \xi^*\|_{E_0} \leq k\|\phi^* - \xi^*\|_{E_0}$. Since $0 \leq k < 1$, we get $\|\phi^* - \xi^*\|_{E_0} = 0$ and then $\phi^* = \xi^*$. Therefore, T has a unique PPF dependent fixed point in $\mathcal{R}_c$. This completes the proof. □

Remark 3.6 If the Razumikhin class $\mathcal{R}_c$ is not topologically closed, then the limit of the sequence $\{\phi_n\}$ in Theorems 3.3, 3.4 and 3.5 may be outside of $\mathcal{R}_c$, which may not be unique.

4 Consequences

In this section, we show that many existing results in the literature can be deduced from and applied easily to our theorems.

4.1 Banach contraction theorem

By applying Theorems 3.3, 3.4 and 3.5, we obtain the following results.

Theorem 4.1 *Let $T : E_0 \to E$, and there exists a real number $k \in [0,1)$ such that*

$$\|T\phi - T\xi\|_E \le k\|\phi - \xi\|_{E_0} \tag{4.1}$$

for all $\phi, \xi \in E_0$.

If there exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference, then T has a unique PPF dependent fixed point in $\mathcal{R}_c$.

Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if a sequence $\{\phi_n\}$ of iterates of T in $\mathcal{R}_c$ is defined by

$$T\phi_{n-1} = \phi_n(c) \tag{4.2}$$

for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathcal{R}_c$.

Proof Let $\alpha : E \times E \to [0,\infty)$ be the mapping defined by $\alpha(x,y) = 1$ for all $x, y \in E$. Then T is an α_c-admissible mapping. It is easy to show that all the hypotheses of Theorems 3.3, 3.4 and 3.5 are satisfied. Consequently, T has a unique PPF dependent fixed point in $\mathcal{R}_c$. □

4.2 PPF dependent coincidence point theorems

In this section, we discuss some relation between PPF dependent fixed point results and PPF dependent coincidence point results. First, we give the concept of PPF dependent coincidence point.

Definition 4.2 Let $S : E_0 \to E_0$ and $T : E_0 \to E$. A point $\phi \in E_0$ is said to be a PPF dependent coincidence point or a coincidence point with PPF dependence of S and T if $T\phi = (S\phi)(c)$ for some $c \in I$.

Definition 4.3 Let $c \in I$ and $S : E_0 \to E_0$, $T : E_0 \to E$, $\alpha : E \times E \to [0,\infty)$. We say that (S,T) is an α_c-admissible pair if for $\phi, \xi \in E_0$,

$$\alpha\big((S\phi)(c), (S\xi)(c)\big) \ge 1 \quad \text{implies} \quad \alpha(T\phi, T\xi) \ge 1.$$

Remark 4.4 It easy to see that if (S,T) is an α_c-admissible pair and S is an identity mapping, then T is also an α_c-admissible mapping.

Now, we indicate that Theorem 3.3 can be utilized to derive a PPF dependent coincidence point theorem.

Theorem 4.5 *Let $S : E_0 \to E_0$, $T : E_0 \to E$, $\alpha : E \times E \to [0,\infty)$ be three mappings satisfying the following conditions*:

(a) *There exists $c \in I$ such that $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference.*
(b) *(S,T) is α_c-admissible.*
(c) *For all $\phi, \xi \in E_0$,*

$$\alpha\big((S\phi)(c), T\phi\big)\alpha\big((S\xi)(c), T\xi\big)\|T\phi - T\xi\|_E \le k\|S\phi - S\xi\|_{E_0},$$

where $k \in [0,1)$.

(d) *If* $\{S\phi_n\}$ *is a sequence in* E_0 *such that* $S\phi_n \to S\phi$ *as* $n \to \infty$ *and* $\alpha((S\phi_n)(c), T\phi_n) \geq 1$ *for all* $n \in \mathbb{N}$, *then* $\alpha((S\phi)(c), T\phi) \geq 1$.
(e) $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.

If there exists $\phi_0 \in \mathcal{R}_c$ *such that* $\alpha(\phi(c), T\phi) \geq 1$, *then S and T have a PPF dependent coincidence point* ω *in* $\mathcal{R}_c$ *such that* $\alpha((S\omega)(c), T\omega) \geq 1$.

Proof Consider the mapping $S : E_0 \to E_0$. We obtain that there exists $F_0 \subseteq E_0$ such that $S(F_0) = S(E_0)$ and $S|_{F_0}$ is one-to-one. Since $T(F_0) \subseteq T(E_0) \subseteq E$, we can define a mapping $\mathcal{A} : S(F_0) \to E$ by

$$\mathcal{A}(S\phi) = T\phi \tag{4.3}$$

for all $\phi \in F_0$. Since $S|_{F_0}$ is one-to-one, then $\mathcal{A}$ is well defined.

From (4.3) and condition (c), we have

$$\alpha\big((S\phi)(c), \mathcal{A}(S\phi)\big)\alpha\big((S\xi)(c), \mathcal{A}(S\xi)\big)\big\|\mathcal{A}(S\phi) - \mathcal{A}(S\xi)\big\|_E \leq k\|S\phi - S\xi\|_{E_0}$$

for all $S\phi, S\xi \in S(E_0)$. This shows that $\mathcal{A}$ satisfies condition (c) of Theorem 3.3.

Now, we use Theorem 3.3 with a mapping $\mathcal{A}$, then there exists a unique PPF dependent fixed point $\varphi \in S(F_0)$ of $\mathcal{A}$, that is, $\mathcal{A}\varphi = \varphi(c)$ and $\alpha(\varphi(c), \mathcal{A}\varphi) \geq 1$. Since $\varphi \in S(F_0)$, we can find $\omega \in F_0$ such that $\varphi = S\omega$. Therefore, we get

$$T\omega = \mathcal{A}(S\omega) = \mathcal{A}\varphi = \varphi(c) = (S\omega)(c)$$

and

$$\alpha\big((S\omega)(c), T\omega\big) = \alpha\big(\varphi(c), \mathcal{A}\varphi\big) \geq 1.$$

This implies that ω is a PPF dependent coincidence point of T and S. This completes the proof. □

Similarly, we can apply Theorems 3.4 and 3.5 to the Theorems 4.6 and 4.7. Then, in order to avoid repetition, the proof is omitted.

Theorem 4.6 *Let* $S : E_0 \to E_0$, $T : E_0 \to E$, $\alpha : E \times E \to [0, \infty)$ *be three mappings satisfying the following conditions*:

(a) *There exists* $c \in I$ *such that* $S(\mathcal{R}_c)$ *is topologically closed and algebraically closed with respect to difference.*
(b) (S, T) *is* α_c*-admissible.*
(c) *For all* $\phi, \xi \in E_0$,

$$\big(\|T\phi - T\xi\|_E + \epsilon\big)^{\alpha((S\phi)(c), T\phi)\alpha((S\xi)(c), T\xi)} \leq k\|S\phi - S\xi\|_{E_0} + \epsilon,$$

where $k \in [0, 1)$ *and* $\epsilon \geq 1$.
(d) *If* $\{S\phi_n\}$ *is a sequence in* E_0 *such that* $S\phi_n \to S\phi$ *as* $n \to \infty$ *and* $\alpha((S\phi_n)(c), T\phi_n) \geq 1$ *for all* $n \in \mathbb{N}$, *then* $\alpha((S\phi)(c), T\phi) \geq 1$.
(e) $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.

If there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi(c), T\phi) \geq 1$, then S and T have a PPF dependent coincidence point ω in $\mathcal{R}_c$ such that $\alpha((S\omega)(c), T\omega) \geq 1$.

Theorem 4.7 *Let $S : E_0 \to E_0$, $T : E_0 \to E$, $\alpha : E \times E \to [0, \infty)$ be three mappings satisfying the following conditions*:

(a) *There exists $c \in I$ such that $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference.*

(b) *(S, T) is α_c-admissible.*

(c) *For all $\phi, \xi \in E_0$,*

$$\left(\alpha\left((S\phi)(c), T\phi\right)\alpha\left((S\xi)(c), T\xi\right) - 1 + \epsilon'\right)^{\|T\phi - T\xi\|_E} \leq \epsilon^{k\|S\phi - S\xi\|_{E_0}},$$

where $k \in [0, 1)$ and $1 < \epsilon \leq \epsilon'$.

(d) *If $\{S\phi_n\}$ is a sequence in E_0 such that $S\phi_n \to S\phi$ as $n \to \infty$ and $\alpha((S\phi_n)(c), T\phi_n) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha((S\phi)(c), T\phi) \geq 1$.*

(e) *$S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.*

If there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi(c), T\phi) \geq 1$, then S and T have a PPF dependent coincidence point ω in $\mathcal{R}_c$ such that $\alpha((S\omega)(c), T\omega) \geq 1$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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