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F-cone metric spaces over Banach algebra

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Abstract

The paper deals with the achievement of introducing the notion of F-cone metric spaces over Banach algebra as a generalization of N_p-cone metric space over Banach algebra and N_b-cone metric space over Banach algebra and studying some of its topological properties. Also, here we define generalized Lipschitz and expansive maps for such spaces. Moreover, we investigate some fixed points for mappings satisfying such conditions in the new framework. Subsequently, as an application of our results, we provide an example. Our results generalize some well-known results in the literature.

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1 Introduction

Partial metric spaces were introduced by Matthews [1] in 1994. He studied a partial metric space as a part of the denotational semantics of dataflow networks and showed that the Banach contraction principle can be generalized to the partial metric context for applications in program verification. Especially, it has the property that differentiates it from other spaces, that is, the self-distance of any point may not be zero, also a convergent sequence need not have unique limit in these spaces.

On the other hand, in 1989, the concept of b-metric spaces was introduced by Bakhtin [2] as a generalization of metric spaces. He showed the contraction mapping principle in a b-metric space that generalizes the prominent Banach contraction principle in metric spaces.

In the same spirit, recently, Huang and Zhang [3] replaced the set of real numbers by ordering Banach space and defined a cone metric space as a generalization of the metric space. The authors proved some fixed point theorems of contractive mappings on cone metric spaces. They also defined the Cauchy sequence and convergence of a sequence in such spaces in terms of interior points of the underlying cone. After that, in [4], Rezapour and Hamlbarani generalized some results of [5] by omitting the assumption of normality. For fixed point theorems on cone metric spaces, readers may refer to [6–9] and the references therein.

Malviya et al. [10] introduced the concept of N-cone metric spaces, which is a new generalization of the generalized G-cone metric space [11] and the generalized D^*-metric space [12]. They proved fixed point theorems for asymptotically regular maps and sequences. Afterwards, in [13], the authors defined contractive maps in N-cone metric spaces and proved various fixed point theorems for such maps.

Despite these features, some authors demonstrated that the fixed point results proved on cone metric spaces are the straightforward outcome of the corresponding results of usual metric spaces where the real-valued metric function d^* is defined by a nonlinear scalarization function ξ_e (see [14]) or by a Minkowski functional q_e (see [5]).

Due to the concrete reasons mentioned above, researchers were losing their interest in a cone metric space. Fortunately, Liu and Xu [15] introduced the approach of cone metric spaces over Banach algebras by replacing Banach spaces E by Banach algebras A as the underlying spaces of cone metric spaces. They verified some fixed point theorems of generalized Lipschitz mappings with weaker conditions on the generalized Lipschitz constant k by means of the spectral radius. Not long ago, Xu and Radenović [16] deleted the normality of cones and greatly generalized the main results of [15]. In particular, some authors have shown recently some fixed point results given in [17–19].

Following these ideas, very recently, Fernandez et al. [6] introduced the notion of partial cone metric spaces over Banach algebra as a generalization of partial metric spaces and cone metric spaces over Banach algebra, which was selected for Young Scientist Award 2016, M.P., India (see [20]).

Recently, proceeding in this direction, Fernandez et al. introduced the structure of N_p-cone metric space over Banach algebra [21] as a generalization of N-cone metric space over Banach algebra [22] and partial metric space and N_b-cone metric space over Banach algebra [23] as a generalization of N-cone metric space over Banach algebra [22] and b-metric space, respectively.

Inspired and encouraged by the previous works, we present seven sections in this paper. For the reader's convenience, we recall in Section 2 some definitions that will be used in the sequel. In Section 3, after the preliminaries, we introduce F-cone metric spaces over Banach algebra which generalize N_p-cone metric spaces over Banach algebra and N_b-cone metric spaces over Banach algebra. In Section 4, we discuss the topological properties. In Section 5, we introduce the notions of generalized Lipschitz and expansive maps. Section 6 is devoted to deriving the existence of fixed point theorems for such spaces by using the mentioned contractive conditions. Finally, in Section 7, we define expansive maps and investigate the existence and uniqueness of the fixed point in the new framework. Our main theorems extend and unify existing results in the recent literature. We also give illustrative examples that verify our results.

2 Preliminaries

We begin with the following definition as a recall from [15].

Let A always be a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined subject to the following properties (for all $x, y, z \in A$, $\alpha \in R$):

1. $(xy)z = x(yz)$;
2. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. $\|xy\| \leq \|x\|\|y\|$.

Throughout this paper, we shall assume that a Banach algebra has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer the reader to [24].

The following proposition is given in [24].

Proposition 2.1 *Let A be a Banach algebra with a unit e, and $x \in A$. If the spectral radius $\rho(x)$ of x is less than 1, i.e.,*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf\|x^n\|^{\frac{1}{n}} < 1,$$

then $e - x$ is invertible. Actually,

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Remark 2.2 From [24] we see that the spectral radius $\rho(x)$ of x satisfies $\rho(x) \leq \|x\|$ for all $x \in A$, where A is a Banach algebra with a unit e.

Remark 2.3 (see [16]) In Proposition 2.1, if the condition $'\rho(x) < 1'$ is replaced by $\|x\| \leq 1$, then the conclusion remains true.

Remark 2.4 (see [16]) If $\rho(x) < 1$, then $\|x\|^n \to 0$ $(n \to \infty)$.

Lemma 2.5 (see [25]) *If E is a real Banach space with a solid cone P and if $\theta \preccurlyeq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*

A subset P of A is called a cone of A if

1. P is nonempty closed and $\{\theta, e\} \subset P$;
2. $\alpha P + \beta P \subset P$ for all nonnegative real numbers α, β;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra A. For a given cone $P \subset A$, we can define a partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preccurlyeq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. If $\operatorname{int} P \neq \varnothing$, then P is called a solid cone.

The cone P is called normal if there is a number $M > 0$ such that, for all $x, y \in A$,

$$\theta \preccurlyeq x \preccurlyeq y \quad \Rightarrow \quad \|x\| \leq M\|y\|.$$

The least positive number satisfying the above is called the normal constant of P [3].

In the following we always assume that A is a Banach algebra with a unit e, P is a solid cone in A and $\preccurlyeq$ is the partial ordering with respect to P.

Definition 2.6 ([3, 15]) Let X be a nonempty set. Suppose that the mapping $d : X \times X \to A$ satisfies

1. $\theta \prec d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \preccurlyeq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space over Banach algebra A.

For other definitions and related results on cone metric space over Banach algebra, we refer to [15].

Definition 2.7 ([2]) Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric on X if, for all $x,y,z \in X$, the following conditions hold:

1. $d(x,y) = 0$ if and only if $x = y$;
2. $d(x,y) = d(y,x)$;
3. $d(x,z) \leq s[d(x,y) + d(y,z)]$.

In this case, the pair (X,d) is called a b-metric space.

For more definitions and results on b-metric spaces, we refer the reader to [26].

Definition 2.8 ([1]) A partial metric on a nonempty set X is a function $p : X \times X \to R^+$ such that for all $x,y,z \in X$, the following conditions hold:

1. $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;
2. $p(x,x) \leq p(x,y)$;
3. $p(x,y) = p(y,x)$;
4. $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X,p) is called a partial metric space. It is clear that if $p(x,y) = 0$, then from (1) and (2) $x = y$. But if $x = y$, $p(x,y)$ may not be 0.

Definition 2.9 ([10]) Let X be a nonempty set. A function $N : X^3 \to A$ is called N-cone metric on X if for any $x,y,z,a \in X$, the following conditions hold:

(N_1) $0 \leq N(x,x,x)$;
(N_2) $N(x,y,z) = \theta$ iff $x = y = z$;
(N_3) $N(x,y,z) \leq N(x,x,a) + N(y,y,a) + N(z,z,a)$.

Then the pair (X,N) is called an N-cone metric space over Banach algebra A.

Definition 2.10 ([23]) An N_b-cone metric on a nonempty set X is a function $N_b : X^3 \to A$ such that for all $x,y,z,a \in X$:

(N_{b_1}) $\theta \preccurlyeq N_b(x,y,z)$;
(N_{b_2}) $N_b(x,y,z) = \theta$ iff $x = y = z$;
(N_{b_3}) $N_b(x,y,z) \preccurlyeq s[N_b(x,x,a) + N_b(y,y,a) + N_b(z,z,a)]$.

The pair (X,N_b) is called an N_b-cone metric space over Banach algebra A. The number $s \geq 1$ is called the coefficient of (X,N_b).

Definition 2.11 ([21]) An N_p-cone metric on a nonempty set X is a function $N_p : X^3 \to A$ such that for all $x,y,z,a \in X$:

(N_p1) $x = y = z \Leftrightarrow N_p(x,x,x) = N_p(y,y,y) = N_p(z,z,z) = N_p(x,y,z)$;
(N_p2) $\theta \preccurlyeq N_p(x,x,x) \preccurlyeq N_p(x,x,y) \preccurlyeq N_p(x,y,z)$, for all $x,y,z \in X$ with $x \neq y \neq z$;
(N_p3) $N_p(x,y,z) \preccurlyeq N_p(x,x,a) + N_p(y,y,a) + N_p(z,z,a) - N_p(a,a,a)$.

The pair (X,N_p) is called an N_p-cone metric space over Banach algebra A.

3 *F*-cone metric space over Banach algebra

In this section, we define a new structure, i.e., F-cone metric spaces over Banach algebra.

Definition 3.1 Let X be a nonempty set. A function $F : X^3 \to A$ is called F-cone metric on X if for any $x, y, z, a \in X$, the following conditions hold:

(F_1) $x = y = z$ iff $F(x,x,x) = F(y,y,y) = F(z,z,z) = F(x,y,z)$;
(F_2) $\theta \preccurlyeq F(x,x,x) \preccurlyeq F(x,x,y) \preccurlyeq F(x,y,z)$, for all $x, y, z \in X$ with $x \neq y \neq z$;
(F_3) $F(x,y,z) \preccurlyeq s[F(x,x,a) + F(y,y,a) + F(z,z,a)] - F(a,a,a)$.

Then the pair (X,F) is called an F-cone metric space over Banach algebra A. The number $s \geq 1$ is called the coefficient of (X,F).

Remark 3.2 In an F-cone metric space over Banach algebra (X,F), if $x, y, z \in X$ and $F(x,y,z) = \theta$, then $x = y = z$, but the converse may not be true.

Example 3.3 Let $A = C_1^R[0,1]$ and define a norm on A by $\|x\| = \|x\|_\infty + \|x'\|_\infty$ for $x \in A$. Define multiplication in A as just pointwise multiplication. Then A is a real unit Banach algebra with unit $e = 1$. Set $P = \{x \in A : x \geq 0\}$ is a cone in A. Moreover, P is not normal (see [4]). Let $X = [0,\infty)$. Define a mapping $F : X^3 \to A$ by $F(x,y,z)(t) = ((\max\{x,z\})^2 + (\max\{y,z\})^2, (\max\{x,z\})^2 + (\max\{y,z\})^2)e^t$ for all $x, y, z \in X$, and let $\alpha > 0$ be any constant. Then (X,F) is an F-cone metric space over Banach algebra A with the coefficient $s = 2$. But it is not an N_p-cone metric space over Banach algebra since the triangle inequality is not satisfied; neither it is an N_b-cone metric space over Banach algebra A since for any $x > 0$, we have $N_b(x,x,x)(t) = 2x^2.e^t \neq \theta$.

Lemma 3.4 *Let (X,F) be an F-cone metric space over Banach algebra A. Then*
(a) *if $F(x,y,z) = \theta$, then $x = y = z$.*
(b) *if $x \neq y$, then $F(x,x,y) > \theta$.*

Proof The proof is obvious. □

Proposition 3.5 *If (X,F) is an F-cone metric space over Banach algebra, then for all $x, y, z \in X$, we have $F(x,x,y) = F(y,y,x)$.*

Definition 3.6 Let (X,F) be an F-cone metric space over Banach algebra A. Then, for $x \in X$ and $c > \theta$, the F-balls with center x and radius $c > \theta$ are

$$B_F(x,c) = \{y \in X : F(x,x,y) \ll F(x,x,x) + c\}.$$

4 Topology on *F*-cone metric space over Banach algebra

In this section, we define the topology of F-cone metric space over Banach algebra and study its topological properties.

Definition 4.1 Let (X,F) be an F-cone metric space over Banach algebra A with coefficient $s \geq 1$. For each $x \in X$ and each $\theta \ll c$, put $B_F(x,c) = \{y \in X : F(x,x,y) \ll F(x,x,x) + c\}$ and put $B = \{B_F(x,c) : x \in X \text{ and } \theta \ll c\}$. Then B is a subbase for some topology τ on X.

Remark 4.2 Let (X,F) be an F-cone metric space over Banach algebra A. In this paper, τ denotes the topology on X, B denotes a subbase for the topology on τ and $B_F(x,c)$ denotes the F-ball in (X,F), which are described in Definition 4.1. In addition, U denotes the base generated by the subbase B.

Theorem 4.3 *Let (X,F) be an F-cone metric space over Banach algebra A, and let P be a solid cone in A. Let $k \in P$ be an arbitrarily given vector, then (X,F) is a Hausdorff space.*

Proof Let (X,F) be an F-cone metric space over Banach algebra, and let $x,y \in X$ with $x \neq y$.

Let $F(x,x,y) = c$.

Suppose $U = B(x, \frac{c}{3})$ and $V = B(y, \frac{c}{3})$.

Then $x \in U$ and $y \in V$.

We claim that $U \cap V = \phi$.

If not, there exists $z \in U \cap V$.

But then

$$F(x,x,z) \prec \frac{c}{3s} \quad \text{and} \quad F(y,y,z) \prec \frac{c}{3s}.$$

So, we get

$$\begin{aligned} c = F(x,x,y) &\preccurlyeq s\bigl[F(x,x,z) + F(x,x,z) + F(y,y,z)\bigr] - F(z,z,z) \\ &\prec s\bigl[2F(x,x,z) + F(y,y,z)\bigr] \\ &\preccurlyeq s\left[\frac{2c}{3s} + \frac{c}{3s}\right] \\ &\prec c, \end{aligned}$$

i.e., $c < c$, which is a contradiction.

Hence $U \cap V = \phi$ and X is a Hausdorff space.

Now, we define θ-Cauchy sequence and convergent sequence in an F-cone metric space over Banach algebra A. □

Definition 4.4 Let (X,F) be an F-cone metric space over Banach algebra A. A sequence $\{x_n\}$ in (X,F) converges to a point $x \in X$ whenever for every $c \gg \theta$ there is a natural number N such that $F(x_n,x,x) \ll c$ for all $n \geq N$. We denote this by

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x \quad (n \to \infty).$$

Definition 4.5 Let (X,F) be an F-cone metric space over Banach algebra A. A sequence $\{x_n\}$ in X is said to be a θ-Cauchy sequence in (X,F) if $\{F(x_n,x_m,x_m)\}$ is a c-sequence in A, i.e., if for every $c \gg \theta$ there exists $n_0 \in N$ such that $F(x_n,x_m,x_m) \ll c$ for all $n,m \geq n_0$.

Definition 4.6 Let (X,F) be an F-cone metric space over Banach algebra A. Then X is said to be θ-complete if every θ-Cauchy sequence $\{x_n\}$ in (X,F) converges to $x \in X$ such that $F(x,x,x) = \theta$.

Definition 4.7 Let (X,F) and (X',F') be an F-cone metric space over Banach algebra A. Then a function $f : X \to X'$ is said to be continuous at a point $x \in X$ if and only if it is sequentially continuous at x, that is, whenever $\{x_n\}$ is convergent to x, we have $\{fx_n\}$ is convergent to $f(x)$.

5 Generalized Lipschitz maps

In this section, we define generalized Lipschitz maps in F-cone metric spaces over Banach algebra.

Definition 5.1 Let (X,F) be an F-cone metric space over Banach algebra A and P be a cone in A. A map $T : X \to X$ is said to be a generalized Lipschitz mapping if there exists a vector $k \in P$ with $\rho(k) < 1$ for all $x, y \in X$ such that

$$F(Tx, Tx, Ty) \preccurlyeq kF(x,x,y).$$

Example 5.2 Let the Banach algebra A and the cone P be the same ones as those in Example 3.3, and let $X = R^+$. Define a mapping $F : X^3 \to A$ as in Example 3.3. Then (X,F) is an F-cone metric space over Banach algebra A. Now define the mapping $T : X \to X$ by $T(x) = \frac{x}{3} \cos \frac{x}{3}$. Since $u \cos u \le u$ for each $u \in [0,\infty)$, for all $x, y \in X$, we have

$$\begin{aligned}
&F(Tx, Tx, Ty)\\
&\quad= \left(\left(\max\{Tx, Ty\}\right)^2 + \left(\max\{Tx, Ty\}\right)^2, \alpha\left(\left(\max\{Tx, Ty\}\right)^2 + \left(\max\{Tx, Ty\}\right)^2\right)\right) \cdot e^t\\
&\quad= 2\left[\left(\max\{Tx, Ty\}\right)^2, \alpha\left(\max\{Tx, Ty\}\right)^2\right] \cdot e^t\\
&\quad= 2\left[\left(\max\left\{\frac{x}{3}\cos\frac{x}{3}, \frac{y}{3}\cos\frac{y}{3}\right\}\right)^2, \alpha\left(\max\left\{\frac{x}{3}\cos\frac{x}{3}, \frac{y}{3}\cos\frac{y}{3}\right\}\right)^2\right] \cdot e^t\\
&\quad\preccurlyeq 2\left[\left(\max\left\{\frac{x}{3}, \frac{y}{3}\right\}\right)^2, \alpha\left(\max\left\{\frac{x}{3}, \frac{y}{3}\right\}\right)^2\right] \cdot e^t\\
&\quad\preccurlyeq \frac{2}{9}\left[\left(\max\{x,y\}\right)^2, \alpha\left(\max\{x,y\}\right)^2\right] \cdot e^t\\
&\quad= \frac{1}{9}\left[\left(\max\{x,y\}\right)^2 + \left(\max\{x,y\}\right)^2, \alpha\left(\left(\max\{x,y\}\right)^2 + \left(\max\{x,y\}\right)^2\right)\right] \cdot e^t\\
&\quad\preccurlyeq \frac{1}{9}F(x,x,y)(t),
\end{aligned}$$

where $k = \frac{1}{9}$. Clearly, T is a generalized Lipschitz map in X.

Now we review some facts on c-sequence theory.

Definition 5.3 ([27]) Let P be a solid cone in a Banach space E. A sequence $\{u_n\} \subset P$ is said to be a c-sequence if for each $c \gg \theta$ there exists a natural number N such that $u_n \ll c$ for all $n > N$.

Lemma 5.4 ([28]) *If E is a real Banach space with a solid cone P and $\{u_n\} \subset P$ is a sequence with $\|u_n\| \to 0$ $(n \to \infty)$, then u_n is a c-sequence.*

Lemma 5.5 ([24]) *Let A be a Banach algebra with a unit $e, k \in A$, then $\lim_{n\to\infty} \|k^n\|^{\frac{1}{n}}$ exists and the spectral radius $\rho(k)$ satisfies*

$$\rho(k) = \lim_{n\to\infty} \left\|k^n\right\|^{\frac{1}{n}} = \inf\left\|k^n\right\|^{\frac{1}{n}}.$$

If $\rho(k) < |\lambda|$, then $(\lambda e - k)$ is invertible in A; moreover,

$$(\lambda e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}},$$

where λ is a complex constant.

Lemma 5.6 ([24]) *Let A be a Banach algebra with a unit $e, a, b \in A$. If a commutes with b, then*

$$\rho(a+b) \leq \rho(a) + \rho(b),$$
$$\rho(ab) \leq \rho(a)\rho(b).$$

Lemma 5.7 ([28]) *If E is a real Banach space with a solid cone P*

(1) *If $a, b, c \in E$ and $a \leq b \ll c$, then $a \ll c$.*
(2) *If $a \in P$ and $a \ll c$ for each $c \gg \theta$, then $a = \theta$.*

Lemma 5.8 ([16]) *Let P be a solid cone in a Banach algebra A. Suppose that $k \in P$ and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.*

Lemma 5.9 ([28]) *Let A be a Banach algebra with a unit e and $k \in A$. If λ is a complex constant and $\rho(k) < |\lambda|$, then*

$$\rho\left((\lambda e - k)^{-1}\right) \leq \frac{1}{|\lambda| - \rho(k)}.$$

Lemma 5.10 ([28]) *Let A be a Banach algebra with a unit e and P be a solid cone in A. Let $a, k, l \in P$ hold $l \preccurlyeq k$ and $a \preccurlyeq la$. If $\rho(k) < 1$, then $a = \theta$.*

6 Applications to fixed point theory

In this section, we prove some famous fixed point theorems satisfying generalized Lipschitz maps in the framework of F-cone metric space over Banach algebra A.

Theorem 6.1 *Let (X, F) be a θ-complete F-cone metric space over Banach algebra A and suppose that $T : X \to X$ is a mapping satisfying the following condition:*

$$F(Tx, Tx, Ty) \preccurlyeq kF(x, x, y), \tag{6.1}$$

where $\rho(k) < 1$. Then T has a unique fixed point. For each $x \in X$, the sequence of iterates $\{T^n x\}_{n\geq 1}$ converges to the fixed point.

Proof For each $x_0 \in X$ and $n \geq 1$, set $x_1 = Tx_0$ and $x_{n+1} = T^{n+1}x_0$. Then

$$\begin{aligned}
F(x_n, x_n, x_{n+1}) &= F(Tx_{n-1}, Tx_{n-1}, Tx_n) \\
&\preccurlyeq kF(x_{n-1}, x_{n-1}, x_n) \\
&\preccurlyeq k^2 F(x_{n-2}, x_{n-2}, x_{n-1}) \\
&\vdots \\
&\preccurlyeq k^n F(x_0, x_0, x_1).
\end{aligned}$$

So, for $m > n$,

$$\begin{aligned}
F(x_n, x_n, x_m) &\preccurlyeq s\bigl[F(x_n, x_n, x_{n+1}) + F(x_n, x_n, x_{n+1}) + F(x_m, x_m, x_{n+1}) - F(x_{n+1}, x_{n+1}, x_{n+1})\bigr] \\
&\preccurlyeq s\bigl[2F(x_n, x_n, x_{n+1}) + F(x_{n+1}, x_{n+1}, x_m)\bigr] \\
&= 2sF(x_n, x_n, x_{n+1}) + sF(x_{n+1}, x_{n+1}, x_m) \\
&\preccurlyeq 2sF(x_n, x_n, x_{n+1}) + s^2\bigl[F(x_{n+1}, x_{n+1}, x_{n+2}) + F(x_{n+1}, x_{n+1}, x_{n+2}) \\
&\quad + F(x_m, x_m, x_{n+2}) - F(x_{n+2}, x_{n+2}, x_{n+2})\bigr] \\
&\preccurlyeq 2sF(x_n, x_n, x_{n+1}) + 2s^2F(x_{n+1}, x_{n+1}, x_{n+2}) + s^2F(x_{n+2}, x_{n+2}, x_m) \\
&\preccurlyeq 2sF(x_n, x_n, x_{n+1}) + 2s^2F(x_{n+1}, x_{n+1}, x_{n+2}) + 2s^3F(x_{n+2}, x_{n+2}, x_{n+3}) \\
&\quad + - - - + 2s^{m-n}F(x_{m-1}, x_{m-1}, x_m) \\
&\preccurlyeq 2sk^nF(x_0, x_0, x_1) + 2s^2k^{n+1}F(x_0, x_0, x_1) + 2s^3k^{n+2}F(x_0, x_0, x_1) \\
&\quad + - - - + 2s^{m-n}k^{m-1}F(x_0, x_0, x_1) \\
&\preccurlyeq \bigl(2s^nk^n + 2s^{n+1}k^{n+1} + 2s^{n+2}k^{n+2} + - - - + 2s^{m-1}k^{m-1}\bigr)F(x_0, x_0, x_1) \\
&= 2s^nk^n\bigl[e + (sk) + (sk)^2 + - - - + (sk)^{m-n-1}\bigr]F(x_0, x_0, x_1) \\
&\preccurlyeq 2(sk)^n(e - sk)^{-1}F(x_0, x_0, x_1).
\end{aligned}$$

By Remark 2.4, $\|(sk)^n \cdot F(x_0, x_0, x_1)\| \leq \|(sk)^n\|\|F(x_0, x_0, x_1)\| \to 0$. By Lemma 5.4, we have $\{(sk)^nF(x_0, x_0, x_1)\}$ is a c-sequence. Next, by using Lemmas 5.7 and 5.8, we conclude that $\{x_n\}$ is a θ-Cauchy sequence in X.

By the θ-completeness of X, there exists $u \in X$ such that

$$\begin{aligned}
\lim_{n\to\infty} F(x_n, x_n, u) &= \lim_{n\to\infty} F(x_n, x_n, x_m) \\
&= F(u, u, u) = \theta.
\end{aligned}$$

Furthermore, one has

$$\begin{aligned}
F(u, u, Tu) &\preccurlyeq s\bigl[F(u, u, Tx_n) + F(u, u, Tx_n) + F(Tu, Tu, Tx_n) - F(Tx_n, Tx_n, Tx_n)\bigr] \\
&\preccurlyeq s\bigl[2F(u, u, Tx_n) + F(Tu, Tu, Tx_n)\bigr] \\
&= s\bigl[2F(u, u, x_{n+1}) + F(u, u, x_n)\bigr].
\end{aligned}$$

Now that $\{F(u,u,x_{n+1})\}$ and $\{F(u,u,x_n)\}$ are c-sequences, by using Lemmas 5.7 and 5.8, we conclude that $Tu = u$. Thus u is a fixed point of T.

Finally, we prove the uniqueness of the fixed point. In fact, if v is another fixed point,

$$\begin{aligned} F(u,u,v) &= F(Tu,Tu,Tv) \\ &\preccurlyeq kF(u,u,v). \end{aligned}$$

That is,

$$(e-k)F(u,u,v) \preccurlyeq \theta.$$

Multiplying both sides above by

$$(e-k)^{-1} = \sum_{i=0}^{\infty} k^i \geq 0,$$

we get $F(u,u,v) \preccurlyeq \theta$. Thus, $F(u,u,v) = \theta$, which implies that $u = v$, a contradiction.

Hence, the fixed point is unique. □

Corollary 6.2 *Let (X,F) be a θ-complete F-cone metric space over Banach algebra A. Suppose that a mapping $T : X \to X$ satisfies, for some positive integer n,*

$$F\bigl(T^n x, T^n x, T^n y\bigr) \preccurlyeq kF(x,x,y) \tag{6.2}$$

for all $x,y \in X$, where k is a vector with $\rho(k) < \frac{1}{s}$. Then T has a unique fixed point in X.

Proof From Theorem 6.1, T^n has a unique fixed point x^*. But $T^n(Tx^*) = T(T^n x^*) = Tx^*$. So, Tx^* is also a fixed point of T^n. Hence $Tx^* = x^*$, x^* is a fixed point of T. Since the fixed point of T is also a fixed point of T^n, then the fixed point of T is unique. □

We now prove Chatterjee's fixed point theorem in the new space.

Theorem 6.3 *Let (X,F) be a θ-complete F-cone metric space over a Banach algebra A, and let P be the underlying cone with $k \in P$ with $\rho(k) < \frac{1}{s+1}$. Suppose that a mapping $T : X \to X$ satisfies the generalized Lipschitz condition*

$$F(Tx,Tx,Ty) \preccurlyeq k\bigl[F(Tx,Tx,x) + F(Ty,Ty,y)\bigr] \tag{6.3}$$

for all $x,y \in X$. Then T has a unique fixed point in X. And for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof Let $x_0 \in X$ be arbitrarily given and set $x_n = T^n x$, $n \geq 1$. We have

$$\begin{aligned} F(x_{n+1},x_{n+1},x_n) &= F(Tx_n,Tx_n,Tx_{n-1}) \\ &\preccurlyeq k\bigl[F(Tx_n,Tx_n,x_n) + F(Tx_{n-1},Tx_{n-1},x_{n-1})\bigr] \\ &\preccurlyeq k\bigl[F(x_{n+1},x_{n+1},x_n) + F(x_n,x_n,x_{n-1})\bigr], \end{aligned}$$

which implies

$$(e-k)F(x_{n+1},x_{n+1},x_n) \preccurlyeq kF(x_{n-1},x_{n-1},x_n). \tag{6.3.1}$$

Note that $\rho(k) < (s+1)\rho(k) < 1$.

Then by Lemma 5.5 it follows that $(e-k)$ is invertible.

Multiplying both sides of (6.3.1) by $(e-k)^{-1}$, we get

$$F(x_{n+1},x_{n+1},x_n) \preccurlyeq (e-k)^{-1} \cdot kF(x_{n-1},x_{n-1},x_n).$$

As is shown in the proof of Theorem 6.1, $\{x_n\}$ is a θ-Cauchy sequence, and by the θ-completeness of X, there exists $z \in X$ such that

$$\begin{aligned}\lim_{n\to\infty} F(x_n,x_n,z) &= \lim_{n,m\to\infty} F(x_n,x_n,x_m)\\ &= F(z,z,z)\\ &= \theta.\end{aligned}$$

Put $h = (e-k)^{-1} \cdot k$, $\therefore F(x_{n+1},x_{n+1},x_n) \preccurlyeq hF(x_{n-1},x_{n-1},x_n)$

$$\begin{aligned}\rho(h) &= \rho\big[(e-k)^{-1} \cdot k\big]\\ &\le \rho(e-k)^{-1} \cdot \rho(k)\\ &\le \frac{\rho(k)}{1-\rho(k)} < \frac{1}{s}.\end{aligned}$$

We shall show that z is a fixed point of T.

We have

$$\begin{aligned}F(z,z,Tz) &\preccurlyeq s\big[F(z,z,Tx_n) + F(z,z,Tx_n) + F(Tz,Tz,Tx_n) - F(Tx_n,Tx_n,Tx_n)\big]\\ &\preccurlyeq s\big[2F(z,z,Tx_n) + F(Tz,Tz,Tx_n)\big]\\ &\preccurlyeq s\big[2F(z,z,x_{n+1}) + k\big(F(Tz,Tz,z) + F(Tx_n,Tx_n,x_n)\big)\big]\\ &= 2sF(z,z,x_{n+1}) + skF(z,z,Tz) + skF(x_{n+1},x_{n+1},x_n),\end{aligned}$$

which implies that

$$(e-sk)F(z,z,Tz) \preccurlyeq 2sF(z,z,x_{n+1}) + skF(x_{n+1},x_{n+1},x_n).$$

Since $\rho(sk) < (s+1)\rho(k) < 1$, it concludes by Lemma 5.5 that $(e-sk)$ is invertible. So,

$$F(z,z,Tz) \preccurlyeq (e-sk)^{-1} \cdot \big[2sF(z,z,x_{n+1}) + skF(x_{n+1},x_{n+1},x_n)\big].$$

Now that $\{F(z,z,x_{n+1})\}$ and $\{F(x_{n+1},x_{n+1},x_n)\}$ are c-sequences, then by Lemmas 5.7 and 5.8, it concludes that $Tz = z$. Then z is a fixed point of T.

Finally, we prove the uniqueness of the fixed point. In fact, if v is another fixed point, then

$$\begin{aligned} F(v,v,z) &= F(Tv,Tv,Tz) \\ &\preccurlyeq k\bigl[F(Tv,Tv,v)+F(Tz,Tz,z)\bigr] \\ &= k\bigl[F(v,v,v)+F(z,z,z)\bigr]. \end{aligned}$$

So, $F(v,v,z) \preccurlyeq \theta$, which is a contradiction. Hence $F(v,v,z) = \theta$.

So, $v = z$.

Hence the fixed point is unique. □

Example 6.4 Let the Banach algebra A and the cone P be the same ones as those in Example 3.3, and let $X = R^{+}$. Define a mapping $F : X^3 \to A$ as in Example 3.3. We make a conclusion that (X,F) is a θ-complete F-cone metric space over Banach algebra A. Now define the mapping $T : X \to X$ by $T(x) = \cos\frac{x}{2} - 1$.

Then

$$\begin{aligned} &F(Tx,Tx,Ty)(t) \\ &\quad = \bigl(\bigl(\max\{Tx,Ty\}\bigr)^2 + \bigl(\max\{Tx,Ty\}\bigr)^2, \alpha\bigl(\bigl(\max\{Tx,Ty\}\bigr)^2 \\ &\qquad + \bigl(\max\{Tx,Ty\}\bigr)^2\bigr)\bigr)e^t \\ &\quad = 2\bigl(\bigl(\max\{Tx,Ty\}\bigr)^2, \alpha\bigl(\max\{Tx,Ty\}\bigr)^2\bigr)e^t \\ &\quad = 2\left(\left(\max\left\{\cos\frac{x}{2}-1, \cos\frac{y}{2}-1\right\}\right)^2, \alpha\left(\max\left\{\cos\frac{x}{2}-1, \cos\frac{y}{2}-1\right\}\right)^2\right)e^t \\ &\quad < 2\left(\left(\max\left\{\cos\frac{x}{2}, \cos\frac{y}{2}\right\}\right)^2, \alpha\left(\max\left\{\cos\frac{x}{2}, \cos\frac{y}{2}\right\}\right)^2\right)e^t \\ &\quad \preccurlyeq 2\left(\left(\max\left\{\frac{x}{2}, \frac{y}{2}\right\}\right)^2, \alpha\left(\max\left\{\frac{x}{2}, \frac{y}{2}\right\}\right)^2\right)e^t \\ &\quad = \frac{1}{4}\bigl[2\bigl(\max\{x,y\}\bigr)^2, 2\alpha\bigl(\max\{x,y\}\bigr)^2\bigr]e^t \\ &\quad = \frac{1}{4}F(x,x,y)(t), \end{aligned}$$

where $k = \frac{1}{4}$, then all the conditions of Theorem 6.1 hold trivially good and 0 is the unique fixed point of T. Clearly, T is a generalized Lipschitz map in X.

7 Expansive mapping on *F*-cone metric space over Banach algebra

In this section, we define expansive maps in F-cone metric spaces over Banach algebra.

Definition 7.1 Let (X,F) be an F-cone metric space over Banach algebra A and P be a cone in A. A map $T : X \to X$ is said to be an expansive mapping where $k, k^{-1} \in P$ are called the generalized Lipschitz constants with $\rho(k^{-1}) < 1$ for all $x, y \in X$ such that

$$F(Tx,Tx,Ty) \succcurlyeq kF(x,x,y).$$

Example 7.2 Let the Banach algebra A and the cone P be the same ones as those in Example 3.3, and let $X = R^+$. Define a mapping $F : X^3 \to A$ as in Example 3.3. Then (X, F) is an F-cone metric space over Banach algebra A. Now define the mapping $T : X \to X$ by $T(x) = 2x$. Then, for all $x, y \in X$, we have

$$\begin{aligned}
&F(Tx, Tx, Ty)(t)\\
&\quad= \bigl(\bigl(\max\{Tx, Ty\}\bigr)^2 + \bigl(\max\{Tx, Ty\}\bigr)^2, \alpha\bigl(\bigl(\max\{Tx, Ty\}\bigr)^2 + \bigl(\max\{Tx, Ty\}\bigr)^2\bigr)\bigr)e^t\\
&\quad= 2\bigl(\bigl(\max\{Tx, Ty\}\bigr)^2, \alpha\bigl(\max\{Tx, Ty\}\bigr)^2\bigr)e^t\\
&\quad= 2\bigl(\bigl(\max\{2x, 2y\}\bigr)^2, \alpha\bigl(\max\{2x, 2y\}\bigr)^2\bigr)e^t\\
&\quad= 8\bigl(\bigl(\max\{x, y\}\bigr)^2, \alpha\bigl(\max\{x, y\}\bigr)^2\bigr)e^t\\
&\quad\succcurlyeq 4\bigl[\bigl(\bigl(\max\{x, y\}\bigr)^2 + \bigl(\max\{x, y\}\bigr)^2\bigr), \alpha\bigl(\bigl(\max\{x, y\}\bigr)^2 + \bigl(\max\{x, y\}\bigr)^2\bigr)\bigr]e^t\\
&\quad\succcurlyeq 4F(x, x, y)(t),
\end{aligned}$$

where $k = 4$. Clearly, T is an expansive map in X.

Now we present a fixed point theorem for such maps.

Theorem 7.3 *Let (X, F) be a θ-complete F-cone metric space over Banach algebra, and let P be an underlying solid cone with $k \in P$ with $\rho((e + k + sk - a)(b + c - 2sk)^{-1}) < \frac{1}{s}$. Let f and g be two surjective selfmaps of X satisfying*

$$F(fx, fx, gy) + k\bigl[F(x, x, gy) + F(y, y, fx)\bigr] \succcurlyeq aF(x, x, fx) + bF(y, y, gy) + cF(x, x, y) \tag{7.3.1}$$

for all $x, y \in X$, and then f and g have a unique common fixed point in X.

Proof We define a sequence x_n as follows for $n = 0, 1, 2, 3, \ldots$:

$$x_{2n} = fx_{2n+1}, \qquad x_{2n+1} = gx_{2n+2}.$$

If $x_{2n} = x_{2n+1} = x_{2n+2}$ for some n, then we say that x_{2n} is a fixed point of f and g. Therefore, we suppose that no two consecutive terms of the sequence $\{x_n\}$ are equal.

Now, putting $x = x_{2n+1}$ and $y = x_{2n+2}$ in (7.3.1), we get

$$\begin{aligned}
&F(fx_{2n+1}, fx_{2n+1}, gx_{2n+2}) + k\bigl[F(x_{2n+1}, x_{2n+1}, gx_{2n+2}) + F(x_{2n+2}, x_{2n+2}, fx_{2n+1})\bigr]\\
&\quad\succcurlyeq aF(x_{2n+1}, x_{2n+1}, fx_{2n+1}) + bF(x_{2n+2}, x_{2n+2}, gx_{2n+2}) + cF(x_{2n+1}, x_{2n+1}, x_{2n+2}),\\
&F(x_{2n}, x_{2n}, x_{2n+1}) + k\bigl[F(x_{2n+1}, x_{2n+1}, x_{2n+1}) + F(x_{2n+2}, x_{2n+2}, x_{2n})\bigr]\\
&\quad\succcurlyeq aF(x_{2n+1}, x_{2n+1}, x_{2n}) + bF(x_{2n+2}, x_{2n+1}, x_{2n+1}) + cF(x_{2n+1}, x_{2n+1}, x_{2n+2}),\\
&aF(x_{2n+1}, x_{2n+1}, x_{2n}) + bF(x_{2n+2}, x_{2n+1}, x_{2n+1}) + cF(x_{2n+1}, x_{2n+1}, x_{2n+2})\\
&\quad\preccurlyeq F(x_{2n}, x_{2n}, x_{2n+1}) + k\bigl[F(x_{2n+1}, x_{2n+1}, x_{2n}) + s\bigl(F(x_{2n+2}, x_{2n+2}, x_{2n+1}),\\
&\qquad + F(x_{2n+2}, x_{2n+2}, x_{2n+1}) + F(x_{2n}, x_{2n}, x_{2n+1}) - F(x_{2n+1}, x_{2n+1}, x_{2n+1})\bigr)\bigr] \quad \bigl[\text{By } (F_2)\bigr],\\
&aF(x_{2n+1}, x_{2n+1}, x_{2n}) + bF(x_{2n+2}, x_{2n+1}, x_{2n+1}) + cF(x_{2n+1}, x_{2n+1}, x_{2n+2})
\end{aligned}$$

$$\begin{aligned}
&\preccurlyeq F(x_{2n},x_{2n},x_{2n+1}) + kF(x_{2n+1},x_{2n+1},x_{2n}) + 2skF(x_{2n+2},x_{2n+2},x_{2n+1})\\
&\quad + skF(x_{2n},x_{2n},x_{2n+1}) - skF(x_{2n},x_{2n},x_{2n+1}),\\
&(b+c-2sk)F(x_{2n+1},x_{2n+1},x_{2n+2}) \preccurlyeq (e+k+sk-a)F(x_{2n},x_{2n},x_{2n+1}).
\end{aligned}$$

Put $b+c-2sk=r$, then

$$rF(x_{2n+1},x_{2n+1},x_{2n+2}) \preccurlyeq (e+k+sk-a)F(x_{2n},x_{2n},x_{2n+1}). \tag{7.3.2}$$

Since r is invertible, to multiply r^{-1} on both sides of (7.3.2),

$$F(x_{2n+1},x_{2n+1},x_{2n+2}) \preccurlyeq hF(x_{2n},x_{2n},x_{2n+1}),$$

where $h=(e+k+sk-a)(b+c-2sk)^{-1}$.

Note that $\rho(h) < \frac{1}{s}$.

Hence by the proof of Theorem 6.1, we can easily see that the sequence $\{x_n\}$ is a θ-Cauchy sequence. Moreover, by the θ-completeness of X, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} F\bigl(x_n,x_n,x^*\bigr) = \lim_{n,m\to\infty} F(x_n,x_n,x_m) = F\bigl(x^*,x^*,x^*\bigr) = \theta.$$

Since f and g are surjective maps and hence there exist two points y and y' in X such that $x^*=fy$ and $x^*=gy'$.

Consider

$$\begin{aligned}
F\bigl(x_{2n},x_{2n},x^*\bigr) &= F\bigl(fx_{2n+1},fx_{2n+1},gy'\bigr)\\
&\succcurlyeq -k\bigl[F\bigl(x_{2n+1},x_{2n+1},gy'\bigr) + F\bigl(y',y',fx_{2n+1}\bigr)\bigr] + aF(x_{2n+1},x_{2n+1},fx_{2n+1})\\
&\quad + bF\bigl(y',y',gy'\bigr) + cF\bigl(x_{2n+1},x_{2n+1},y'\bigr).
\end{aligned}$$

Then

$$\begin{aligned}
F\bigl(x_{2n},x_{2n},x^*\bigr) \succcurlyeq{}& -kF\bigl(x_{2n+1},x_{2n+1},x^*\bigr) - kF\bigl(y',y',x_{2n}\bigr) + aF(x_{2n+1},x_{2n+1},x_{2n})\\
&+ bF\bigl(y',y',x^*\bigr) + cF\bigl(x_{2n+1},x_{2n+1},y'\bigr).
\end{aligned}$$

Since

$$F\bigl(y',y',x^*\bigr) \preccurlyeq s\bigl[2F\bigl(y',y',x_{2n+1}\bigr) + F\bigl(x^*,x^*,x_{2n+1}\bigr) - F(x_{2n+1},x_{2n+1},x_{2n+1})\bigr],$$

so

$$\begin{aligned}
&-kF\bigl(x_{2n+1},x_{2n+1},x^*\bigr) - kF\bigl(y',y',x_{2n}\bigr) + aF(x_{2n+1},x_{2n+1},x_{2n}) + cF\bigl(x_{2n+1},x_{2n+1},y'\bigr)\\
&\quad\preccurlyeq F\bigl(x_{2n},x_{2n},x^*\bigr) - b\bigl[2sF\bigl(y',y',x_{2n+1}\bigr) + sF\bigl(x^*,x^*,x_{2n+1}\bigr) - sF(x_{2n+1},x_{2n+1},x_{2n+1})\bigr],\\
&-kF\bigl(x_{2n+1},x_{2n+1},x^*\bigr) + aF(x_{2n+1},x_{2n+1},x_{2n}) + cF\bigl(x_{2n+1},x_{2n+1},y'\bigr)\\
&\quad\preccurlyeq ks\bigl[2F(x_{2n},x_{2n},x_{2n+1}) + F\bigl(x_{2n+1},x_{2n+1},y'\bigr) - F(x_{2n+1},x_{2n+1},x_{2n+1})\bigr]
\end{aligned}$$

$$\begin{aligned}&+ s\bigl[2F(x_{2n}, x_{2n}, x_{2n+1}) + F\bigl(x_{2n+1}, x_{2n+1}, x^*\bigr)\bigr]\\&- b\bigl[2sF\bigl(y', y', x_{2n+1}\bigr) + sF\bigl(x^*, x^*, x_{2n+1}\bigr) - F(x_{2n+1}, x_{2n+1}, x_{2n+1})\bigr],\end{aligned}$$

which implies

$$\begin{aligned}&(2sb + c - sk)F\bigl(x_{2n+1}, x_{2n+1}, y'\bigr)\\&\quad\preccurlyeq (s - bs + k)F\bigl(x_{2n+1}, x_{2n+1}, x^*\bigr) + (2sk + 2s - a)F(x_{2n+1}, x_{2n+1}, x_{2n}).\end{aligned}$$

Since $2sb + c - sk = r$ is invertible, we have

$$\begin{aligned}&rF\bigl(x_{2n+1}, x_{2n+1}, y'\bigr)\\&\quad\preccurlyeq (s - bs + k) \cdot F\bigl(x_{2n+1}, x_{2n+1}, x^*\bigr) + (2sk + 2s - a) \cdot F(x_{2n+1}, x_{2n+1}, x_{2n}),\\&F\bigl(x_{2n+1}, x_{2n+1}, y'\bigr)\\&\quad\preccurlyeq r^{-1}\bigl\{(s - bs + k)F\bigl(x_{2n+1}, x_{2n+1}, x^*\bigr) + (2sk + 2s - a)F(x_{2n+1}, x_{2n+1}, x_{2n})\bigr\}.\end{aligned}$$

Now that $\{F(x_{2n+1}, x_{2n+1}, x^*)\}$ and $\{F(x_{2n+1}, x_{2n+1}, x_{2n})\}$ are c-sequences, then by using Lemmas 5.7 and 5.8, we conclude that $gx_{n+2} = y'$.

Finally, we prove the uniqueness of the fixed point. In fact, if y^* is another common fixed point of f and g, that is, $fy^* = y^*$ and $gy^* = y^*$,

$$\begin{aligned}&F\bigl(x, x, y^*\bigr)\\&\quad= F\bigl(Tx, Tx, Ty^*\bigr)\\&\quad\succcurlyeq -k\bigl[F\bigl(x, x, gy^*\bigr) + F\bigl(y^*, y^*, fx\bigr)\bigr] + aF(x, x, fx) + bF\bigl(y^*, y^*, gy^*\bigr) + cF\bigl(x, x, y^*\bigr)\\&\Rightarrow\quad F\bigl(x, x, y^*\bigr) \succcurlyeq -k\bigl[F\bigl(x, x, y^*\bigr) + F\bigl(y^*, y^*, x\bigr)\bigr] + aF(x, x, x) + bF\bigl(y^*, y^*, y^*\bigr)\\&\qquad\qquad + cF\bigl(x, x, y^*\bigr)\\&\Rightarrow\quad F\bigl(x, x, y^*\bigr) \succcurlyeq (-2k + c)F\bigl(x, x, y^*\bigr)\end{aligned}$$

or

$$\begin{aligned}&(c - 2k)F\bigl(x, x, y^*\bigr) \preccurlyeq F\bigl(x, x, y^*\bigr),\\&(c - 2k - e)F\bigl(x, x, y^*\bigr) \preccurlyeq \theta,\end{aligned}$$

which means $F(x, x, y^*) = \theta$, which implies that $x = y^*$, a contradiction. Hence the fixed point is unique. □

Corollary 7.4 *Let (X, F) be a θ-complete F-cone metric space over Banach algebra, and let P be an underlying solid cone, where $c \in P$ is a generalized Lipschitz constant with $\rho(c)^{-1} < \frac{1}{s}$. Let f and g be two surjective selfmaps of X satisfying*

$$F(fx, fx, gy) \succcurlyeq cF(x, x, y). \tag{7.3}$$

Then f and g have a unique common fixed point in X.

Proof If we put $k, a, b = \theta$ in Theorem 7.3, then we get the above Corollary 7.4. □

Corollary 7.5 *Let (X, F) be a θ-complete F-cone metric space over Banach algebra, and let P be an underlying solid cone, where $c \in P$ is a generalized Lipschitz constant with $\rho(c)^{-1} < \frac{1}{s}$. Let f be a surjective selfmap of X satisfying*

$$F(fx, fx, fy) \succcurlyeq cF(x, x, y). \tag{7.4}$$

Then f has a unique fixed point in X.

Proof If we put $f = g$ in Corollary 7.4, then we get the above Corollary 7.5 which is an extension of Theorem 1 of Wang et al. [29] in an F-cone metric space over Banach algebra. □

Corollary 7.6 *Let (X, F) be a θ-complete F-cone metric space over Banach algebra, and let P be an underlying solid cone, where $c \in P$ is a generalized Lipschitz constant with $\rho(c)^{-1} < \frac{1}{s}$. Let f be a surjective selfmap of X, and suppose that there exists a positive integer n satisfying*

$$F\bigl(f^n x, f^n x, f^n y\bigr) \succcurlyeq cF(x, x, y). \tag{7.5}$$

Then f has a unique fixed point in X.

Proof From Corollary 7.5 f^n has a unique fixed point z. But $f^n(fz) = f(f^n z) = fz$, so fz is also a fixed point of f^n. Hence $fz = z$, z is a fixed point of f. Since the fixed point of f is also a fixed point of f^n, the fixed point of f is unique. □

Corollary 7.7 *Let (X, F) be a θ-complete F-cone metric space over Banach algebra, and let P be an underlying solid cone, where $a, b, c, -a \in P$ are generalized Lipschitz constants with $\rho[(e - a)(b + c)^{-1}] < \frac{1}{s}$. Let f and g be two surjective selfmaps of X satisfying*

$$F(fx, fx, gy) \succcurlyeq aF(x, x, fx) + bF(y, y, gy) + cF(x, x, y). \tag{7.6}$$

Then f and g have a unique common fixed point in X.

Proof If we put $k = \theta$ in Theorem 7.3, then we get the above Corollary 7.7. □

Corollary 7.8 *Let (X, F) be a θ-complete F-cone metric space over Banach algebra, and let P be an underlying solid cone, where $a, b, c, -a \in P$ are generalized Lipschitz constants with $\rho[(e - a)(b + c)^{-1}] < \frac{1}{s}$. Let f be a surjective selfmap of X satisfying*

$$F(fx, fx, fy) \succcurlyeq aF(x, x, fx) + bF(y, y, fy) + cF(x, x, y). \tag{7.7}$$

Then f has a unique fixed point in X.

Proof If we put $f = g$ in Corollary 7.7, then we get the above Corollary 7.8 which is an extension of Theorem 2 of Wang et al. [29] in an F-cone metric space over Banach algebra. □

8 Conclusion

In this paper, we introduce an F-cone metric space over Banach algebra which generalizes an N_p-cone metric space over Banach algebra and an N_b-cone metric space over Banach algebra. We introduce the concept of generalized Lipschitz and expansive mapping in the new structure. Also we derive the existence and uniqueness of some fixed point theorems for such spaces. Our main theorems extend and unify the existing results in the recent literature. Example is constructed to support our result.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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