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Best proximity points and stability results for controlled proximal contractive set valued mappings

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Abstract

In this paper, we introduce first the concept of a Pompeiu-Hausdorff b-metric-like space. We also establish some best proximity points and stability results for controlled proximal contractive set valued mappings in the class of b-metric-like spaces and partial b-metric spaces. Moreover, we provide some examples and many nice consequences from our obtained results.

MSC: 47H10; 54H25

Keywords: Pompeiu-Hausdorff b-metric-like; best proximity point; controlled proximal contraction; stability

1 Introduction and preliminaries

Markins [1] and Nadler [2] initiated the study of fixed point theorems for set valued operators. Since then, several other papers have been concerned with the study of multi-valued operators in variant (generalized) metric space. We cite for example, Ali *et al.* [3, 4], Aydi *et al.* [5, 6], Berinde and Berinde [7], Berinde and Pãcurar [8], Boriceanu *et al.* [9], Bota [10], Ćirić [11], Ćirić and Ume [12, 13], Czerwik [14], Daffer and Kaneko [15], Jleli *et al.* [16], Mizoguchi and Takahashi [17], *etc.* In this paper, we are interested first to initiate the concept of a Pompeiu-Hausdorff b-metric-like and to prove some best proximity points and stability results.

On the other hand, metric-like spaces were considered by Hitzler and Seda [18] under the name of dislocated metric spaces. In 2013, Alghamdi *et al.* [19] generalized the notion of a b-metric [14] by introducing the concept of a b-metric-like and proved some related fixed point results. After that, Hussain *et al.* [20] established some fixed point theorems in the setting of b-metric-like spaces.

Definition 1.1 Let X be a nonempty set and $s \geq 1$ be a given real. A function $\sigma : X \times X \to \mathbb{R}^+$ is said to be a b-metric-like (or a dislocated b-metric) on X if for any $x, y, z \in X$, the following conditions hold:

(bm_1) $\sigma(x,y) = 0 \Rightarrow x = y$;
(bm_2) $\sigma(x,y) = \sigma(y,x)$;
(bm_3) $\sigma(x,z) \leq s(\sigma(x,y) + \sigma(y,z))$.

The pair (X,σ) is then called a b-metric-like space.

Let (X,σ) be a b-metric-like space. An open σ-ball $\{B_\sigma(x,\varepsilon) : x \in X, \varepsilon > 0\}$ is defined as $B_\sigma(x,\varepsilon) = \{y \in X : |\sigma(x,y) - \sigma(x,x)| < \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if

$$\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x). \tag{1.1}$$

Mention that the limit for a convergent sequence is not unique in general. $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\to\infty} \sigma(x_n,x_m)$ exists and is finite. We say that (X,σ) is complete if and only if each Cauchy sequence in X is convergent.

Lemma 1.2 *Let (X,σ) be a b-metric-like space and $\{x_n\}$ be a sequence that converges to u with $\sigma(u,u) = 0$. Then, for each $y,z \in X$, one has*

$$\frac{1}{s}\sigma(u,z) \le \liminf_{n\to\infty} \sigma(x_n,z) \le \limsup_{n\to\infty} \sigma(x_n,z) \le s\sigma(u,z) \quad \textit{and} \quad \sigma(z,z) \le 2s\sigma(z,y).$$

In 2015, Aydi *et al.* [21] introduced the following concept.

Definition 1.3 Let (X,d) be a rectangular b-metric space. We say that (X,d) satisfies the property (S_C) if for every sequence $\{x_n\}$ in X and all $x,y \in X$, we have

$$\lim_{n\to\infty} d(x_n,x) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} d(x_n,y) = d(x,y).$$

We extend Definition 1.3 to the class of b-metric-like spaces.

Definition 1.4 Let (X,σ) be a b-metric-like space. We say that (X,σ) satisfies the property (G_C) if for all sequences $\{x_n\}$, $\{y_n\}$ in X and all $x,y \in X$, we have

$$\lim_{n\to\infty} \sigma(x_n,x) = \lim_{n\to\infty} \sigma(y_n,y) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \sigma(x_n,y_n) = \sigma(x,y).$$

Remark 1.5

1. If (X,d) is a rectangular b-metric space satisfying the property (G_C), then it also satisfies the property (S_C). Indeed, let $\{x_n\}$ be a sequence in X and $x,y \in X$ such that $\lim_{n\to\infty} d(x_n,x) = 0$. Take $\{y_n\}$ in X such that $y_n = y$ for all $n \ge 0$. Then $d(y_n,y) = d(y,y) = 0$, and so $\lim_{n\to\infty} d(y_n,y) = 0$. Since (X,d) satisfies the property (G_C), it follows that $\lim_{n\to\infty} d(x_n,y_n) = d(x,y)$, that is, $\lim_{n\to\infty} d(x_n,y) = d(x,y)$, and so (X,d) satisfies the property (S_C).
2. Let (X,σ) be a b-metric-like space satisfying the property (G_C). Take $\{x_n\}$ a sequence in X and $x,y \in X$ such that $\sigma(y,y) = 0$ and $\lim_{n\to\infty} \sigma(x_n,x) = 0$. Then $\lim_{n\to\infty} \sigma(x_n,y) = \sigma(x,y)$.

The following examples make effective use of the property (G_C).

Example 1.6 Let $X = [0,1]$. Consider the mapping $\sigma : X \times X \to [0,\infty)$ defined by $\sigma(x,y) = (x+y-xy)^2$ for all $x,y \in X$. Then (X,σ) is a b-metric-like space with $s = 2$. Let $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} \sigma(x_n,x) = \lim_{n\to\infty} \sigma(y_n,y) = 0.$$

It follows that $\sigma(x,x) = \sigma(y,y) = 0$, and so $x = y = 0$. Then we get

$$\lim_{n\to\infty} x_n^2 = \lim_{n\to\infty} y_n^2 = 0.$$

This leads to

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0.$$

Hence,

$$\lim_{n\to\infty} \sigma(x_n,y_n) = \lim_{n\to\infty} (x_n + y_n - x_n y_n)^2 = 0 = \sigma(0,0).$$

Consequently, (X,σ) satisfies the property (G_C).

Example 1.7 Let $X = \{0,1,2\}$. Consider the mapping $\sigma : X \times X \to [0,\infty)$ defined by

$$\sigma(0,0) = 0, \qquad \sigma(1,1) = \sigma(2,2) = 2, \qquad \sigma(0,1) = \sigma(1,0) = 4,$$
$$\sigma(1,2) = \sigma(2,1) = 2, \qquad \sigma(0,2) = \sigma(2,0) = 2.$$

Then (X,σ) is a b-metric-like space with $s = 2$. Let $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} \sigma(x_n,x) = \lim_{n\to\infty} \sigma(y_n,y) = 0.$$

It follows that $\sigma(x,x) = \sigma(y,y) = 0$, and so $x = y = 0$. Moreover, there exists $N \in \mathbb{N}$, such that, for all $n \geq N$,

$$\sigma(x_n,0) \leq \frac{1}{2} \quad \text{and} \quad \sigma(y_n,0) \leq \frac{1}{2}.$$

Therefore

$$\sigma(x_n,0) = 0 \quad \text{and} \quad \sigma(y_n,0) = 0, \quad \forall n \geq N.$$

Thus, for all $n \geq N$, we have $x_n = y_n = 0$. This yields $\sigma(x_n,y_n) = \sigma(0,0)$ for all $n \geq N$, and so $\lim_{n\to\infty} \sigma(x_n,y_n) = \sigma(x,y)$. Hence, (X,σ) satisfies the property (G_C).

Lemma 1.8 *Let (X,σ) be a b-metric-like space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x,y \in X$ such that* $\lim_{n\to\infty} \sigma(x_n,x) = \lim_{n\to\infty} \sigma(y_n,y) = 0$. *Then one has*

$$s^{-2}\sigma(x,y) \leq \liminf_{n\to\infty} \sigma(x_n,y_n) \leq \limsup_{n\to\infty} \sigma(x_n,y_n) \leq s^2\sigma(x,y).$$

We also have the following useful lemma.

Lemma 1.9 *Any metric-like space satisfies the property (G_C).*

Proof It suffices to take $s = 1$ in Lemma 1.8. □

Recently, Aydi *et al.* [21, 22] introduced the concept of a Pompeiu-Hausdorff metric-like. The aim of the first part of paper is to extend this concept to the class of b-metric-like spaces and then to prove some results on best proximity points and stability for controlled proximal contractions, so generalizing the very recent paper of Kiran *et al.* [23]. In the second part of paper, the analogous of above results in the class of partial b-metric spaces is studied.

From now on, let (X,σ) be a b-metric-like space. As in [21, 22, 24], let $C_b(X)$ be the family of all nonempty, closed and bounded subsets of the b-metric-like space (X,σ), induced by the b-metric-like σ. For $A,B \in C_b(X)$ and $x \in X$, define

$$\begin{aligned} &\sigma(x,A) = \inf\bigl\{\sigma(x,a) : a \in A\bigr\}, \\ &\delta_\sigma(A,B) = \sup\bigl\{\sigma(a,B) : a \in A\bigr\}, \\ &\delta_\sigma(B,A) = \sup\bigl\{\sigma(b,A) : b \in B\bigr\}. \end{aligned}$$

Also

$$H^b_\sigma(A,B) = \max\bigl\{\delta_\sigma(A,B), \delta_\sigma(B,A)\bigr\}. \tag{1.2}$$

The above H^b_σ is called a Pompeiu-Hausdorff b-metric-like. For A and B two nonempty subsets of a b-metric-like space (X,σ), define

$$\begin{aligned} &\sigma(A,B) = \inf\bigl\{\sigma(a,b) : a \in A, b \in B\bigr\}, \\ &A_0 = \bigl\{a \in A : \sigma(a,b) = \sigma(A,B), \text{for some } b \in B\bigr\}, \\ &B_0 = \bigl\{b \in B : \sigma(a,b) = \sigma(A,B), \text{for some } a \in A\bigr\}. \end{aligned}$$

As in [25], the concept of a weak P-property is stated as follows.

Definition 1.10 Let A and B be nonempty subsets of a b-metric-like space (X,σ) with $A_0 \neq \emptyset$. The pair (A,B) is said to have the weak P-property if and only if

$$\begin{cases} \sigma(x_1,y_1) = \sigma(A,B), \\ \sigma(x_2,y_2) = \sigma(A,B) \end{cases} \quad \Rightarrow \quad \sigma(x_1,x_2) \le \sigma(y_1,y_2),$$

where $x_1,x_2 \in A_0$ and $y_1,y_2 \in B_0$.

Example 1.11 Let $X = \{(1,2),(0,1),(1,3),(3,1)\}$ be endowed with the b-metric-like $\sigma((x_1, x_2),(y_1,y_2)) = (x_1 + x_2 + y_1 + y_2)^2$ for all $(x_1,x_2),(y_1,y_2) \in X$. Let $A = \{(1,2),(0,1)\}$ and $B = \{(1,3),(3,1)\}$. Clearly,

$$\sigma\bigl((0,1),(1,3)\bigr) = 25 = \sigma(A,B) \quad \text{and} \quad \sigma\bigl((0,1),(3,1)\bigr) = \sigma(A,B).$$

Also

$$\sigma\bigl((0,1),(0,1)\bigr) = 4 < 64 = \sigma\bigl((1,3),(3,1)\bigr).$$

Moreover, $A_0 \neq \emptyset$. Hence, the pair (A,B) satisfies the weak P-property.

Example 1.12 Let A and B be nonempty subsets of a b-metric-like space (X,σ) with $A_0 \neq \emptyset$ and $\sigma(A,B) = 0$. Then the pair (A,B) satisfies the weak P-property.

On the other hand, the definition of a best proximity point is as follows.

Definition 1.13 Let (X,σ) be a b-metric-like space. Consider A and B two nonempty subsets of X. An element $a \in X$ is said to be a best proximity point of $T : A \to B$ if

$$\sigma(a, Ta) = \sigma(A,B).$$

It is clear that a fixed point coincides with a best proximity point if $\sigma(A,B) = 0$. For more results on best proximity points, see for example [26–31].

In this paper, we give first some properties of H^b_σ. Second, we establish some existence results on best proximity points and some stability results for controlled proximal set valued contractive mappings in the setting of two (generalized) metric spaces. We will support the obtained theorems by some concrete examples. We also provide many interesting consequences and corollaries.

2 Properties and preliminaries

First, we present some useful properties of the Pompeiu-Hausdorff b-metric-like H^b_σ.

Lemma 2.1 [21, 22] *Let (X,σ) be a b-metric-like space and A any nonempty set in (X,σ), then*

$$\textit{if } \sigma(a,A) = 0, \quad \textit{then } a \in \bar{A}. \tag{2.1}$$

Lemma 2.2 *Let (X,σ) be a b-metric-like space. For $x \in X$ and $A,B,C \in C_b(X)$, we have*

(i) $H^b_\sigma(A,A) = \delta_\sigma(A,A) = \sup\{\sigma(a,A) : a \in A\}$;
(ii) $H^b_\sigma(A,B) = H^b_\sigma(B,A)$;
(iii) $H^b_\sigma(A,B) = 0$ *implies that* $A = B$;
(iv) $H^b_\sigma(A,B) \leq s(H^b_\sigma(A,C) + H^b_\sigma(C,B))$;
(v) $\sigma(x,A) \leq s(\sigma(x,y) + \sigma(y,A))$.

Proof (i)-(iii) are clear.

(iv) Let $a \in A$, $b \in B$, and $c \in C$. By a triangular inequality

$$\sigma(a,b) \leq s\bigl(\sigma(a,c) + \sigma(c,b)\bigr).$$

The points b and c are arbitrary, so

$$\sigma(a,B) \leq s\bigl(\sigma(a,c) + \sigma(c,B)\bigr) \leq s\bigl(\sigma(a,c) + \delta_\sigma(C,B)\bigr) \leq s\bigl(\sigma(a,C) + \delta_\sigma(C,B)\bigr).$$

Again, a is arbitrary, so

$$\delta_\sigma(A,B) \leq s\bigl(\delta_\sigma(A,C) + \delta_\sigma(C,B)\bigr) \leq sH^b_\sigma(A,C) + sH^b_\sigma(C,B).$$

Similarly, by symmetry of H_σ^b, we have

$$\delta_\sigma(B,A) \leq s\big(H_\sigma^b(A,C) + H_\sigma^b(C,B)\big).$$

Combining the two above inequalities, we get (iv).

(v) For $a \in A$ and $x, y \in X$, we have $\sigma(x,A) \leq \sigma(x,a) \leq s(\sigma(x,y) + \sigma(y,a))$. Again, a is arbitrary, then

$$\sigma(x,A) \leq s\big(\sigma(x,y) + \sigma(y,A)\big). \qquad \square$$

The following two lemmas are very essential for best proximity points and stability results stated in the next section. The proofs are very classical.

Lemma 2.3 *Let (X,σ) be a b-metric-like space. Let $A, B \in C_b(X)$ and $h > 1$. For any $x \in A$, there exists $y = y(a) \in B$ such that*

$$\sigma(x,y) \leq hH_\sigma^b(A,B). \tag{2.2}$$

Lemma 2.4 *Let (X,σ) be a b-metric-like space. Let $A, B \in C_b(X)$ and $a \in A$. Then, for all $\varepsilon > 0$, there exists a point $y \in B$ such that $\sigma(a,y) \leq H_\sigma^b(A,B) + \varepsilon$.*

3 Best proximity points and stability results on the class of *b*-metric-like spaces

3.1 Best proximity points

First, we need the following definition.

Definition 3.1 Let A and B be nonempty subsets of a b-metric-like space (X,σ) such that $A_0 \neq \emptyset$. Let $x_0 \in A_0$ and $r > 0$. A mapping $T : A \to C_b(B)$ is called a proximal contraction on $\overline{B}_\sigma(x_0,r)$, if there exists $\alpha \in (0, \frac{1}{s})$ such that

$$H_\sigma^b(Tx, Ty) \leq \alpha\sigma(x,y), \tag{3.1}$$

for all $x, y \in \overline{B}_\sigma(x_0,r) \cap A$.

Our first main result is the following theorem.

Theorem 3.2 *Let A and B be nonempty closed subsets of a complete b-metric-like space (X,σ) and $r > 0$. Let $T : A \to C_b(B)$ be a multi-valued mapping. Suppose that*

(i) $A_0 \neq \emptyset$;

(ii) *for each $x \in A_0$, we have $Tx \subseteq B_0$;*

(iii) *the pair (A,B) satisfies the weak P-property;*

(iv) *there exists $x_0 \in A_0$ such that T is a proximal contraction on $\overline{B}_\sigma(x_0,r)$ and* $\delta_\sigma(Tx_0, \{x_0\}) + \sigma(A,B) \leq \frac{1}{2s^3 - s^2}(1 - \sqrt{\alpha s})r$;

(v) *(X,σ) satisfies the property (G_C).*

Then T has a best proximity point in $\overline{B}_\sigma(x_0,r) \cap A$. We also have $\sigma(x^\star, x^\star) = 0$.

Proof By assumption (iv), there exists $x_0 \in A_0$ such that T is a proximal contraction on $\overline{B}_\sigma(x_0,r)$ and

$$\delta_\sigma\big(Tx_0, \{x_0\}\big) + \sigma(A,B) \leq \frac{1}{2s^3 - s^2}(1 - \sqrt{\alpha s})r.$$

Let $y_0 \in Tx_0$. By condition (ii), we have $Tx_0 \subseteq B_0$. Then there exists $x_1 \in A_0$ such that

$$\sigma(x_1, y_0) = \sigma(A, B). \tag{3.2}$$

We have

$$\begin{aligned}\sigma(x_0, x_1) &\le s\bigl[\sigma(x_0, y_0) + \sigma(y_0, x_1)\bigr]\\ &\le s\bigl[\delta_\sigma\bigl(Tx_0, \{x_0\}\bigr) + \sigma(A, B)\bigr]\\ &\le \frac{1}{2s^2 - s}(1 - \sqrt{\alpha s})r.\end{aligned} \tag{3.3}$$

On the other hand, we have

$$\sigma(x_0, x_0) - \sigma(x_0, x_1) \le (2s - 1)\sigma(x_0, x_1).$$

Also

$$\sigma(x_0, x_1) - \sigma(x_0, x_0) \le \sigma(x_0, x_1) \le (2s - 1)\sigma(x_0, x_1).$$

Then

$$\begin{aligned}\bigl|\sigma(x_0, x_1) - \sigma(x_0, x_0)\bigr| &\le (2s - 1)\sigma(x_0, x_1)\\ &\le \frac{(2s - 1)}{2s^2 - s}(1 - \sqrt{\alpha s})r\\ &= s^{-1}(1 - \sqrt{\alpha s})r < r.\end{aligned}$$

Thus, $x_1 \in \overline{B}_\sigma(x_0, r) \cap A_0$. By Lemma 2.3, there exists $y_1 \in Tx_1$ such that

$$\sigma(y_0, y_1) \le \frac{1}{\sqrt{\alpha s}} H_\sigma^b(Tx_0, Tx_1). \tag{3.4}$$

So, by (3.1), we get

$$\sigma(y_0, y_1) \le \sqrt{\frac{\alpha}{s}}\sigma(x_0, x_1). \tag{3.5}$$

Since $y_1 \in Tx_1 \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$\sigma(x_2, y_1) = \sigma(A, B). \tag{3.6}$$

From condition (iii), (3.2), and (3.6)

$$\sigma(x_1, x_2) \le \sigma(y_0, y_1). \tag{3.7}$$

Therefore,

$$\sigma(x_1, x_2) \le \sqrt{\frac{\alpha}{s}}\sigma(x_0, x_1). \tag{3.8}$$

We have

$$\begin{aligned}
\left|\sigma(x_0,x_2)-\sigma(x_0,x_0)\right| &\leq (2s-1)\sigma(x_0,x_2)\\
&\leq s(2s-1)\left[\sigma(x_0,x_1)+\sigma(x_1,x_2)\right]\\
&\leq s(2s-1)\left[\sigma(x_0,x_1)+s\sigma(x_1,x_2)\right]\\
&\leq s(2s-1)\left[1+s\sqrt{\frac{\alpha}{s}}\right]\sigma(x_0,x_1)\\
&\leq s(2s-1)[1+\sqrt{\alpha s}]\frac{1}{2s^2-s}(1-\sqrt{\alpha s})r\\
&= (1-\alpha s)r < r.
\end{aligned}$$

Then $x_2 \in \overline{B}_\sigma(x_0,r)\cap A_0$. Again, by Lemma 2.3, there exists $y_2 \in Tx_2$ such that

$$\sigma(y_1,y_2) \leq \frac{1}{\sqrt{\alpha s}}H_\sigma^b(Tx_1,Tx_2). \tag{3.9}$$

So, by (3.1), we get

$$\sigma(y_1,y_2) \leq \sqrt{\frac{\alpha}{s}}\sigma(x_1,x_2). \tag{3.10}$$

Since $y_2 \in Tx_2 \subseteq B_0$, then there exists $x_3 \in A_0$ such that

$$\sigma(x_3,y_2) = \sigma(A,B). \tag{3.11}$$

By condition (iii), (3.8), and (3.10)

$$\sigma(x_2,x_3) \leq \sigma(y_1,y_2) \leq \sqrt{\frac{\alpha}{s}}\sigma(x_1,x_2) \leq \left(\sqrt{\frac{\alpha}{s}}\right)^2\sigma(x_0,x_1). \tag{3.12}$$

We have

$$\begin{aligned}
\left|\sigma(x_0,x_3)-\sigma(x_0,x_0)\right| &\leq (2s-1)\sigma(x_0,x_3)\\
&\leq (2s-1)\left[s\sigma(x_0,x_1)+s^2\sigma(x_1,x_2)+s^2\sigma(x_2,x_3)\right]\\
&\leq (2s-1)\left[s\sigma(x_0,x_1)+s^2\sigma(x_1,x_2)+s^3\sigma(x_2,x_3)\right]\\
&\leq s(2s-1)\left[1+s\sqrt{\frac{\alpha}{s}}+s^2\left(\sqrt{\frac{\alpha}{s}}\right)^2\right]\sigma(x_0,x_1)\\
&\leq s(2s-1)\left[1+\sqrt{\alpha s}+(\sqrt{\alpha s})^2\right]\frac{1}{2s^2-s}(1-\sqrt{\alpha s})r\\
&= \left(1-(\sqrt{\alpha s})^3\right)r < r.
\end{aligned}$$

Then $x_3 \in \overline{B}_\sigma(x_0,r)\cap A_0$.

Continuing this process, we complete two sequences $\{x_n\} \subseteq \overline{B}_\sigma(x_0,r)\cap A_0$ and $\{y_n\} \subseteq B_0$ such that

$$\begin{cases}
\sigma(x_n,y_{n-1}) = \sigma(A,B),\\
\sigma(x_n,x_{n+1}) \leq \sigma(y_{n-1},y_n) \leq (\sqrt{\frac{\alpha}{s}})^n\sigma(x_0,x_1),\\
y_n \in Tx_n, \quad \text{for all } n = 1,2,\ldots.
\end{cases}$$

For $m > n$, we have

$$\begin{aligned}\sigma(x_n,x_m) &\leq \sum_{k=n}^{m-1} s^k\sigma(x_k,x_{k+1}) \leq \sum_{k=n}^{m-1}(\sqrt{s\alpha})^k\sigma(x_0,x_1)\\ &\leq \sum_{k=n}^{\infty}(\sqrt{s\alpha})^k\sigma(x_0,x_1) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

We supposed that $0 < \alpha s < 1$, so $\lim_{n,m\to\infty}\sigma(x_n,x_m) = 0$. Hence, $\{x_n\}$ is a Cauchy sequence in $\overline{B}_\sigma(x_0,r) \cap A$. A similar reasoning shows that $\lim_{n,m\to\infty}\sigma(y_n,y_m) = 0$ and so $\{y_n\}$ is a Cauchy sequence in B. Since $\overline{B}_\sigma(x_0,r) \cap A$ and B are closed subsets of the complete b-metric-like space (X,σ), there exist $x^\star \in \overline{B}_\sigma(x_0,r) \cap A$ and $y^\star \in B$ such that

$$\begin{aligned}&\lim_{n\to\infty}\sigma\left(x_n,x^\star\right) = \sigma\left(x^\star,x^\star\right) = \lim_{n,m\to\infty}\sigma(x_n,x_m) = 0 \quad \text{and}\\ &\lim_{n\to\infty}\sigma\left(y_n,y^\star\right) = \sigma\left(y^\star,y^\star\right) = \lim_{n,m\to\infty}\sigma(y_n,y_m) = 0.\end{aligned}$$

Since, for all $n \geq 1$, we have $\sigma(x_n,y_{n-1}) = \sigma(A,B)$ and by condition (v), (X,σ) satisfies the property (G_C), by letting $n \to \infty$, we conclude that

$$\sigma\left(x^\star,y^\star\right) = \sigma(A,B).$$

On the other hand, since $y_n \in Tx_n$, we have, for all $n \geq 1$,

$$\begin{aligned}\sigma\left(y^\star,Tx^\star\right) &\leq s\sigma\left(y^\star,y_n\right) + s\sigma\left(y_n,Tx^\star\right) \leq s\sigma\left(y^\star,y_n\right) + sH_\sigma^b\left(Tx_n,Tx^\star\right)\\ &\leq s\sigma\left(y^\star,y_n\right) + s\alpha\sigma\left(x_n,x^\star\right).\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\sigma\left(y^\star,Tx^\star\right) \leq 0,$$

and so $\sigma(y^\star,Tx^\star) = 0$. By Lemma 2.1, we have $y^\star \in \overline{Tx^\star} = Tx^\star$. Also, we have

$$\sigma(A,B) \leq \sigma\left(x^\star,Tx^\star\right) \leq \sigma\left(x^\star,y^\star\right) = \sigma(A,B).$$

Thus, $x^\star$ is a best proximity point of T. Moreover, we have $\sigma(x^\star,x^\star) = 0$. □

The following example illustrates Theorem 3.2.

Example 3.3 Let $X = [0,\infty) \times [0,\infty)$. Consider the mapping $\sigma : X \times X \to [0,\infty)$ as follows:

$$\sigma\left((x_1,x_2),(y_1,y_2)\right) = \begin{cases}(|x_1-y_1| + |x_2-y_2|)^2 & \text{if } (x_1,x_2),(y_1,y_2) \in [0,10]^2,\\ (x_1+x_2+y_1+y_2)^2 & \text{if not.}\end{cases}$$

It is easy to see that (X,σ) a complete b-metric-like space with $s = 2$.

Take $A = \{1\} \times [0,10]$ and $B = \{0\} \times [0,10]$. Define the mapping $T : A \to C_b(B)$ by

$$T(1,x) = \begin{cases} \{(0,0),(0,\frac{x}{2})\} & \text{if } 0 \le x \le 8, \\ \{0\} \times [0,1] & \text{if } 8 < x \le 10. \end{cases}$$

Note that for all $(1,x) \in A$, we have $T(1,x)$ is closed and is bounded in (X,σ). Remark that $\sigma(A,B) = 1$, $A_0 = A$ and $B_0 = B$. So, for each $(1,x) \in A_0$, we have $T(1,x) \subseteq B_0$. Moreover, A and B are closed subsets of X. Consider the ball $B_\sigma(x_0,r)$ with $x_0 = (1,0)$ and $r = 82$. Now, let $(1,x_1),(1,x_2) \in A$ and $(0,y_1),(0,y_2) \in B$ such that

$$\begin{cases} \sigma((1,x_1),(0,y_1)) = \sigma(A,B) = 1, \\ \sigma((1,x_2),(0,y_2)) = \sigma(A,B) = 1. \end{cases}$$

Necessarily, $(x_1 = y_1 \in [0,10])$ and $(x_2 = y_2 \in [0,10])$. In this case,

$$\sigma\bigl((1,x_1),(1,x_2)\bigr) = \sigma\bigl((0,y_1),(0,y_2)\bigr),$$

that is, the pair (A,B) has the weak P-property.

Now, we shall show that T is a proximal contraction on $\overline{B}_\sigma(x_0,r)$ with $\alpha = \frac{1}{4}$.

It is easy to see that $\overline{B}_\sigma(x_0,r) \cap A = \{1\} \times [0,\sqrt{82}-2]$.

Let $(1,x)$ and $(1,y) \in \overline{B}_\sigma(x_0,r) \cap A$. Then $x,y \in [0,\sqrt{82}-2] \subseteq [0,8]$. In this case, we have

$$T(1,x) = \left\{(0,0),\left(0,\frac{x}{2}\right)\right\}, \qquad T(1,y) = \left\{(0,0),\left(0,\frac{y}{2}\right)\right\}.$$

Then

$$\begin{aligned} &\delta_\sigma\bigl(T(1,x),T(1,y)\bigr) \\ &\quad = \max\left\{\sigma\left((0,0),\left\{(0,0),\left(0,\frac{y}{2}\right)\right\}\right),\sigma\left(\left(0,\frac{x}{2}\right),\left\{(0,0),\left(0,\frac{y}{2}\right)\right\}\right)\right\} \\ &\quad = \min\left\{\frac{x^2}{4},\frac{(x-y)^2}{4}\right\} \le \frac{(x-y)^2}{4}. \end{aligned}$$

Similarly, we have

$$\delta_\sigma\bigl(T(1,y),T(1,x)\bigr) \le \frac{(x-y)^2}{4}.$$

This yields

$$\begin{aligned} H_\sigma^b\bigl(T(1,x),T(1,y)\bigr) &= \max\bigl\{\delta_\sigma\bigl(T(1,x),T(1,y)\bigr),\delta_\sigma\bigl(T(1,y),T(1,x)\bigr)\bigr\} \\ &\le \frac{(x-y)^2}{4} = \alpha\sigma\bigl((1,x),(1,y)\bigr). \end{aligned}$$

We also have $\delta_\sigma(Tx_0,\{x_0\}) + \sigma(A,B) = 2 \le \frac{1}{2s^3-s^2}(1-\sqrt{\alpha s})r$. Furthermore, (X,σ) satisfies the (G_C) property. In fact, let $\{(x_n,y_n)\}$, $\{(z_n,t_n)\}$ in X and $(x,y),(z,t) \in X$ such that

$$\lim_{n\to\infty} \sigma\bigl((x_n,y_n),(x,y)\bigr) = \lim_{n\to\infty} \sigma\bigl((z_n,t_n),(z,t)\bigr) = 0.$$

Then $\sigma((x,y),(x,y)) = \sigma((z,t),(z,t)) = 0$. It follows that $(x,y),(z,t) \in [0,10]^2$. There also exists $N \in \mathbb{N}$ such that $(x_n,y_n),(z_n,t_n) \subset [0,10]^2$ for all $n \geq N$. This yields, for all $n \geq N$,

$$\sigma\bigl((x_n,y_n),(x,y)\bigr) = \bigl(|x_n - x| + |y_n - y|\bigr)^2 \quad \text{and}$$
$$\sigma\bigl((z_n,t_n),(z,t)\bigr) = \bigl(|z_n - z| + |t_n - t|\bigr)^2.$$

So

$$\lim_{n\to\infty} |x_n - x| = \lim_{n\to\infty} |y_n - y| = \lim_{n\to\infty} |z_n - z| = \lim_{n\to\infty} |t_n - t| = 0.$$

Thus

$$\begin{aligned}\lim_{n\to\infty} \sigma\bigl((x_n,y_n),(z_n,t_n)\bigr) &= \lim_{n\to\infty}\bigl(|x_n - z_n| + |y_n - t_n|\bigr)^2 \\ &= \bigl(|x - z| + |y - t|\bigr)^2 = \sigma\bigl((x,y),(z,t)\bigr).\end{aligned}$$

Therefore, all conditions of Theorem 3.2 are verified. So, T has a best proximity point, which is $x^* = (1,0)$. It also verifies $\sigma(x^*,x^*) = 0$.

As consequences of our first result, we give the following immediate corollaries.

Corollary 3.4 *Let A and B be nonempty closed subsets of a complete metric-like space (X,σ) and $r > 0$. Let $T : A \to C_b(B)$ be a multi-valued mapping. Suppose that*
(i) $A_0 \neq \emptyset$;
(ii) *for each $x \in A_0$, we have $Tx \subseteq B_0$;*
(iii) *the pair (A,B) satisfies the weak P-property;*
(iv) *there exists $x_0 \in A_0$ such that T is a proximal contraction on $\overline{B}_\sigma(x_0,r)$ and $\delta_\sigma(Tx_0,\{x_0\}) + \sigma(A,B) \leq (1-\sqrt{\alpha})r$.*

Then T has a best proximity point in $\overline{B}_\sigma(x_0,r) \cap A$. We also have $\sigma(x^\star,x^\star) = 0$.

Proof It suffices to take $s = 1$ in Theorem 3.2. By Lemma 1.9, (X,σ) satisfies the property (G_C). □

Corollary 3.5 *Let A and B be nonempty closed subsets of a complete metric-like space (X,σ) and $r > 0$. Let $T : A \to B$ be a given mapping. Suppose that*
(i) $A_0 \neq \emptyset$;
(ii) *for each $x \in A_0$, we have $Tx \in B_0$;*
(iii) *the pair (A,B) satisfies the weak P-property;*
(iv) *there exists $x_0 \in A_0$ such that T is a proximal contraction on $\overline{B}_\sigma(x_0,r)$ and $\sigma(x_0,Tx_0) + \sigma(A,B) \leq \frac{1}{2s^3-s^2}(1-\sqrt{\alpha s})r$;*
(v) *(X,σ) satisfies the property (G_C).*

Then T has a best proximity point in $\overline{B}_\sigma(x_0,r) \cap A$. We also have $\sigma(x^\star,x^\star) = 0$.

Proof It suffices to take $s = 1$ and T as a single-valued mapping in Theorem 3.2. □

Corollary 3.6 *Let A and B be nonempty closed subsets of a complete metric space (X,d) and $r > 0$. Let $T : A \to C_b(B)$ be a multi-valued mapping. Suppose that*

(i) $A_0 \neq \emptyset$;
(ii) *for each* $x \in A_0$, *we have* $Tx \subseteq B_0$;
(iii) *the pair* (A, B) *satisfies the weak P-property*;
(iv) *there exists* $x_0 \in A_0$ *such that* T *is a proximal contraction on* $\overline{B}_d(x_0, r)$ *and* $\delta_d(Tx_0, \{x_0\}) + d(A, B) \leq (1 - \sqrt{\alpha})r$.

Then T *has a best proximity point in* $\overline{B}_d(x_0, r) \cap A$.

If we choose $A = B = X$, then we have the following fixed point theorem.

Corollary 3.7 *Let* (X, σ) *be a complete b-metric-like space*, $r > 0$, *and* $T : X \to C_b(X)$ *be a multi-valued mapping. Suppose there exist* $x_0 \in X$ *and* $\alpha \in (0, \frac{1}{s})$ *such that*

$$H^b_\sigma(Tx, Ty) \leq \alpha\sigma(x, y),$$

for all $x, y \in \overline{B}_\sigma(x_0, r)$ *and* $\delta_\sigma(Tx_0, \{x_0\}) \leq \frac{1}{2s^3 - s^2}(1 - \sqrt{\alpha s})r$. *Then* T *has a fixed point.*

Proof Following the proof of Theorem 3.2, we construct two sequences $\{x_n\} \subseteq \overline{B}_\sigma(x_0, r)$ and $\{y_n\} \subseteq X$ such that

$$\begin{cases} \sigma(x_n, y_{n-1}) = \sigma(X, X), \\ \sigma(x_n, x_{n+1}) \leq \sigma(y_{n-1}, y_n) \leq (\sqrt{\frac{\alpha}{s}})^n \sigma(x_0, x_1), \\ y_n \in Tx_n, \quad \text{for all } n = 1, 2, \ldots. \end{cases}$$

Moreover, there exist $x^\star \in \overline{B}_\sigma(x_0, r)$ and $y^\star \in X$ such that

$$\lim_{n\to\infty} \sigma\left(x_n, x^\star\right) = \sigma\left(x^\star, x^\star\right) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0 \quad \text{and}$$
$$\lim_{n\to\infty} \sigma\left(y_n, y^\star\right) = \sigma\left(y^\star, y^\star\right) = \lim_{n,m\to\infty} \sigma(y_n, y_m) = 0.$$

We have, for all $n \geq 1$,

$$\begin{aligned} \sigma\left(x^\star, y^\star\right) &\leq s\sigma\left(x^\star, x_n\right) + s\sigma\left(x_n, y^\star\right) \leq s\sigma\left(x^\star, x_n\right) + s^2\sigma(x_n, y_{n-1}) + s^2\sigma\left(y_{n-1}, y^\star\right) \\ &= s\sigma\left(x^\star, x_n\right) + s^2\sigma(A, B) + s^2\sigma\left(y_{n-1}, y^\star\right). \end{aligned}$$

Letting $n \to \infty$, we obtain

$$\sigma\left(x^\star, y^\star\right) \leq s\sigma\left(x^\star, x^\star\right) + s^2\sigma(X, X) + s^2\sigma\left(y^\star, y^\star\right) = s^2\sigma(X, X). \tag{3.13}$$

Also, for all $n \geq 1$,

$$\sigma(X, X) = \sigma(x_n, y_{n-1}) \leq s\sigma\left(x_n, x^\star\right) + s^2\sigma\left(x^\star, y^\star\right) + s^2\sigma\left(y^\star, y_{n-1}\right).$$

We pass to the limit $n \to \infty$,

$$\sigma(X, X) \leq s^2\sigma\left(x^\star, y^\star\right). \tag{3.14}$$

Combining (3.13) and (3.14), we get

$$s^{-2}\sigma(X,X) \le \sigma\big(x^\star, y^\star\big) \le s^2\sigma(X,X). \tag{3.15}$$

On the other hand, since $y_n \in Tx_n$, we have, for all $n \ge 1$,

$$\begin{aligned}\sigma\big(y^\star, Tx^\star\big) &\le s\sigma\big(y^\star, y_n\big) + s\sigma\big(y_n, Tx^\star\big) \le s\sigma\big(y^\star, y_n\big) + sH_\sigma^b\big(Tx_n, Tx^\star\big)\\ &\le s\sigma\big(y^\star, y_n\big) + s\alpha\sigma\big(x_n, x^\star\big).\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\sigma\big(y^\star, Tx^\star\big) \le 0,$$

and so $\sigma(y^\star, Tx^\star) = 0$. By Lemma 2.1, we have $y^\star \in \overline{Tx^\star} = Tx^\star$. Again

$$\sigma(X,X) \le \sigma\big(x^\star, Tx^\star\big) \le \sigma\big(x^\star, y^\star\big) \le s^2\sigma(X,X).$$

We also have $\sigma(x^\star, x^\star) = 0$. Thus, $\sigma(X,X) \le \sigma(x^\star, x^\star) = 0$, and so $\sigma(X,X) = 0$. It follows that $\sigma(x^\star, Tx^\star) = 0$. By Lemma 2.1, we get $x^\star \in \overline{Tx^\star} = Tx^\star$. Here, we do not need the conditions (i), (ii), (iii) and (v) of Theorem 3.2. □

3.2 Stability results

In this paragraph, we extend and generalize the stability results due to Kiran *et al.* [23] to b-metric-like spaces.

Let A and B be nonempty subsets of a b-metric-like space (X,σ) and $T : A \to C_b(B)$ be a multi-valued mapping. Take the set $B(T) = \{a \in A : \sigma(A,B) = \sigma(a, Ta)\}$. It corresponds to the set of best proximity points of T.

Theorem 3.8 *Let A and B be nonempty closed subsets of a complete b-metric-like space (X,σ) and $r_1, r_2 > 0$. Let $T_i : A \to C_b(B)$, $i = 1, 2$, be two multi-valued mappings. Suppose that*

(i) $A_0 \neq \emptyset$;
(ii) *for each $x \in A_0$, we have $T_i x \subseteq B_0$, $i = 1, 2$;*
(iii) *the pair (A,B) satisfies the weak P-property;*
(iv) *(X,σ) satisfies the property (G_C);*
(v) *for each $i = 1, 2$, there exists $a_i \in A_0$ such that T_i is a proximal contraction on $\overline{B}_\sigma(a_i, r) \cap A$ with the same Lipschitz constant $\alpha \in (0, \frac{1}{s})$, that is,*

$$H_\sigma^b(T_i x, T_i y) \le \alpha\sigma(x,y), \tag{3.16}$$

for all $x, y \in \overline{B}_\sigma(a_i, r) \cap A$ and $\delta_\sigma(T_i a_i, \{a_i\}) + \sigma(A,B) \le \frac{1}{2s^3 - s^2}(1 - \sqrt{\alpha s})r_i$.

Then

$$H_\sigma^b\big(B(T_1), B(T_2)\big) \le \frac{s^4}{1 - \sqrt{\alpha s}}\Big[\sup_{x \in A} H_\sigma^b(T_1 x, T_2 x) + \big(1 + s^{-1}\big)\sigma(A,B)\Big]. \tag{3.17}$$

Proof Let $\varepsilon > 0$ and $x_0 \in B(T_1)$, then there exists $z_0 \in T_1x_0$ such that

$$\sigma(x_0, z_0) \le \sigma(x_0, T_1x_0) + \varepsilon = \sigma(A,B) + \varepsilon. \tag{3.18}$$

By Lemma 2.4, there exists $y_0 \in T_2x_0$ such that

$$\sigma(z_0, y_0) \le H_\sigma^b(T_1x_0, T_2x_0) + \varepsilon. \tag{3.19}$$

Then, from (3.18) and (3.19), we get

$$\begin{aligned}\sigma(x_0, y_0) &\le s\bigl[\sigma(x_0, z_0) + \sigma(z_0, y_0)\bigr] \\ &\le s\bigl[H_\sigma^b(T_1x_0, T_2x_0) + \sigma(A,B) + 2\varepsilon\bigr].\end{aligned} \tag{3.20}$$

Since $y_0 \in T_2x_0 \subseteq B_0$, there exists $x_1 \in A_0$ such that

$$\sigma(x_1, y_0) = \sigma(A,B). \tag{3.21}$$

By Lemma 2.3, there exists $y_1 \in T_2x_1$ such that

$$\sigma(y_0, y_1) \le \frac{1}{\sqrt{\alpha s}} H_\sigma^b(T_2x_0, T_2x_1). \tag{3.22}$$

Without loss generality, we take $a_2 = x_0$ and $r_2 = r$ such that

$$\delta_\sigma\bigl(T_2x_0, \{x_0\}\bigr) + \sigma(A,B) \le \frac{1}{2s^3 - s^2}(1 - \sqrt{\alpha s})r.$$

As (3.3), we have

$$\begin{aligned}\bigl|\sigma(x_0, x_1) - \sigma(x_0, x_0)\bigr| &\le (2s-1)\sigma(x_0, x_1) \\ &\le \frac{(2s-1)}{2s^2 - s}(1 - \sqrt{\alpha s})r = s^{-1}(1 - \sqrt{\alpha s})r < r.\end{aligned}$$

Thus, $x_1 \in \overline{B}_\sigma(x_0, r) \cap A_0$. By Lemma 2.3, there exists $y_1 \in T_2x_1$ such that

$$\sigma(y_0, y_1) \le \frac{1}{\sqrt{\alpha s}} H_\sigma^b(T_2x_0, T_2x_1). \tag{3.23}$$

So, we get

$$\sigma(y_0, y_1) \le \sqrt{\frac{\alpha}{s}}\sigma(x_0, x_1). \tag{3.24}$$

Again, $y_1 \in T_2x_1 \subseteq B_0$, hence there exists $x_2 \in A_0$ such that

$$\sigma(x_2, y_1) = \sigma(A,B). \tag{3.25}$$

By condition (iii), it follows that

$$\sigma(x_1, x_2) \le \sigma(y_0, y_1). \tag{3.26}$$

Applying (3.24),

$$\sigma(x_1,x_2) \le \sqrt{\frac{\alpha}{s}}\sigma(x_0,x_1). \tag{3.27}$$

Repeating the same process and similar to the proof of Theorem 3.2, we construct two sequences $\{x_n\} \subseteq \overline{B}_\sigma(x_0,r) \cap A_0$ and $\{y_n\} \subseteq B_0$ such that

$$\begin{cases} \sigma(x_n,y_{n-1}) = \sigma(A,B), \\ \sigma(x_n,x_{n+1}) \le \sigma(y_{n-1},y_n) \le (\sqrt{\frac{\alpha}{s}})^n\sigma(x_0,x_1), \\ y_n \in T_2x_n, \quad \text{for all } n = 1,2,\ldots. \end{cases}$$

It follows that $\lim_{n,m\to\infty}\sigma(x_n,x_m) = 0$. Thus, $\{x_n\}$ is a Cauchy sequence in $\overline{B}_\sigma(x_0,r) \cap A$. A similar reasoning shows that $\lim_{n,m\to\infty}\sigma(y_n,y_m) = 0$ and so $\{y_n\}$ is a Cauchy sequence in B. Since $\overline{B}_\sigma(x_0,r) \cap A$ and B are closed subsets of a complete b-metric-like space (X,σ), there exist $u \in \overline{B}_\sigma(x_0,r) \cap A$ and $v \in B$ such that

$$\lim_{n\to\infty}\sigma(x_n,u) = \sigma(u,u) = \lim_{n,m\to\infty}\sigma(x_n,x_m) = 0 \quad \text{and}$$
$$\lim_{n\to\infty}\sigma(y_n,v) = \sigma(v,v) = \lim_{n,m\to\infty}\sigma(y_n,y_m) = 0.$$

Similarly, we have $u \in T_2u$ and $\sigma(A,B) = \sigma(u,T_2u)$. Thus, $u \in B(T_2)$.

On the other hand, for all $n \ge 1$

$$\begin{aligned} \sigma(x_0,u) &\le s\sigma(x_0,x_n) + s\sigma(x_n,u) \le s^2\sigma(x_0,x_1) + s^2\sigma(x_1,x_n) + s\sigma(x_n,u) \\ &\vdots \\ &\le s^2\sigma(x_0,x_1) + s^3\sigma(x_1,x_2) + \cdots + s^{n+1}\sigma(x_{n-1},x_n) + s\sigma(x_n,u) \\ &= s^2\sum_{k=0}^{n-1}s^k\sigma(x_k,x_{k+1}) + s\sigma(x_n,u) \\ &\le s^2\sum_{k=0}^{\infty}(\sqrt{s\alpha})^k\sigma(x_0,x_1) + s\sigma(x_n,u). \end{aligned}$$

Letting $n \to \infty$, we obtain

$$\sigma(x_0,u) \le s^2\sum_{k=0}^{\infty}(\sqrt{s\alpha})^k\sigma(x_0,x_1) = \frac{s^2}{1-\sqrt{s\alpha}}\sigma(x_0,x_1).$$

Thus, from (3.20),

$$\begin{aligned} \sigma(x_0,u) &\le \frac{s^3}{1-\sqrt{s\alpha}}\bigl[\sigma(x_0,y_0) + \sigma(y_0,x_1)\bigr] \\ &\le \frac{s^3}{1-\sqrt{s\alpha}}\bigl(s\bigl[H_\sigma^b(T_1x_0,T_2x_0) + \sigma(A,B) + 2\varepsilon\bigr] + \sigma(A,B)\bigr) \\ &= \frac{s^4}{1-\sqrt{s\alpha}}\bigl[H_\sigma^b(T_1x_0,T_2x_0) + \bigl(1+s^{-1}\bigr)\sigma(A,B) + 2\varepsilon\bigr]. \end{aligned}$$

Similarly, if $y_0 \in B(T_2)$, then there exists $u' \in B(T_1)$ such that

$$\sigma\big(y_0, u'\big) \le \frac{s^4}{1-\sqrt{s\alpha}}\big[H_\sigma^b(T_1 y_0, T_2 y_0) + \big(1+s^{-1}\big)\sigma(A,B) + 2\varepsilon\big].$$

Consequently, we obtain

$$H_\sigma^b\big(B(T_1), B(T_2)\big) \le \frac{s^4}{1-\sqrt{s\alpha}}\Big[\sup_{x\in A} H_\sigma^b(T_1 x, T_2 x) + \big(1+s^{-1}\big)\sigma(A,B) + 2\varepsilon\Big].$$

The real $\varepsilon > 0$ is arbitrary, so the proof is completed, that is, (3.17) is satisfied. □

We provide the following example.

Example 3.9 Let $X = [0,\infty) \times [0,\infty)$ be endowed with the b-metric-like $\sigma : X \times X \to [0,\infty)$ defined by

$$\sigma\big((x_1,x_2),(y_1,y_2)\big) = \begin{cases} (|x_1 - y_1| + |x_2 - y_2|)^2 & \text{if } (x_1,x_2),(y_1,y_2) \in [0,10]^2, \\ (x_1 + x_2 + y_1 + y_2)^2 & \text{if not.} \end{cases}$$

Take $A = \{1\} \times [0,10]$ and $B = \{0\} \times [0,10]$. Define the mapping $T_1, T_2 : A \to C_b(B)$ by

$$T_1(1,x) = \begin{cases} \{(0,0),(0,\frac{x}{2})\} & \text{if } 0 \le x \le 8, \\ \{0\} \times [0,1] & \text{if } 8 < x \le 10 \end{cases}$$

and

$$T_2(1,x) = \begin{cases} \{(0,0),(0,\frac{x+8}{2})\} & \text{if } 0 \le x \le 8, \\ \{0\} \times [0,5] & \text{if } 8 < x \le 10. \end{cases}$$

Note that $A_0 = A$ and $B_0 = B$. So, for each $x \in A_0$, we have $Tx \subseteq B_0$. Moreover, A and B are closed subsets of X. Consider the balls $B_\sigma(a_1,r_1)$, $B_\sigma(a_2,r_2)$ with $a_1 = (1,0)$, $a_2 = (1,0.2)$ and $r_1 = 82$, $r_2 = 84$. We know that the pair (A,B) has the weak P-property. Moreover, it is easy to prove that T_i is a proximal contraction on $\overline{B}_\sigma(a_i,r_i)$ for $i = 1,2$ with the same constant $\alpha = \frac{1}{4}$. We also have $\delta_\sigma(Ta_i,\{a_i\}) + \sigma(A,B) \le \frac{1}{2s^3-s^2}(1-\sqrt{\alpha s})r_i$, $i = 1,2$. Furthermore, (X,σ) satisfies the (G_C) property.

Therefore, all conditions of Theorem 3.8 are verified. So, we have

$$H_\sigma^b\big(B(T_1), B(T_2)\big) \le \frac{16\sqrt{2}}{\sqrt{2}-1}\Big[\sup_{x\in A} H_\sigma^b(T_1 x, T_2 x) + \frac{3}{2}\Big].$$

We derive the following interesting consequences from Theorem 3.8.

Corollary 3.10 *Let A and B be nonempty closed subsets of a complete metric-like space (X,σ) and $r_1, r_2 > 0$. Let $T_i : A \to C_b(B)$, $i = 1,2$, be two multi-valued mappings. Suppose that*

(i) $A_0 \neq \emptyset$;

(ii) *for each $x \in A_0$, we have $T_i x \subseteq B_0$, $i = 1,2$;*

(iii) *the pair (A,B) satisfies the weak P-property;*

(iv) *for each $i = 1,2$, there exists $a_i \in A_0$ such that T_i is a proximal contraction on $\overline{B}_\sigma(a_i, r) \cap A$ with the same Lipschitz constant $\alpha \in (0,1)$, that is,*

$$H_\sigma(T_i x, T_i y) \leq \alpha\sigma(x,y), \tag{3.28}$$

for all $x, y \in \overline{B}_\sigma(a_i, r) \cap A$ and $\delta_\sigma(T_i a_i, \{a_i\}) + \sigma(A,B) \leq (1 - \sqrt{\alpha})r_i$.

Then

$$H_\sigma\bigl(B(T_1), B(T_2)\bigr) \leq \frac{1}{1-\sqrt{\alpha}}\Bigl[\sup_{x\in A} H_\sigma(T_1x, T_2x) + 2\sigma(A,B)\Bigr]. \tag{3.29}$$

Proof It suffices to consider $s = 1$ in Theorem 3.8. □

Corollary 3.11 *Let (X,σ) be a complete b-metric-like space, $r_1, r_2 > 0$, and let $T_i : X \to C_b(X)$, $i = 1,2$, be two multi-valued mappings. Suppose there exist $\alpha \in (0, s^{-1})$ and $a_i \in X$ such that, for each $i = 1,2$, we have*

$$H^b_\sigma(T_i x, T_i y) \leq \alpha\sigma(x,y), \tag{3.30}$$

for all $x, y \in \overline{B}_\sigma(a_i, r)$ and $\delta_\sigma(T_i a_i, \{a_i\}) \leq \frac{1}{2s^3-s^2}(1-\sqrt{\alpha s})r_i$. Then

$$H^b_\sigma\bigl(F(T_1), F(T_2)\bigr) \leq \frac{s^4}{1-\sqrt{s\alpha}} \sup_{x\in A} H_\sigma(T_1x, T_2x), \tag{3.31}$$

where $F(T_i)$ is the set of fixed points of T_i, $i = 1,2$.

Proof It suffices to consider $A = B = X$ in Theorem 3.8. Here, we do not need the conditions (i), (ii), and (iii) of Theorem 3.8. □

Corollary 3.12 *Let A and B be nonempty closed subsets of a complete metric space (X,d) and $r_1, r_2 > 0$. Let $T_i : A \to C_b(B)$, $i = 1,2$, be two multi-valued mappings. Suppose that*

(i) *$A_0 \neq \emptyset$;*

(ii) *for each $x \in A_0$, we have $T_i x \subseteq B_0$, $i = 1,2$;*

(iii) *the pair (A,B) satisfies the weak P-property;*

(iv) *for each $i = 1,2$, there exists $a_i \in A_0$ such that T_i is a proximal contraction on $\overline{B}_d(a_i, r) \cap A$ with the same Lipschitz constant $\alpha \in (0,1)$, that is,*

$$H(T_i x, T_i y) \leq \alpha d(x,y), \tag{3.32}$$

for all $x, y \in \overline{B}_d(a_i, r) \cap A$ and $\delta_d(T_i a_i, \{a_i\}) + d(A,B) \leq (1 - \sqrt{\alpha})r_i$.

Then

$$H\bigl(B(T_1), B(T_2)\bigr) \leq \frac{1}{1-\sqrt{\alpha}}\Bigl[\sup_{x\in A} H(T_1x, T_2x) + 2d(A,B)\Bigr]. \tag{3.33}$$

Proof It suffices to consider σ as a metric in Corollary 3.10. □

4 Best proximity points and stability results on the class of partial b-metric spaces

In 2014, Shukla [32] introduced a generalized metric space called a partial b-metric space and established the Banach contraction principle as well as the Kannan type fixed point theorem in partial b-metric spaces.

Definition 4.1 [32] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $b : X \times X \to \mathbb{R}^+$ is called a partial b-metric on X if for all $x, y, z \in X$, the following conditions are satisfied:

(Pb1) $b(x,x) = b(x,y) = b(y,y)$, then $x = y$;

(Pb2) $b(x,x) \leq b(x,y)$;

(Pb3) $b(x,y) = b(y,x)$;

(Pb4) $b(x,z) + b(y,y) \leq s[b(x,y) + b(y,z)]$.

The pair (X,b) is then called a partial b-metric space.

Remark 4.2 Each partial b-metric space is a b-metric-like space, but the converse is not true.

Example 4.3 Let $X = [0,\infty)$. Consider the mapping $\sigma : X \times X \to [0,\infty)$ defined by $\sigma(x,y) = (x+y)^2$ for all $x, y \in X$. Then (X,σ) is a b-metric-like space with $s = 2$, but it is not a partial b-metric space since $\sigma(x,x) > \sigma(x,y)$ for all $x > y$.

Lemma 4.4 *Let (X,b) be a partial b-metric space. We have*

(1) *if $b(x,y) = 0$, then $x = y$,*

(2) *if $x \neq y$, then $b(x,y) > 0$.*

Remark 4.5 If b is a partial b-metric, then $B_b(x,\varepsilon) = \{y \in X : b(x,y) - b(x,x) < \varepsilon\}$.

Very recently, Felhi [33] introduced the concept of a partial Pompeiu-Hausdorff b-metric and he obtained some fixed point results.

Remark 4.6 If b is a partial b-metric, for simplicity we denote $H_b = H_b^b$ (defined as in (1.2)).

Following [33], we have the following lemmas.

Lemma 4.7 [33] *Let (X,b) be a partial b-metric space with coefficient $s \geq 1$. For $A \in C_b(X)$ ($C_b(X)$ is the set of bounded and closed subsets in the partial b-metric space) and $x \in X$, we have*

$$b(x,A) = b(x,x) \quad \text{if and only if} \quad x \in \bar{A} = A, \tag{4.1}$$

where $\bar{A}$ is the closure of A.

Lemma 4.8 [33] *Let (X,b) be a partial b-metric space with coefficient $s \geq 1$. For $A, B, C \in C_b(X)$, we have*

(i) $H_b(A,A) \leq H_b(A,B)$;

(ii) $H_b(A,B) = H_b(B,A)$;

(iii) $H_b(A,B) \leq s[H_b(A,C) + H_b(C,B)] - \inf_{c \in C} b(c,c)$.

4.1 Best proximity results

The main result of this paragraph is the analogous of Theorem 3.2 on the class of partial b-metric spaces. It is stated as follows.

Theorem 4.9 *Let A and B be nonempty closed subsets of a complete partial b-metric space (X,b) and $r>0$. Let $T:A\to C_b(B)$ be a multi-valued mapping. Suppose that*

(i) $A_0 \neq \emptyset$;
(ii) *for each $x\in A_0$, we have $Tx\subseteq B_0$*;
(iii) *the pair (A,B) satisfies the weak P-property*;
(iv) *there exists $x_0\in A_0$ such that T is a proximal contraction on $\overline{B}_b(x_0,r)$ and $\delta_b(Tx_0,\{x_0\})+b(A,B)\le s^{-2}(1-\sqrt{\alpha s})r$*;
(v) *(X,b) satisfies the property (G_C).*

Then T has a best proximity point in $\overline{B}_b(x_0,r)\cap A$. We also have $b(x^\star,x^\star)=0$.

Proof By assumption (iv), there exists $x_0\in A_0$ such that T is a proximal contraction on $\overline{B}_b(x_0,r)$ and $\delta_b(Tx_0,\{x_0\})+b(A,B)\le s^{-2}(1-\sqrt{\alpha s})r$.

Let $y_0\in Tx_0$. By condition (ii), we have $Tx_0\subseteq B_0$. Then there exists $x_1\in A_0$ such that

$$b(x_1,y_0)=b(A,B). \tag{4.2}$$

We have

$$\begin{aligned} b(x_0,x_1)-b(x_0,x_0) &\le b(x_0,x_1)\le s\bigl[b(x_0,y_0)+b(y_0,x_1)\bigr]-b(y_0,y_0)\\ &\le s\bigl[\delta_b\bigl(Tx_0,\{x_0\}\bigr)+b(A,B)\bigr]\\ &\le s\bigl[s^{-2}(1-\sqrt{\alpha s})r\bigr]=s^{-1}(1-\sqrt{\alpha s})r<r. \end{aligned} \tag{4.3}$$

Then $x_1\in\overline{B}_b(x_0,r)\cap A_0$. By Lemma 2.3, there exists $y_1\in Tx_1$ such that

$$b(y_0,y_1)\le\frac{1}{\sqrt{\alpha s}}H_b(Tx_0,Tx_1). \tag{4.4}$$

So, by (3.1), we get

$$b(y_0,y_1)\le\sqrt{\frac{\alpha}{s}}\,b(x_0,x_1). \tag{4.5}$$

Since $y_1\in Tx_1\subseteq B_0$, there exists $x_2\in A_0$ such that

$$b(x_2,y_1)=b(A,B). \tag{4.6}$$

By condition (iii), (4.2), and (4.6)

$$b(x_1,x_2)\le b(y_0,y_1). \tag{4.7}$$

The above inequality together with (4.7) implies that

$$b(x_1,x_2)\le\sqrt{\frac{\alpha}{s}}\,b(x_0,x_1). \tag{4.8}$$

Using (4.3), we have

$$\begin{aligned} b(x_0,x_2)-b(x_0,x_0) &\leq b(x_0,x_2) \leq sb(x_0,x_1)+sb(x_1,x_2)-b(x_1,x_1) \\ &\leq sb(x_0,x_1)+s^2 b(x_1,x_2) \leq s\left[1+s\sqrt{\frac{\alpha}{s}}\right]b(x_0,x_1) \\ &\leq s[1+\sqrt{\alpha s}]s^{-1}(1-\sqrt{\alpha s})r=(1-\alpha s)r<r. \end{aligned}$$

Then $x_2 \in \overline{B}_b(x_0,r)\cap A_0$. Again, by Lemma 2.3, there exists $y_2 \in Tx_2$ such that

$$b(y_1,y_2) \leq \frac{1}{\sqrt{\alpha s}}H_b(Tx_1,Tx_2). \tag{4.9}$$

So, by (3.1), we get

$$b(y_1,y_2) \leq \sqrt{\frac{\alpha}{s}}b(x_1,x_2). \tag{4.10}$$

Again, $y_2 \in Tx_2 \subseteq B_0$, so there exists $x_3 \in A_0$ such that

$$b(x_3,y_2)=b(A,B). \tag{4.11}$$

From condition (iii), (4.10), and (4.8)

$$b(x_2,x_3) \leq b(y_1,y_2) \leq \sqrt{\frac{\alpha}{s}}b(x_1,x_2) \leq \left(\sqrt{\frac{\alpha}{s}}\right)^2 b(x_0,x_1). \tag{4.12}$$

We have

$$\begin{aligned} b(x_0,x_3)-b(x_0,x_0) &\leq b(x_0,x_3) \leq sb(x_0,x_1)+s^2 b(x_1,x_2)+s^2 b(x_2,x_1) \\ &\leq sb(x_0,x_1)+s^2 b(x_1,x_2)+s^3 b(x_2,x_1) \\ &\leq s\left[1+s\sqrt{\frac{\alpha}{s}}+s^2\left(\sqrt{\frac{\alpha}{s}}\right)^2\right]b(x_0,x_1) \\ &\leq s\bigl[1+\sqrt{\alpha s}+(\sqrt{\alpha s})^2\bigr]s^{-1}(1-\sqrt{\alpha s})r=\bigl(1-(\sqrt{\alpha s})^3\bigr)r<r. \end{aligned}$$

Then $x_3 \in \overline{B}_b(x_0,r)\cap A_0$.

Continuing this process, we construct two sequences $\{x_n\} \subseteq \overline{B}_b(x_0,r)\cap A_0$ and $\{y_n\} \subseteq B_0$ such that

$$\begin{cases} b(x_n,y_{n-1})=b(A,B), \\ b(x_n,x_{n+1}) \leq b(y_{n-1},y_n) \leq (\sqrt{\frac{\alpha}{s}})^n b(x_0,x_1), \\ y_n \in Tx_n, \quad \text{for all } n=1,2,\ldots. \end{cases}$$

As in the proof of Theorem 3.2, there exist $x^\star \in \overline{B}_b(x_0,r)\cap A$ and $y^\star \in B$ such that

$$\begin{aligned} &\lim_{n\to\infty} b\bigl(x_n,x^\star\bigr)=b\bigl(x^\star,x^\star\bigr)=\lim_{n,m\to\infty} b(x_n,x_m)=0 \quad \text{and} \\ &\lim_{n\to\infty} b\bigl(y_n,y^\star\bigr)=b\bigl(y^\star,y^\star\bigr)=\lim_{n,m\to\infty} b(y_n,y_m)=0. \end{aligned}$$

By the same strategy, we see that $x^\star$ is a best proximity point of T and $b(x^\star,x^\star)=0$. □

As consequences, we may provide the following corollaries.

Corollary 4.10 *Let A and B be nonempty closed subsets of a complete partial b-metric space (X,b) and $r>0$. Let $T:A\to B$ be a given single-valued mapping. Suppose that*
- (i) $A_0\neq\emptyset$;
- (ii) *for each $x\in A_0$, we have $Tx\in B_0$;*
- (iii) *the pair (A,B) satisfies the weak P-property;*
- (iv) *there exists $x_0\in A_0$ such that T is a proximal contraction on $\overline{B}_b(x_0,r)$ and $b(x_0,Tx_0)+b(A,B)\le s^{-2}(1-\sqrt{\alpha s})r$;*
- (v) *(X,b) satisfies the property (G_C).*

Then T has a best proximity point in $\overline{B}_b(x_0,r)\cap A$. We also have $b(x^\star,x^\star)=0$.

In the setting of b-metric spaces, we have the following.

Corollary 4.11 *Let A and B be nonempty closed subsets of a complete b-metric space (X,d), $r>0$, and $T:A\to C_b(B)$ be a multi-valued mapping. Suppose that*
- (i) $A_0\neq\emptyset$;
- (ii) *for each $x\in A_0$, we have $Tx\subseteq B_0$;*
- (iii) *the pair (A,B) satisfies the weak P-property;*
- (iv) *there exists $x_0\in A_0$ such that T is a proximal contraction on $\overline{B}_d(x_0,r)$ and $\delta_d(Tx_0,\{x_0\})+d(A,B)\le s^{-2}(1-\sqrt{\alpha s})r$;*
- (v) *(X,d) satisfies the property (G_C).*

Then T has a best proximity point in $\overline{B}_d(x_0,r)\cap A$.

Corollary 4.12 *Let (X,d) be a complete b-metric space and $T:X\to C_b(X)$ be a multi-valued contractive non-self-mapping, that is,*

$$H(Tx,Ty)\le\alpha d(x,y),$$

for some $\alpha\in(0,\frac{1}{s})$ and for all $x,y\in\overline{B}_d(x_0,r)$ and $\delta_d(Tx_0,\{x_0\})\le s^{-2}(1-\sqrt{\alpha s})r$. Then T has a fixed point.

Corollary 4.13 ([2], Theorem 1) *Let (X,d) be a complete metric space and $T:X\to C_b(X)$ be such that*

$$H(Tx,Ty)\le\alpha d(x,y),$$

for some $\alpha\in(0,1)$ and for all $x,y\in X$. Then T has a fixed point.

Corollary 4.14 ([26], Theorem 2.1) *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that $A_0\neq\emptyset$ and (A,B) satisfies the P-property. Let $T:A\to 2^B$ be a multi-valued contraction non-self-mapping, that is,*

$$H(Tx,Ty)\le\alpha d(x,y),$$

for some $\alpha\in(0,1)$ and for all $x,y\in A$. If $T(x)$ is bounded and is closed in B for all $x\in A$, and $T(x_0)\subseteq B_0$ for each $x_0\in A$, then T has a best proximity point in A.

4.2 Stability results

As Theorem 3.8, we state the following stability result.

Theorem 4.15 *Let A and B be nonempty closed subsets of a complete partial b-metric space (X,b) and $r_1, r_2 > 0$. Let $T_i : A \to C_b(B)$ with $i = 1,2$, be two multi-valued mappings. Suppose that*

(i) $A_0 \neq \emptyset$;
(ii) *for each $x \in A_0$, we have $T_i x \subseteq B_0$, $i = 1,2$;*
(iii) *the pair (A,B) satisfies the weak P-property;*
(iv) *(X,b) satisfies the property (G_C);*
(v) *for each $i = 1,2$, there exists $a_i \in A_0$ such that T_i is a proximal contraction on $\overline{B}_b(a_i,r) \cap A$ with the same Lipschitz constant $\alpha \in (0,\frac{1}{s})$, that is,*

$$H_b(T_i x, T_i y) \leq \alpha b(x,y), \tag{4.13}$$

for all $x,y \in \overline{B}_b(a_i,r) \cap A$ and $\delta_b(T_i a_i, \{a_i\}) + b(A,B) \leq s^{-2}(1-\sqrt{\alpha s})r_i$.

Then

$$H_b\bigl(B(T_1),B(T_2)\bigr) \leq \frac{s^4}{1-\sqrt{\alpha s}}\Bigl[\sup_{x\in A} H_b(T_1 x, T_2 x) + \bigl(1+s^{-1}\bigr)b(A,B)\Bigr]. \tag{4.14}$$

Proof The proof is similar to that of Theorem 3.8. □

Corollary 4.16 *Let (X,d) be a complete b-metric space. Take $r_1, r_2 > 0$. Let $T_i : X \to C_b(X)$, $i = 1,2$, be two multi-valued mappings. Suppose there exist $\alpha \in (0,s^{-1})$ and $a_i \in X$ such that, for each $i = 1,2$,*

$$H_b(T_i x, T_i y) \leq \alpha d(x,y), \tag{4.15}$$

for all $x,y \in \overline{B}_d(a_i,r)$ and $\delta_d(T_i a_i, \{a_i\}) \leq s^{-2}(1-\sqrt{\alpha s})r_i$. Then

$$H_b\bigl(F(T_1),F(T_2)\bigr) \leq \frac{s^4}{1-\sqrt{s\alpha}}\sup_{x\in A} H_b(T_1 x, T_2 x). \tag{4.16}$$

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The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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