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Viscosity approximations methods for (ψ,φ)-weakly contractive mappings

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Abstract

In this paper, we study viscosity approximations with (ψ,φ)-weakly contractive mappings. We show that Moudafi's viscosity approximations follow from Browder and Halpern type convergence theorems. Our results generalize a number of convergence theorems including a strong convergence theorem of Song and Liu (Fixed Point Theory Appl. 2009:824374, 2009).

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1 Introduction and preliminaries

Let (M,d) be a metric space and $f: M \to M$ a self-mapping. A point $z \in M$ is said to be a fixed point of f if $f(z) = z$. Throughout this paper, $F(f)$ denotes the set of fixed points of f, $\mathbb{N}$ the set of natural numbers and M a metric space (M,d).

A mapping $f: M \to M$ is a contraction if there exists $r \in [0,1)$ such that for all $x, y \in M$,

$$d\big(f(x),f(y)\big) \le rd(x,y). \tag{1.1}$$

The classical *Banach contraction principle* (BCP) states that 'Every contraction of a complete metric space has a unique fixed point.' In 1969, Boyd and Wong [2] obtained the following interesting generalization of the BCP.

Theorem 1.1 *Let $f: M \to M$ a self-mapping of a complete metric space M such that for all $x, y \in M$,*

$$d\big(f(x),f(y)\big) \le \alpha\big(d(x,y)\big), \tag{1.2}$$

where $\alpha: [0,\infty) \to [0,\infty)$ is upper semicontinuous from the right and $\alpha(t) < t$ for all $t > 0$. Then f has a unique fixed point in M.

The mapping $f: M \to M$ satisfying (1.2) is called a nonlinear contraction [2].

The mapping $f: M \to M$ is called weakly contractive, if

$$d\big(f(x),f(y)\big) \le d(x,y) - \varphi\big(d(x,y)\big) \tag{1.3}$$

for all $x, y \in M$, where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

We note that (1.3) follows from Tasković [3, 4]. For an earlier work in this direction, we refer to Krasnosel'skiĭ *et al.* [5] and Dugundji and Granas [6]. Also, these mappings have been studied by Aĺber and Guerre-Delabriere [7] and Rhoades [8] as mentioned by Jachymski [9] (see also [10]).

In this paper, we use the following class of mappings satisfying the so-called (ψ, φ)-condition (see for details [11–19]).

A mapping $f : M \to M$ is called (ψ, φ)-weakly contractive if

$$\psi\big(d\big(f(x), f(y)\big)\big) \le \psi\big(d(x, y)\big) - \varphi\big(d(x, y)\big) \tag{1.4}$$

for all $x, y \in M$, where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$.

Remark 1.2 We remark that if $\varphi(t) = (1 - r)t$ with $r \in (0, 1)$, then (1.3) reduces to (1.1). If $\psi(t) = t$, then (1.4) recovers (1.3). In fact, weakly contractive mappings are also related closely to nonlinear contractions. If α is continuous and $\varphi(t) = t - \alpha(t)$ then (1.3) turns into (1.2). We have the following irreversible implications (see [13], Example 2.2).

$$(1.1) \Rightarrow (1.2) \Rightarrow (1.3) \Rightarrow (1.4).$$

Thus (ψ, φ)-weakly contractive mappings are more general than its predecessors as listed above.

Theorem 1.3 ([13], Theorem 2.1) *Every (ψ, φ)-weakly contractive mapping of a complete metric space has a unique fixed point.*

It was observed by Đorić [20] that the continuity of φ can be relaxed to lower semicontinuity in Theorem 1.3.

Definition 1.4 Let Y be a nonempty subset of a Banach space X. A mapping $f : Y \to Y$ is said to be nonexpansive if for all $x, y \in Y$,

$$\big\|f(x) - f(y)\big\| \le \|x - y\|.$$

Let X be a real Banach space with its dual space X^* and Y be a nonempty closed convex subset of X. Let $\langle x, x^* \rangle$ be the dual pairing between $x \in X$ and $x^* \in X^*$, and $J : X \to 2^{X^*}$ be the normalized duality mapping on X defined by

$$J(x) = \big\{x^* \in X^* : \big\langle x, x^* \big\rangle = \|x\|^2 = \big\|x^*\big\|^2\big\}$$

for all $x \in X$. Then X is said to be *smooth* or to have a *Gâteaux differentiable norm* if $\lim_{t \to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in X$ with $\|x\| = \|y\| = 1$. A Banach space X is said to be uniformly smooth whenever given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = 1$ and $\|y\| \le \delta$, then $\|x + y\| + \|x - y\| < 2 + \varepsilon\|y\|$.

Definition 1.5 [21] Let Y be a nonempty closed convex subset of a Banach space X and Z a nonempty subset of Y. A retraction from Y to Z is a continuous mapping $P : Y \to Z$ such that $P(x) = x$ for $x \in Z$. A retraction P from Y to Z is sunny if P satisfies the property: $P(P(x) + t(x - P(x))) = P(x)$ for all $x \in Y$ and $t > 0$, whenever $P(x) + t(x - P(x)) \in Y$. A retraction P from Y to Z is sunny nonexpansive if P is both sunny and nonexpansive [22–24].

A well-known way to find a fixed point of a nonexpansive mapping is to use a contraction to approximate it (Browder [25, 26]). More precisely, fix $z \in Y$ and define a mapping $f_t : Y \to Y$ by $f_t(x) = tz + (1 - t)S(x)$ for all $x \in Y$ and given $t \in (0,1)$. It is easy to see that f_t is a contraction on Y and the BCP ensures that f_t has a unique fixed point $u_t \in Y$, that is,

$$u_t = tz + (1 - t)S(u_t). \tag{1.5}$$

In 1967, Halpern [27] introduced the following iteration for an arbitrary $z \in Y$ and a sequence $\{\alpha_n\} \subset (0,1)$:

$$u_0 \in Y, \qquad u_{n+1} = \alpha_n z + (1 - \alpha_n)S(u_n) \tag{1.6}$$

for $n \in \mathbb{N}$, where $S : Y \to Y$ is a nonexpansive mapping.

In the case $F(S) \neq \emptyset$, Browder [25] (respectively, Halpern [27]) showed that $\{u_t\}$ (respectively, $\{u_n\}$) converges strongly to the fixed point of S that is nearest to z in a Hilbert space. A number of extensions and generalizations of their results have appeared in [1, 28–34] and elsewhere.

Theorem 1.6 [28] *Let Y be a bounded closed convex subset of a uniformly smooth Banach space X and $S : Y \to Y$ a nonexpansive mapping. Define a net $\{x_\alpha\}$ in Y by*

$$x_\alpha = \alpha z + (1 - \alpha)S(x_\alpha)$$

for $\alpha \in (0,1)$, where $z \in Y$ is fixed. Then $\{x_\alpha\}$ converges strongly to $P(z)$ as $\alpha \to 0^+$, where P is the unique sunny nonexpansive retraction from Y onto $F(S)$.

Theorem 1.7 [31, 32] *Let X, Y, S, P and z be as in Theorem* 1.6. *Define a sequence $\{u_n\}$ in Y by*

$$u_1 \in Y, \qquad u_{n+1} = \alpha_n z + (1 - \alpha_n)S(u_n)$$

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying

$$\text{(C1)} \quad \lim_{n\to\infty} \alpha_n = 0, \qquad \text{(C2)} \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \text{(C3)} \quad \lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1.$$

Then $\{u_n\}$ converges strongly to $P(z)$.

In 2000, Moudafi [35] generalized Browder's and Halpern's theorems and proved that in a real Hilbert space H, for a given $u_0 \in Y \subseteq H$, the sequence $\{u_n\}$ generated by the algorithm

$$u_{n+1} = \alpha_n f(u_n) + (1 - \alpha_n)S(u_n) \tag{1.7}$$

for $n \in \mathbb{N} \cup \{0\}$, where $f : Y \to Y$ is a contraction, $S : Y \to Y$ a nonexpansive mapping and $\{\alpha_n\} \subseteq (0,1)$, satisfying certain conditions, converges strongly to a fixed point of S in Y, which is the unique solution to the following variational inequality:

$$\bigl\langle (I - f)x^*, x^* - x \bigr\rangle \geq 0, \quad \forall x \in F(S).$$

Moudafi's generalizations are called viscosity approximations. These methods can be applied to convex optimization, linear programming, monotone inclusions, and elliptic differential equations [30]. In 2004, Xu [36] extended Moudafi's results from Hilbert spaces to more general Banach spaces. Suzuki [30] used Meir-Keeler type contractions f in (1.7) to find fixed points of S in Banach spaces. Recently, Song and Liu [1] considered the following viscosity approximations:

$$v_n = \alpha_n f(v_n) + (1 - \alpha_n)S_n(v_n);$$
$$u_{n+1} = \alpha_n f(u_n) + (1 - \alpha_n)S_n(u_n)$$

for $n \in \mathbb{N}$, where $S_n : Y \to Y$ is a sequence of nonexpansive mappings and $f : Y \to Y$ is a weakly contractive mapping.

In this paper, motivated by Moudafi [35], Kopecká and Reich [37], Suzuki [30] and Song and Liu [1], we study viscosity approximations with a more general class of weakly contractive mappings. We show that Moudafi's viscosity approximations can be obtained from Browder and Halpern type convergence results.

2 Convergence results

Throughout this section, $\psi, \varphi : [0,\infty) \to [0,\infty)$ are continuous and strictly increasing functions such that

$$\psi(t) = 0 = \varphi(t) \quad \text{if and only if} \quad t = 0.$$

Our main results are prefaced by the following lemmas and propositions.

Lemma 2.1 [24, 33] *Let Y be a nonempty convex subset of a smooth Banach space X and Z a nonempty subset of Y. Let J be the duality mapping from X into X^*, and $P : Y \to Z$ a retraction. Then P is both sunny and nonexpansive if and only if*

$$\bigl\langle x - P(x), J\bigl(y - P(x)\bigr) \bigr\rangle \leq 0$$

for all $x \in Y$ and $y \in Z$.

Lemma 2.2 [38] *Let $\{\alpha_n\}$ be a sequence of positive reals and $\{\beta_n\}$ a sequence of nonnegative reals such that*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \textit{and} \quad \lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0.$$

Further, consider a sequence of nonnegative reals $\{\ell_n\}$ *and the recursive inequality*

$$\ell_{n+1} \le \ell_n - \alpha_n \xi(\ell_n) + \beta_n$$

for $n \in \mathbb{N} \cup \{0\}$, *where* $\xi(\ell)$ *is continuous strictly increasing for* $\ell \ge 0$ *and* $\xi(0) = 0$. *Then*

(1) $\lim_{n\to\infty} \ell_n = 0$;

(2) *there exists a subsequence* $\{\ell_{n_k}\}$ *of* $\{\ell_n\}$ *such that*

$$\ell_{n_k} \le \xi^{-1}\left(\frac{1}{\sum_{m=0}^{n_k}\alpha_m} + \frac{\beta_{n_k}}{\alpha_{n_k}}\right),$$
$$\ell_{n_k+1} \le \xi^{-1}\left(\frac{1}{\sum_{m=0}^{n_k}\alpha_m} + \frac{\beta_{n_k}}{\alpha_{n_k}}\right) + \beta_{n_k},$$
$$\ell_n \le \ell_{n_k+1} - \sum_{m=n_{k+1}}^{n-1}\frac{\alpha_m}{\theta_m}, \quad n_k + 1 < n < n_{k+1}, \theta_m = \sum_{i=0}^{m}\alpha_i,$$
$$\ell_{n+1} \le \ell_0 - \sum_{m=1}^{n}\frac{\alpha_m}{\theta_m} \le \ell_0, \quad 1 \le n < n_k - 1,$$
$$1 \le n_k \le s_{\max} = \max\left\{s, \sum_{m=0}^{s}\frac{\alpha_m}{\theta_m} \le \ell_0\right\}.$$

Definition 2.3 [30] Let $\{S_n\}$ be sequence of nonexpansive mappings on a closed convex subset Y of a Banach space X and $F = \bigcap_{n=0}^{\infty} F(S_n) \ne \emptyset$.

(A) Let $\{\alpha_n\}$ be a sequence in $(0,1]$ with $\lim_{n\to\infty}\alpha_n = 0$. Then $(X, Y, \{S_n\}, \{\alpha_n\})$ is said to satisfy Browder property if for each $z \in Y$, a sequence $\{v_n\}$ defined by

$$v_n = \alpha_n z + (1 - \alpha_n)S_n(v_n), \tag{2.1}$$

for $n \in \mathbb{N}$, converges strongly.

(B) Let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then $(X, Y, \{S_n\}, \{\alpha_n\})$ is said to satisfy Halpern's property if for each $z \in Y$, a sequence $\{v_n\}$ defined by

$$v_1 \in Y, \qquad v_{n+1} = \alpha_n z + (1 - \alpha_n)S_n(v_n), \tag{2.2}$$

for $n \in \mathbb{N}$, converges strongly.

It is well known that if X is a Hilbert space, Y is bounded and $\{S_n\}$ is a constant sequence S, then $(X, Y, \{S_n\}, \{1/n\})$ has both the Browder and the Halpern properties (*cf.* [24, 31, 32, 34]).

Example 2.4 Let $X = [0,\infty)$ equipped with the norm $\|\cdot\|$ defined by $\|x\| = |x|$ and $Y = [0,1]$ a closed convex subset of X. Define a sequence of nonexpansive mappings $S_n : Y \to Y$ by $S_n(x) = \frac{x}{n}$ for all $x \in Y$ and $n \in \mathbb{N}$. Let $\{\alpha_n\}$ be a sequence in $(0,1]$ defined by $\alpha_n = \frac{1}{n^2+1}$. Then

(i) It is easy to see that the quadruple $(X, Y, \{S_n\}, \{\alpha_n\})$ satisfies the Browder property and the sequence $\{v_n\}$ defined by

$$v_n = \alpha_n z + (1 - \alpha_n) S_n(v_n),$$

for each $z \in Y$ and $n \in \mathbb{N}$, converges strongly to 0.

(ii) However, the quadruple $(X, Y, \{S_n\}, \{\alpha_n\})$ does not satisfy the Halpern property, as the series $\sum_{n=1}^{\infty} \alpha_n$ is a convergent series.

(iii) If we take $\alpha_n = \frac{1}{n}$, then the quadruple $(X, Y, \{S_n\}, \{\alpha_n\})$ satisfies both the Browder and the Halpern properties. We note that $\{S_n\}$ is not a constant sequence here.

Proposition 2.5 ([30], Proposition 4) *Let the quadruple $(X, Y, \{S_n\}, \{\alpha_n\})$ satisfies Browder's property and $\{v_n\}$ is a sequence in Y, defined by* (2.1). *If $P(z) = \lim_{n\to\infty} v_n$ for each $z \in Y$ then P is a nonexpansive mapping on Y.*

Proposition 2.6 ([30], Proposition 5) *Let the quadruple $(X, Y, \{S_n\}, \{\alpha_n\})$ satisfies Halpern's property and $\{v_n\}$ is a sequence in Y, defined by* (2.2). *If $P(z) = \lim_{n\to\infty} v_n$ for each $z \in Y$ then*

- *$P : Y \to Y$ is a nonexpansive mapping*;
- *$P(z)$ does not depend on the initial point v_1.*

Proposition 2.7 *Let Y be a closed convex subset of a smooth Banach space X and Z a nonempty subset of Y. Let $S : Y \to Y$ be a nonexpansive mapping, $P : Y \to Z$ a unique sunny nonexpansive retraction, and $T : Y \to Y$ a (ψ, φ)-weakly contractive mapping with ψ convex. Then*

(a) *the composite mapping $S \circ T$ is (ψ, φ)-weakly contractive on Y*;

(b) *the mapping $S_t = tT + (1 - t)S$ for $t \in (0, 1)$ is (ψ, ϕ)-weakly contractive on Y and u_t is the unique solution of the fixed point equation*

$$u_t = tT(u_t) + (1 - t)S(u_t),$$

where $\phi(s) = t\varphi(s)$ for each fixed $t \in (0, 1)$;

(c) *$P(T(z)) = z$ if and only if $z \in Y$ is the unique solution of the variational inequality*

$$\langle T(z) - z, J(y - z)\rangle \le 0 \tag{2.3}$$

for all $y \in Z$.

Proof

(a) For any $x, y \in Y$ we have

$$\|S(T(x)) - S(T(y))\| \le \|T(x) - T(y)\|.$$

Since ψ is nondecreasing and T is a (ψ, φ)-weakly contractive, the above inequality reduces to

$$\begin{aligned}\psi(\|S(T(x)) - S(T(y))\|) &\le \psi(\|T(x) - T(y)\|) \\ &\le \psi(\|x - y\|) - \varphi(\|x - y\|).\end{aligned}$$

Therefore the mapping $S \circ T$ is a (ψ, φ)-weakly contractive.

(b) Let $x, y \in Y$. Then for each fixed $t \in (0,1)$, we have

$$\begin{aligned} \|S_t(x) - S_t(y)\| &= \|(tT(x) + (1-t)S(x)) - (tT(y) + (1-t)S(y))\| \\ &\le (1-t)\|S(x) - S(y)\| + t\|T(x) - T(y)\| \\ &\le (1-t)\|x - y\| + t\|T(x) - T(y)\|. \end{aligned}$$

Since ψ is nondecreasing, the above inequality reduces to

$$\psi(\|S_t(x) - S_t(y)\|) \le \psi((1-t)\|x-y\| + t\|T(x) - T(y)\|).$$

Convexity of ψ implies

$$\psi(\|S_t(x) - S_t(y)\|) \le (1-t)\psi(\|x-y\|) + t\psi(\|T(x) - T(y)\|).$$

Since T is (ψ, φ)-weakly contractive, we have

$$\begin{aligned} \psi(\|S_t(x) - S_t(y)\|) &\le (1-t)\psi(\|x-y\|) + t[\psi(\|x-y\|) - \varphi(\|x-y\|)] \\ &= \psi(\|x-y\|) - t\varphi(\|x-y\|). \end{aligned}$$

Let $\phi(s) = t\varphi(s)$. Then

$$\psi(\|S_t(x) - S_t(y)\|) \le \psi(\|x-y\|) - \phi(\|x-y\|).$$

Thus, the mappings S_t is (ψ, ϕ)-weakly contractive and by Theorem 1.3, S_t has a unique fixed point u_t in Y.

(c) By (a) and Proposition 2.5, the mapping $P \circ T$ is (ψ, ϕ)-weakly contractive. By Theorem 1.3, $P \circ T$ has a unique fixed point $P(T(z)) = z \in Z$. By Lemma 2.1, such a $z \in Z$ satisfies (2.3). Next, we show that the variational inequality (2.3) has a unique solution. Let $w \in Y$ be another solution of (2.3). Then

$$\langle T(w) - w, J(z-w) \rangle \le 0 \tag{2.4}$$

and

$$\langle T(z) - z, J(w-z) \rangle \le 0. \tag{2.5}$$

Adding (2.4) and (2.5)

$$\begin{aligned} 0 &\ge \langle w - z - (T(w) - T(z)), J(w-z) \rangle \\ &= \|w - z - (T(w) - T(z))\| \|w-z\| \\ &\ge \|w-z\|^2 - \|T(w) - T(z)\| \|w-z\| \\ &= \|w-z\| \{\|w-z\| - \|T(w) - T(z)\|\}, \end{aligned}$$

which implies that

$$\|w-z\| - \|T(w)-T(z)\| \leq 0 \quad \text{or} \quad \|w-z\| \leq \|T(w)-T(z)\|.$$

Since ψ is nondecreasing and T is a (ψ,φ)-weakly contractive, we have

$$\begin{aligned}\psi(\|w-z\|) &\leq \psi(\|T(w)-T(z)\|)\\ &\leq \psi(\|w-z\|) - \varphi(\|w-z\|).\end{aligned}$$

Therefore $\varphi(\|w-z\|) \leq 0$, and $w = z$. □

Now we present our first convergence result.

Theorem 2.8 *Suppose the quadruple* $(X, Y, \{S_n\}, \{\alpha_n\})$ *satisfies Browder's property and* $T : Y \to Y$ *is a* (ψ,φ)*-weakly contractive mapping with* ψ *convex. For each* $z \in Y$*, put* $P(z) = \lim_{n\to\infty} v_n$*, where* $\{v_n\}$ *is a sequence in* Y *defined by* (2.1)*. Then the sequence* $\{u_n\} \subset Y$ *defined by*

$$u_n = \alpha_n T(u_n) + (1-\alpha_n)S_n(u_n),$$

for $n \in \mathbb{N}$*, converges strongly to* $P(T(z)) = z \in Y$*.*

Proof Proposition 2.7(b) ensures the existence and uniqueness of $\{u_n\}$. It follows from Proposition 2.7(a) and Proposition 2.5 that $P \circ T$ is a (ψ,φ)-weakly contractive mapping on Y. Therefore, by Theorem 1.3 there exists a unique element $z \in Y$ such that $P(T(z)) = z$. Define a sequence $\{v_n\}$ in Y by

$$v_n = \alpha_n T(z) + (1-\alpha_n)S_n(v_n)$$

for $n \in \mathbb{N}$. Then it is easy to see that $\{v_n\}$ converges strongly to $P(T(z))$.

Now, for $n \in \mathbb{N}$,

$$\begin{aligned}\|u_n - v_n\| &\leq (1-\alpha_n)\|S_n(u_n) - S_n(v_n)\| + \alpha_n\|T(u_n) - T(z)\|\\ &\leq (1-\alpha_n)\|u_n - v_n\| + \alpha_n\|T(u_n) - T(z)\|\end{aligned}$$

or

$$\|u_n - v_n\| \leq \|T(u_n) - T(z)\|.$$

Since ψ is nondecreasing, we have

$$\psi(\|u_n - v_n\|) \leq \psi(\|T(u_n) - T(z)\|).$$

Further, (ψ,φ)-weak contractivity of T implies

$$\begin{aligned}\psi(\|u_n - v_n\|) &\leq \psi(\|u_n - z\|) - \varphi(\|u_n - z\|)\\ &\leq \psi(\|u_n - v_n\| + \|v_n - z\|) - \varphi(\|u_n - z\|).\end{aligned}$$

Since $v_n \to z$ as $n \to \infty$, we get

$$\begin{aligned}\overline{\lim_{n\to\infty}}\ \psi\left(\|u_n - v_n\|\right) &\le \overline{\lim_{n\to\infty}}\ \psi\left(\|u_n - v_n\| + \|v_n - z\|\right) - \underline{\lim_{n\to\infty}}\ \varphi\left(\|u_n - z\|\right)\\ &= \overline{\lim_{n\to\infty}}\ \psi\left(\|u_n - v_n\|\right) - \underline{\lim_{n\to\infty}}\ \varphi\left(\|u_n - z\|\right),\end{aligned}$$

or

$$\underline{\lim_{n\to\infty}}\ \varphi\left(\|u_n - z\|\right) \le 0.$$

The continuity of φ and $\varphi(0) = 0$ imply that

$$\lim_{n\to\infty} \|u_n - z\| = 0.$$

Therefore $\{u_n\}$ converges strongly to z. □

Corollary 2.9 *Let X, Y, $\{S_n\}$, $\{v_n\}$, P, z and $\{\alpha_n\}$ be as in Theorem* 2.8 *and $T : Y \to Y$ a weakly contractive mapping. Then the sequence $\{u_n\} \subset Y$ defined by*

$$u_n = \alpha_n T(u_n) + (1 - \alpha_n)S_n(u_n),$$

for $n \in \mathbb{N}$, converges strongly to $P(T(z)) = z \in Y$.

Proof This follows from Theorem 2.8 when $\psi(t) = t$. □

Example 2.10 Let $(X, \|\cdot\|)$ and Y be as in Example 2.4. Define the mappings $S_n, T : Y \to Y$ by

$$S_n(x) = x \quad \text{for all } x \in Y \text{ and } n \in \mathbb{N} \quad \text{and} \quad T(x) = 1 - \frac{x}{2} \quad \text{for all } x \in Y.$$

Let $\psi, \varphi : [0,\infty) \to [0,\infty)$ be the functions defined by

$$\psi(t) = t \quad \text{and} \quad \varphi(t) = \frac{t}{2}.$$

Then the mapping S_n is nonexpansive for each $n \in \mathbb{N}$ and T is (ψ,φ)-weakly contractive.

Let $\{\alpha_n\}$ be a sequence in $(0,1]$ defined by $\alpha_n = \frac{1}{n+1}$. Then the quadruple $(X, Y, \{S_n\}, \{\alpha_n\})$ satisfies the Browder property and for each $z \in Y$, we have

$$\begin{aligned}v_n &= z\alpha_n + (1 - \alpha_n)S_n(v_n)\\ &= z\frac{1}{n+1} + \left(1 - \frac{1}{1+n}\right)v_n,\end{aligned}$$

or

$$v_n = z.$$

By Theorem 2.8, put $P(z) = \lim_{n\to\infty} v_n = z$. Then P is an identity mapping. Now

$$\begin{aligned} u_n &= \alpha_n T(u_n) + (1-\alpha_n)S_n(u_n) \\ &= \frac{1}{n+1}\left[1 - \frac{u_n}{2}\right] + \left(1 - \frac{1}{1+n}\right)u_n, \end{aligned}$$

or

$$u_n = \frac{2}{3}.$$

Now $\lim_{n\to\infty} u_n = \frac{2}{3} = P(T(\frac{2}{3}))$. Thus the quadruple $(X, Y, \{S_n\}, \{\alpha_n\})$ satisfies all the conditions of Theorem 2.8 and the sequence $\{u_n\}$ strongly converges to $\frac{2}{3}$.

The following theorem is our second convergence result.

Theorem 2.11 *Suppose the quadruple $(X, Y, \{S_n\}, \{\alpha_n\})$ satisfies Halpern's property and $T : Y \to Y$ is a (ψ, φ)-weakly contractive mapping with ψ is convex. Put $P(z) = \lim_{n\to\infty} v_n$ for each $z \in Y$, where $\{v_n\}$ is defined by* (2.2). *Then the sequence $\{u_n\} \subset Y$ defined by*

$$x_1 \in Y \quad \textit{and} \quad u_{n+1} = \alpha_n T(u_n) + (1-\alpha_n)S_n(u_n)$$

for $n \in \mathbb{N}$, converges strongly to a unique point $P(T(z)) = z \in Y$.

Proof By Propositions 2.6 and 2.7(a), the mapping $P \circ T$ is (ψ, φ)-weakly contractive on Y. From Theorem 1.3, there exists a unique $z \in Y$ such that $z = P(T(z))$. For $n \in \mathbb{N}$, we define a sequence $\{v_n\}$ in Y by

$$v_{n+1} = \alpha_n T(z) + (1-\alpha_n)S_n(v_n).$$

Then by the assumption $\{v_n\}$ converges strongly to $P(T(z))$.

Now for $n \in \mathbb{N}$, we have

$$\|u_{n+1} - v_{n+1}\| = \left\|\alpha_n\big(T(u_n) - T(z)\big) + (1-\alpha_n)\big(S_n(u_n) - S_n(v_n)\big)\right\|.$$

Since S_n is nonexpansive and ψ is nondecreasing and convex, we have

$$\begin{aligned} \psi\big(\|u_{n+1} - v_{n+1}\|\big) &= \psi\big(\alpha_n\|T(u_n) - T(z)\| + (1-\alpha_n)\|S_n(u_n) - S_n(v_n)\|\big) \\ &\le \alpha_n\psi\big(\|T(u_n) - T(z)\|\big) + (1-\alpha_n)\psi\big(\|S_n(u_n) - S_n(v_n)\|\big) \\ &\le \alpha_n\psi\big(\|T(u_n) - T(z)\|\big) + (1-\alpha_n)\psi\big(\|u_n - v_n\|\big). \end{aligned}$$

By (ψ, φ)-weak contractivity of T, we get

$$\begin{aligned} \psi\big(\|u_{n+1} - v_{n+1}\|\big) &\le \alpha_n\big[\psi\big(\|u_n - z\|\big) - \varphi\big(\|u_n - z\|\big)\big] + (1-\alpha_n)\psi\big(\|u_n - v_n\|\big) \\ &\le \alpha_n\big[\psi\big(\|u_n - v_n\| + \|v_n - z\|\big) - \varphi\big(\|u_n - v_n\| + \|v_n - z\|\big)\big] \\ &\quad + (1-\alpha_n)\psi\big(\|u_n - v_n\|\big). \end{aligned}$$

The continuity of ψ, φ and $v_n \to z$ as $n \to \infty$ imply that

$$\begin{aligned}
&\lim_{n\to\infty} \psi\big(\|u_{n+1} - v_{n+1}\|\big) \\
&\quad \leq \lim_{n\to\infty} \alpha_n\big[\psi\big(\|u_n - v_n\|\big) - \varphi\big(\|u_n - v_n\|\big) + (1-\alpha_n)\psi\big(\|u_n - v_n\|\big)\big] \\
&\quad = \lim_{n\to\infty}\big[\psi\big(\|u_n - v_n\|\big) - \alpha_n\varphi\big(\|u_n - v_n\|\big)\big] \\
&\quad \leq \lim_{n\to\infty}\big[\psi\big(\|u_n - v_n\|\big) - \alpha_n\varphi\big(\|u_n - v_n\|\big) + \alpha_n\|v_n - z\|\big].
\end{aligned}$$

Thus, for some (sufficiently large) $N_0 \leq n$, we have

$$\psi\big(\|u_{n+1} - v_{n+1}\|\big) \leq \psi\big(\|u_n - v_n\|\big) - \alpha_n\varphi\big(\|u_n - v_n\|\big) + \alpha_n\|v_n - z\|.$$

For $\ell_n = \psi(\|u_n - v_n\|)$, we get the following recursive inequality:

$$\ell_{n+1} \leq \ell_n - \alpha_n\varphi(\ell_n) + \beta_n,$$

where $\varepsilon_n = \|v_n - z\|$ and $\beta_n = \alpha_n\varepsilon_n$. Now, by Lemma 2.2,

$$\lim_{n\to\infty} \psi\big(\|u_n - v_n\|\big) = 0.$$

The continuity of ψ and the fact that $\psi(0) = 0$ imply that

$$\lim_{n\to\infty} \|u_n - v_n\| = 0.$$

By the triangle inequality, we have

$$\lim_{n\to\infty} \|u_n - z\| \leq \lim_{n\to\infty} \|u_n - v_n\| + \lim_{n\to\infty} \|v_n - z\| = 0.$$

Therefore $\{u_n\}$ strong convergence of to $z = P(T(z))$. □

Corollary 2.12 [1] *Let $X, Y, \{S_n\}, \{v_n\}, P, z$ and $\{\alpha_n\}$ be as in Theorem* 2.11 *and $T : Y \to Y$ a weakly contractive mapping. Then the sequence $\{u_n\} \subset Y$ defined by*

$$x_1 \in Y \quad \textit{and} \quad u_{n+1} = \alpha_n T(u_n) + (1-\alpha_n)S_n(u_n),$$

for $n \in \mathbb{N}$, converges strongly to a unique point $P(T(z)) = z \in Y$.

Proof This follows from Theorem 2.11 when $\psi(t) = t$. □

Competing interests
The authors declare that they do not have any competing interests.

Authors' contributions
Each author equally contributed to this paper and read and approved the final manuscript.

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