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Discussion of several contractions by Jachymski's approach

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Abstract

We discuss several contractions of integral type by using Jachymski's approach. We give alternative proofs of recent generalizations of the Banach contraction principle due to Ri (Indag. Math. 27:85-93, 2016) and Wardowski (Fixed Point Theory Appl. 2012:94, 2012).

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Keywords: the Banach contraction principle; Boyd-Wong contraction; Meir-Keeler contraction; Matkowski contraction; contraction of integral type; fixed point

1 Introduction

The Banach contraction principle [3, 4] is an elegant, forceful tool in nonlinear analysis and has many generalizations. See, *e.g.*, [5–10]. For example, Boyd and Wong in [11] proved the following.

Theorem 1 (Boyd and Wong [11]) *Let (X,d) be a complete metric space and let T be a mapping on X. Assume that T is a Boyd-Wong contraction, that is, there exists a function φ from $[0,\infty)$ into itself satisfying the following:*

(i) *φ is upper semicontinuous from the right.*
(ii) *$\varphi(t) < t$ holds for any $t \in (0,\infty)$.*
(iii) *$d(Tx,Ty) \le \varphi \circ d(x,y)$ for any $x,y \in X$.*

Then T has a unique fixed point.

Branciari in [12] introduced contractions of integral type as follows: A mapping T on a metric space (X,d) is a *Branciari contraction* if there exist $r \in [0,1)$ and a locally integrable function f from $[0,\infty)$ into itself such that

$$\int_0^s f(t)\,dt > 0 \quad \text{and} \quad \int_0^{d(Tx,Ty)} f(t)\,dt \le r \int_0^{d(x,y)} f(t)\,dt$$

for all $s > 0$ and $x,y \in X$. We have studied contractions of integral type in [13–15].

In this paper, we discuss several contractions of integral type by using Jachymski's approach. As applications, we give alternative proofs of recent generalizations of the Banach contraction principle due to Ri [1] and Wardowski [2].

2 Preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

Let f be a function from a subset Q of $\mathbb{R}$ into $\mathbb{R}$. Then f is said to satisfy $(\mathrm{UR})_f$ if the following holds:

$(\mathrm{UR})_f$ For any $t \in Q$, there exist $\delta > 0$ and $\varepsilon > 0$ such that $f(s) \le t - \varepsilon$ holds for any $s \in [t, t+\delta) \cap Q$.

We give some lemmas concerning (UR).

Lemma 2 *Let f be a function from a subset Q of $\mathbb{R}$ into $\mathbb{R}$. Then the following are equivalent*:

(i) *f satisfies* $(\mathrm{UR})_f$.
(ii) $\limsup[f(u) : u \to t, u \in Q, t \le u] < t$ *holds for any* $t \in Q$.
(iii) $\limsup[f(u) : u \to t, u \in Q, t < u] < t$ *and* $f(t) < t$ *hold for any* $t \in Q$.

Proof Obvious. □

Lemma 3 *Let f be a function from a subset Q of $\mathbb{R}$ into $\mathbb{R}$ such that $f(t) < t$ for any $t \in Q$. Assume that f is upper semicontinuous from the right. Then f satisfies* $(\mathrm{UR})_f$.

Proof Obvious. □

Lemma 4 *Let f be a function from a subset Q of $\mathbb{R}$ into $\mathbb{R}$ satisfying* $(\mathrm{UR})_f$. *Define a function g from Q into $\mathbb{R}$ by*

$$g(t) = \limsup\bigl[f(u) : u \to t, u \in Q, t \le u\bigr]$$

for $t \in Q$. Define a mapping L from Q into the power set of $\mathbb{R}$, a function ℓ from Q into $[-\infty, \infty)$ and a function h from Q into $\mathbb{R}$ by

$$L(t) = \bigl\{s \in Q : s \le t, \limsup\bigl[g(u) : u \to s, u \in Q, u \le s\bigr] = s\bigr\},$$

$$\ell(t) = \begin{cases} \sup L(t) & \text{if } L(t) \ne \varnothing, \\ -\infty & \text{if } L(t) = \varnothing, \end{cases} \quad \text{and}$$

$$h(t) = \sup\bigl\{g(s) : s \in Q, \ell(t) \le s \le t\bigr\}$$

for $t \in Q$. Define a function φ from Q into $\mathbb{R}$ by

$$\varphi(t) = \frac{h(t) + t}{2}$$

for $t \in Q$. Then the following hold:

(i) *g is upper semicontinuous from the right.*
(ii) *h and φ are right continuous.*
(iii) $f(t) \le g(t) \le h(t) < \varphi(t) < t$ *holds for any* $t \in Q$.

Proof Since f satisfies $(\mathrm{UR})_f$, we have $f(t) \le g(t) < t$ for any $t \in Q$. In order to show (i), we fix $t \in Q$ and let $\{t_n\}$ be a strictly decreasing sequence in Q converging to t. Fix $\varepsilon > 0$. Then for every $n \in \mathbb{N}$, there exists $s_n \in Q$ satisfying $t_n \le s_n \le t_n + 1/n$ and $g(t_n) \le f(s_n) + \varepsilon$. Since $\{s_n\}$ converges to t, we have

$$\limsup_{n\to\infty} g(t_n) \le \limsup_{n\to\infty} f(s_n) + \varepsilon \le g(t) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\limsup_n g(t_n) \le g(t)$. Therefore we have shown (i). We shall show $h(t) < t$ for any $t \in Q$. Arguing by contradiction, we assume $h(t) \ge t$ for some $t \in Q$. Then since $g(t) < t$, there exists a strictly increasing sequence $\{s_n\}$ such that $\lim_n s_n = t$ and $\lim_n g(s_n) = h(t)$. Since $g(s_n) < s_n$ for $n \in \mathbb{N}$, we have $h(t) = t$. Therefore $t \in L(t)$, which implies $h(t) = g(t) < t$. This is a contradiction. So $h(t) < t$ holds. It is obvious that $h(t) < \varphi(t) < t$ for any $t \in Q$. Therefore we have shown (iii). In order to show (ii), we fix $t \in Q$ and $\varepsilon > 0$ with $h(t) + \varepsilon < t$. From (i), there exists $\delta > 0$ such that

$$g(s) \le g(t) + \varepsilon \le h(t) + \varepsilon < t$$

for $s \in (t, t + \delta) \cap Q$. Let $\{t_n\}$ be a strictly decreasing sequence $\{t_n\}$ in Q such that $t_1 < t + \delta$ and $\{t_n\}$ converges to t. Then we note $\ell(t) = \ell(t_n)$ for $n \in \mathbb{N}$. So we have

$$\begin{aligned} h(t) &\le h(t_n) \\ &= \max\bigl\{h(t), \sup\bigl\{g(s) : s \in Q, t < s \le t_n\bigr\}\bigr\} \\ &\le \max\bigl\{h(t), g(t) + \varepsilon\bigr\} \\ &\le h(t) + \varepsilon \end{aligned}$$

for $n \in \mathbb{N}$. Hence

$$h(t) \le \liminf_{n\to\infty} h(t_n) \le \limsup_{n\to\infty} h(t_n) \le h(t) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\lim_n h(t_n) = h(t)$. Thus, h is right continuous. It is obvious that φ is also right continuous. We have shown (ii). □

Remark See Theorem 2 in [7]. Note that the domain of h is Q. We cannot extend the domain of h to $\bigcup[[t, \infty) : t \in Q]$, considering the function f from $(-\infty, 0) \cup (0, \infty)$ into $\mathbb{R}$ defined by

$$f(t) = \begin{cases} -2t & \text{if } t < 0, \\ t/2 & \text{if } t > 0. \end{cases}$$

3 Definitions

We list the following notation in order to simplify the statement of the results of this paper:

(A1) Let D be a subset of $(0, \infty)^2$.

(A2) Let θ be a function from $(0,\infty)$ into $\mathbb{R}$. Put $\Theta = \theta((0,\infty))$ and

$$\Theta_{\leq} = \bigcup \bigl[[t,\infty) : t \in \Theta\bigr].$$

Jachymski in [8] discussed several contractions by using subsets of $[0,\infty)^2$. Since this approach seems to be very reasonable for considering future studies, we use an approach similar to Jachymski's.

Definition 5 Assume (A1).

(1) D is said to be *contractive* (*Cont* for short) [3, 4] if there exists $r \in (0,1)$ such that $u \leq rt$ holds for any $(t,u) \in D$.

(2) D is said to be a *Browder* (*Bro*, for short) [16] if there exists a function φ from $(0,\infty)$ into itself satisfying the following:
 - (2-i) φ is nondecreasing and right continuous.
 - (2-ii) $\varphi(t) < t$ holds for any $t \in (0,\infty)$.
 - (2-iii) $u \leq \varphi(t)$ holds for any $(t,u) \in D$.

(3) D is said to be *Boyd-Wong* (*BW* for short) [11] if there exists a function φ from $(0,\infty)$ into itself satisfying the following:
 - (3-i) φ is upper semicontinuous from the right.
 - (3-ii) $\varphi(t) < t$ holds for any $t \in (0,\infty)$.
 - (3-iii) $u \leq \varphi(t)$ holds for any $(t,u) \in D$.

(4) D is said to be *Meir-Keeler* (*MK* for short) [17] if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $u < \varepsilon$ holds for any $(t,u) \in D$ with $t < \varepsilon + \delta$; see also [18–20].

(5) D is said to be *Matkowski* (*Mat* for short) [21] if there exists a function φ from $(0,\infty)$ into itself satisfying the following:
 - (5-i) φ is nondecreasing.
 - (5-ii) $\lim_n \varphi^n(t) = 0$ for every $t \in (0,\infty)$.
 - (5-iii) $u \leq \varphi(t)$ holds for any $(t,u) \in D$.

(6) D is said to be *CJM* [6, 22–24] if the following hold:
 - (6-i) For any $\varepsilon > 0$, there exists $\delta > 0$ satisfying $u \leq \varepsilon$ holds for any $(t,u) \in D$ with $t < \varepsilon + \delta$.
 - (6-ii) $u < t$ holds for any $(t,u) \in D$.

Remark We know the following implications; see, *e.g.*, [5, 7, 10].

- Cont $\Rightarrow$ Bro $\Rightarrow$ BW $\Rightarrow$ MK $\Rightarrow$ CJM;
- Cont $\Rightarrow$ Bro $\Rightarrow$ Mat $\Rightarrow$ CJM.

We give one proposition on the concept of Boyd-Wong. Note that we can easily obtain similar results on the other concepts.

Proposition 6 *Let T be a mapping on a metric space (X,d) and define a subset D of $(0,\infty)^2$ by*

$$D = \bigl\{\bigl(d(x,y), d(Tx,Ty)\bigr) : x,y \in X\bigr\} \cap (0,\infty)^2. \tag{1}$$

Then T is a Boyd-Wong contraction iff D is Boyd-Wong.

Proof We first note

$$\begin{aligned} D &= \{ (d(x,y), d(Tx,Ty)) : x,y \in X, x \neq y, Tx \neq Ty \} \\ &= \{ (d(x,y), d(Tx,Ty)) : x,y \in X, Tx \neq Ty \} \end{aligned}$$

because $Tx \neq Ty$ implies $x \neq y$. We assume that D is Boyd-Wong. Then there exists φ satisfying (3-i)-(3-iii) in Definition 5. Define a function η from $[0,\infty)$ into itself by $\eta(0) = 0$ and $\eta(t) = \varphi(t)$ for $t \in (0,\infty)$. Then we have $(\mathrm{i})_\eta$ and $(\mathrm{ii})_\eta$ in Theorem 1. If either $x = y$ or $Tx = Ty$ holds, then $d(Tx,Ty) \leq \eta \circ d(x,y)$ obviously holds. Considering this fact, we have $(\mathrm{iii})_\eta$ in Theorem 1. Therefore T is a Boyd-Wong contraction. Conversely, we next assume that T is a Boyd-Wong contraction. Then there exists η satisfying $(\mathrm{i})_\eta$-$(\mathrm{iii})_\eta$ in Theorem 1. Define a function φ from $(0,\infty)$ into itself by

$$\varphi(t) = \max\{\eta(t), t/2\}$$

for any $t \in (0,\infty)$. Then φ satisfies (3-i) and (3-ii) in Definition 5. We also have

$$d(Tx,Ty) \leq \eta \circ d(x,y) \leq \varphi \circ d(x,y)$$

for any $x,y \in X$ with $Tx \neq Ty$. So (3-iii) holds. Therefore D is Boyd-Wong. □

The following are variants of Corollaries 9 and 14 in [14].

Proposition 7 ([14]) *Assume* (A1), (A2) *and the following*:

(i) *θ is nondecreasing and continuous.*
(ii) *There exists an upper semicontinuous function ψ from Θ into $\mathbb{R}$ satisfying $\psi(\tau) < \tau$ for any $\tau \in \Theta$ and $\theta(u) \leq \psi \circ \theta(t)$ for any $(t,u) \in D$.*

Then D is Browder.

Proposition 8 ([14]) *Assume* (A1), (A2), *and the following*:

(i) *θ is nondecreasing.*
(ii) *There exists an upper semicontinuous function ψ from $\Theta_\leq$ into $\mathbb{R}$ satisfying $\psi(\tau) < \tau$ for any $\tau \in \Theta_\leq$ and $\theta(u) \leq \psi \circ \theta(t)$ for any $(t,u) \in D$.*

Then D is CJM.

Remark From the proof in [14], we can weaken (ii) of Proposition 8 to the following:

(ii)′ There exists a function ψ from $\Theta_\leq$ into $\mathbb{R}$ such that ψ is upper semicontinuous from the right, $\psi(\tau) < \tau$ for any $\tau \in \Theta_\leq$ and $\theta(u) \leq \psi \circ \theta(t)$ for any $(t,u) \in D$.

4 Main results

In this section, we prove our main results. We begin with Boyd-Wong.

Proposition 9 *Assume* (A1), (A2), *and the following*:

(i) *θ is nondecreasing and continuous.*
(ii) *There exists a function ψ from Θ into $\mathbb{R}$ satisfying* $(\mathrm{UR})_\psi$ *and $\theta(u) \leq \psi \circ \theta(t)$ for any $(t,u) \in D$.*

Then D is Boyd-Wong.

Proof Define a function θ_+^{-1} from $\mathbb{R}$ into $[0,\infty]$ by

$$\theta_+^{-1}(\tau) = \begin{cases} \sup\{s \in (0,\infty) : \theta(s) \le \tau\} & \text{if } \{s \in (0,\infty) : \theta(s) \le \tau\} \ne \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

We also define a function η from $(0,\infty)$ into $[0,\infty)$ by $\eta = \theta_+^{-1} \circ \psi \circ \theta$. We note

$$\eta(t) = \sup\bigl\{s \in (0,\infty) : \theta(s) \le \psi \circ \theta(t)\bigr\} \quad \text{provided } \eta(t) > 0.$$

Since $\psi(\tau) < \tau$ for any $\tau \in \Theta$, we have $\psi \circ \theta(t) < \theta(t) \le \theta(s)$ for any $t, s \in (0,\infty)$ with $t \le s$. Hence $\eta(t) \le t$ holds for any $t \in (0,\infty)$. Arguing by contradiction, we assume that $(\mathrm{UR})_\eta$ does not hold. Then there exist $t \in (0,\infty)$ and a sequence $\{t_n\}$ in $[t,\infty)$ such that $\{t_n\}$ converges to t and

$$\eta(t_n) > (1 - 1/n)t$$

holds for $n \in \mathbb{N}$. Since $\eta(t_n) > 0$,

$$\sup\bigl\{s \in (0,\infty) : \theta(s) \le \psi \circ \theta(t_n)\bigr\} = \eta(t_n) > (1 - 1/n)t$$

holds. Hence there exists a sequence $\{u_n\}$ in $(0,\infty)$ satisfying

$$\theta(u_n) \le \psi \circ \theta(t_n) < \theta(t_n) \quad \text{and} \quad u_n > (1 - 2/n)t$$

for $n \in \mathbb{N}$. Since θ is nondecreasing, $u_n < t_n$ holds for any $n \in \mathbb{N}$. Thus $\{u_n\}$ also converges to t. Hence by the continuity of θ,

$$\theta(t) \le \limsup_{n\to\infty} \psi \circ \theta(t_n) \le \limsup\bigl[\psi(\tau) : \tau \to \theta(t), \tau \ge \theta(t), \tau \in \Theta\bigr].$$

This contradicts $(\mathrm{UR})_\psi$. Therefore $(\mathrm{UR})_\eta$ holds. For any $(t,u) \in D$, since $\theta(u) \le \psi \circ \theta(t)$, we have

$$u \le \theta_+^{-1} \circ \theta(u) \le \theta_+^{-1} \circ \psi \circ \theta(t) = \eta(t).$$

By Lemma 4, there exists a right continuous function φ from $(0,\infty)$ into itself satisfying $\eta(t) < \varphi(t) < t$. It is obvious that $u \le \eta(t) < \varphi(t)$ for any $(t,u) \in D$. Therefore D is Boyd-Wong. □

Remark There appears θ_+^{-1} in Proposition 2.1 in [15].

We next discuss Meir-Keeler.

Proposition 10 *Assume* (A1), (A2), *and the following*:

(i) θ *is nondecreasing and right continuous.*

(ii) *For any* $\varepsilon \in \Theta$, *there exists* $\delta > 0$ *such that* $\theta(t) < \varepsilon + \delta$ *implies* $\theta(u) < \varepsilon$ *for any* $(t,u) \in D$.

Then D is Meir-Keeler.

Proof Fix $\varepsilon > 0$. Then from (ii), there exists $\alpha > 0$ such that

$$\theta(t) < \theta(\varepsilon) + \alpha \quad \text{implies} \quad \theta(u) < \theta(\varepsilon)$$

for any $(t,u) \in D$. From the right continuity of θ, there exists $\delta > 0$ such that $\theta(\varepsilon + \delta) < \theta(\varepsilon) + \alpha$. Fix $(t,u) \in D$ with $t < \varepsilon + \delta$. Then we have

$$\theta(t) \le \theta(\varepsilon + \delta) < \theta(\varepsilon) + \alpha$$

and hence $\theta(u) < \theta(\varepsilon)$. Therefore $u < \varepsilon$ holds. So D is Meir-Keeler. □

We obtain the following, which is a generalization of Corollary 17 in [14].

Corollary 11 *Assume* (A1), (A2), (i) *of Proposition* 10, *and* (ii) *of Proposition* 9. *Then D is Meir-Keeler.*

Let us discuss Matkowski.

Proposition 12 *Assume* (A1), (A2), *and the following*:
- (i) *θ is nondecreasing and left continuous.*
- (ii) $\min \Theta$ *does not exist.*
- (iii) *There exist a subset Q of $\mathbb{R}$ and a nondecreasing function ψ from Q into Q satisfying* $\Theta \subset Q \subset \Theta_{\le}$,

$$\lim_{n \to \infty} \psi^n(\tau) = \inf \Theta$$

for any $\tau \in Q$ and $\theta(u) \le \psi \circ \theta(t)$ for any $(t,u) \in D$.

Then D is Matkowski.

Proof We first note that $\inf \Theta = \inf Q = \inf \Theta_{\le}$ holds and neither $\min \Theta$, $\min Q$ nor $\min \Theta_{\le}$ does exist. So, from (ii) and (iii), $\psi(\tau) < \tau$ holds for any $\tau \in Q$. Define a function θ_+^{-1} from Q into $(0,\infty]$ by

$$\theta_+^{-1}(\tau) = \sup\{s \in (0,\infty) : \theta(s) \le \tau\}.$$

Since θ is left continuous, we have $\tau < \theta(t)$ implies $\theta_+^{-1}(\tau) < t$. We also have

$$\theta_+^{-1}(\tau) = \max\{s \in (0,\infty) : \theta(s) \le \tau\}$$

provided $\tau < \sup \Theta$. Hence $\theta \circ \theta_+^{-1}(\tau) \le \tau$ provided $\tau < \sup \Theta$. It is obvious that θ_+^{-1} is nondecreasing. Define a function φ from $(0,\infty)$ into itself by $\varphi = \theta_+^{-1} \circ \psi \circ \theta$. Then for any $t \in (0,\infty)$, since $\psi \circ \theta(t) < \theta(t)$, we have $\varphi(t) < t$. Since θ, ψ, and θ_+^{-1} are nondecreasing, φ is also nondecreasing. Noting $\psi \circ \theta(t) < \theta(t) \le \sup \Theta$, we have

$$\varphi^2(t) = \theta_+^{-1} \circ \psi \circ \theta \circ \theta_+^{-1} \circ \psi \circ \theta(t) \le \theta_+^{-1} \circ \psi^2 \circ \theta(t).$$

Continuing this argument, we can prove $\varphi^n(t) \leq \theta_+^{-1} \circ \psi^n \circ \theta(t)$ by induction. Since $\lim_n \psi^n \circ \theta(t) = \inf \Theta$, we have $\lim_n \theta_+^{-1} \circ \psi^n \circ \theta(t) = 0$ from (ii). Therefore we obtain

$$\lim_{n\to\infty} \varphi^n(t) \leq \lim_{n\to\infty} \theta_+^{-1} \circ \psi^n \circ \theta(t) = 0$$

for any $t \in (0,\infty)$. Since $u \leq \theta_+^{-1} \circ \theta(u) \leq \theta_+^{-1} \circ \psi \circ \theta(t)$, we obtain $u \leq \varphi(t)$ for any $(t,u) \in D$. Therefore D is Matkowski. □

5 Counterexamples

In this section, we give counterexamples connected with the results in Section 4.

Example 13 (Example 2.3 in [15], Example 10 in [14]) Define a complete metric space (X,d) by

$$X = [0,1] \cup [2,\infty) \quad \text{and} \quad d(x,y) = \begin{cases} \min\{x+y,2\} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Define a mapping T on X and functions θ and ψ from $(0,\infty)$ into itself by

$$Tx = \begin{cases} 0 & \text{if } x \leq 1, \\ 1 - 1/x & \text{if } x \geq 2, \end{cases} \qquad \theta(t) = \begin{cases} t/2 & \text{if } t < 2, \\ 2 & \text{if } t \geq 2, \end{cases}$$

and $\psi(t) = t/2$. Define D by (1). Then all the assumptions of Propositions 9 and 12 except the left continuity of θ are satisfied. However, D is neither Boyd-Wong nor Matkowski.

Remark By Corollary 11, D is Meir-Keeler. We define E by

$$E = \{(\theta \circ d(x,y), \theta \circ d(Tx,Ty)) : x,y \in X\} \cap (0,\infty)^2. \tag{2}$$

Then $E \subset \{2\} \times (1/4,1)$ holds. Hence E is contractive.

Proof We have

$$\begin{aligned} D &\supset \{(d(x,y), d(Tx,Ty)) : x,y \geq 2, x \neq y\} \\ &= \{(2, 2 - 1/x - 1/y) : x,y \geq 2, x \neq y\} \\ &= \{2\} \times (1,2). \end{aligned}$$

Hence D is neither Boyd-Wong nor Matkowski. □

Example 14 (Example 2.6 in [13], Example 11 in [14]) Define a complete metric space (X,d) by $X = [0,\infty)$ and $d(x,y) = x+y$ for $x,y \in X$ with $x \neq y$. Define a mapping T on X and functions θ and ψ from $(0,\infty)$ into itself by

$$Tx = \begin{cases} 0 & \text{if } x \leq 1, \\ 1 & \text{if } x > 1, \end{cases} \qquad \theta(t) = \begin{cases} t & \text{if } t \leq 1, \\ 2 & \text{if } t > 1, \end{cases}$$

and $\psi(t) = t/2$. Define D by (1). Then all the assumptions of Proposition 10 except the right continuity of θ are satisfied. However, D is not Meir-Keeler. Therefore D is not Boyd-Wong.

Remark By Proposition 12, D is Matkowski. We define E by (2). Then $E = \{(2,1)\}$ holds. Hence E is contractive.

Proof We have

$$\begin{aligned} D &\supset \bigl\{\bigl(d(0,y), d(T0,Ty)\bigr) : y > 1\bigr\} \\ &= \bigl\{(y,1) : y > 1\bigr\} = (1,\infty) \times \{1\}. \end{aligned}$$

Hence D is not Meir-Keeler. □

Example 15 Define a complete metric space (X,d) by $X = \{0,1\}$ and $d(0,1) = 1$. Define a mapping T on X and functions θ and ψ from $(0,\infty)$ into itself by

$$Tx = 1 - x \quad \text{and} \quad \theta(t) = \psi(t) = 1.$$

Define D by (1). Then all the assumptions of Proposition 12 except (ii) are satisfied. However, D is not Matkowski.

Proof Obvious. □

6 Applications

In this section, as applications, we give alternative proofs of some recent generalizations of the Banach contraction principle. Ri in [1] proved the following fixed point theorem.

Theorem 16 (Ri [1]) *Let (X,d) be a complete metric space and let T be a mapping on X. Assume there exists a function ψ from $[0,\infty)$ into itself satisfying the following*:

(R1) $\psi(t) < t$ *for any* $t \in (0,\infty)$.

(R2) $\limsup_{s\to t+0} \psi(s) < t$ *for any* $t \in (0,\infty)$.

(R3) $d(Tx,Ty) \le \psi(d(x,y))$ *for any* $x, y \in X$.

Then T has a unique fixed point.

We give an alternative proof of Theorem 16 by showing that a mapping T in Theorem 16 is a Boyd-Wong contraction.

Proof of Theorem 16 By Lemma 2, the restriction ψ to $(0,\infty)$ satisfies $(\mathrm{UR})_\psi$. Then by Lemma 4, there exists a right continuous function φ from $(0,\infty)$ into itself satisfying $\psi(t) < \varphi(t) < t$ for $t \in (0,\infty)$. Thus T is a Boyd-Wong contraction. So T has a unique fixed point. □

Wardowski in [2] proved a fixed point theorem on F-contraction.

Theorem 17 (Wardowski [2]) *Let (X,d) be a complete metric space and let T be a F-contraction on X, that is, there exist a function F from $(0,\infty)$ into $\mathbb{R}$ and real numbers $\eta \in (0,\infty)$ and $k \in (0,1)$ satisfying the following*:

(F1) *F is strictly increasing.*
(F2) *For any sequence $\{\alpha_n\}$ of positive numbers*, $\lim_n \alpha_n = 0$ *iff* $\lim_n F(\alpha_n) = -\infty$.
(F3) $\lim_{t\to+0} t^k F(t) = 0$ *holds*.
(F4) *If $Tx \neq Ty$, then*

$$F\big(d(Tx,Ty)\big) \le F\big(d(x,y)\big) - \eta$$

holds.

Then T has a unique fixed point.

Remark By (F1), we note that (F2) is equivalent to the following:

(F2)′ $\lim_{t\to+0} F(t) = -\infty$ holds.

We give an alternative proof of Theorem 17 by showing that mappings satisfying (F1) and (F4) are CJM contractions.

Proof of Theorem 17 Define a subset D of $(0,\infty)^2$ by (1). Define θ and ψ by $\theta = F$ and $\psi(\tau) = \tau - \eta$. Then all the assumptions of Proposition 8 hold. So, by Proposition 8, D is CJM. Therefore T has a unique fixed point. □

Remark We assume (F4) and that F is nondecreasing instead of (F1)-(F4). Then D defined by (1) is CJM. Moreover, the following hold:

- If we assume additionally that F is right continuous, then D is Meir-Keeler by Corollary 11.
- If we assume additionally that F is left continuous, then D is Matkowski by Proposition 12.
- If we assume additionally that F is continuous, then D is Browder by Proposition 7.

Competing interests
The author declares that he has no competing interests.

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