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Some applications via fixed point results in partially ordered S_b-metric spaces

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Abstract

In this paper we give some applications to integral equations as well as homotopy theory via fixed point theorems in partially ordered complete S_b-metric spaces by using generalized contractive conditions. We also furnish an example which supports our main result.

Keywords: S_b-metric space; *w*-compatible pairs; S_b-completeness

1 Introduction

Banach contraction principle in metric spaces is one of the most important results in fixed theory and nonlinear analysis in general. Since 1922, when Stefan Banach [1] formulated the concept of contraction and posted a famous theorem, scientists around the world have published new results related to the generalization of a metric space or with contractive mappings (see [1–24]). Banach contraction principle is considered to be the initial result of the study of fixed point theory in metric spaces.

In the year 1989, Bakhtin introduced the concept of b-metric spaces as a generalization of metric spaces [6]. Later several authors proved so many results on b-metric spaces (see [13–16]). Mustafa and Sims defined the concept of a generalized metric space which is called a G-metric space [12]. Sedghi, Shobe and Aliouche gave the notion of an S-metric space and proved some fixed point theorems for a self-mapping on a complete S-metric space [22]. Aghajani, Abbas and Roshan presented a new type of metric which is called G_b-metric and studied some properties of this metric [2].

Recently Sedghi et al. [20] defined S_b-metric spaces using the concept of S-metric spaces [22].

The aim of this paper is to prove some unique fixed point theorems for generalized contractive conditions in complete S_b-metric spaces. Also, we give applications to integral equations as well as homotopy theory. Throughout this paper R, R^+ and N denote the sets of all real numbers, non-negative real numbers and positive integers, respectively.

First we recall some definitions, lemmas and examples.

2 Preliminaries

Definition 2.1 ([22]) Let X be a non-empty set. An S-metric on X is a function $S : X^3 \to [0, +\infty)$ that satisfies the following conditions for each $x, y, z, a \in X$:

$(S1)$: $0 < S(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z \neq x$,

$(S2)$: $S(x,y,z) = 0$ if and only if $x = y = z$,
$(S3)$: $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$ for all $x,y,z,a \in X$.

Then the pair (X,S) is called an S-metric space.

Definition 2.2 ([20]) Let X be a non-empty set and $b \geq 1$ be a given real number. Suppose that a mapping $S_b : X^3 \to [0,\infty)$ is a function satisfying the following properties:

(S_b1) $0 < S_b(x,y,z)$ for all $x,y,z \in X$ with $x \neq y \neq z \neq x$,
(S_b2) $S_b(x,y,z) = 0$ if and only if $x = y = z$,
(S_b3) $S_b(x,y,z) \leq b(S_b(x,x,a) + S_b(y,y,a) + S_b(z,z,a))$ for all $x,y,z,a \in X$.

Then the function S_b is called an S_b-metric on X and the pair (X,S_b) is called an S_b-metric space.

Remark 2.3 ([20]) It should be noted that the class of S_b-metric spaces is effectively larger than that of S-metric spaces. Indeed each S-metric space is an S_b-metric space with $b = 1$.

The following example shows that an S_b-metric on X need not be an S-metric on X.

Example 2.4 ([20]) Let (X,S) be an S-metric space and $S_*(x,y,z) = S(x,y,z)^p$, where $p > 1$ is a real number. Note that S_* is an S_b-metric with $b = 2^{2(p-1)}$. Also, (X,S_*) is not necessarily an S-metric space.

Definition 2.5 ([20]) Let (X,S_b) be an S_b-metric space. Then, for $x \in X$, $r > 0$, we define the open ball $B_{S_b}(x,r)$ and the closed ball $B_{S_b}[x,r]$ with center x and radius r as follows, respectively:

$$B_{S_b}(x,r) = \{y \in X : S_b(y,y,x) < r\},$$
$$B_{S_b}[x,r] = \{y \in X : S_b(y,y,x) \leq r\}.$$

Lemma 2.6 ([20]) *In an S_b-metric space, we have*

$$S_b(x,x,y) \leq bS_b(y,y,x)$$

and

$$S_b(y,y,x) \leq bS_b(x,x,y).$$

Lemma 2.7 ([20]) *In an S_b-metric space, we have*

$$S_b(x,x,z) \leq 2bS_b(x,x,y) + b^2S_b(y,y,z).$$

Definition 2.8 ([20]) If (X,S_b) is an S_b-metric space, a sequence $\{x_n\}$ in X is said to be:

(1) S_b-Cauchy sequence if, for each $\epsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $S_b(x_n,x_n,x_m) < \epsilon$ for each $m,n \geq n_0$.
(2) S_b-convergent to a point $x \in X$ if, for each $\epsilon > 0$, there exists a positive integer n_0 such that $S_b(x_n,x_n,x) < \epsilon$ or $S_b(x,x,x_n) < \epsilon$ for all $n \geq n_0$, and we denote $\lim_{n\to\infty} x_n = x$.

Definition 2.9 ([20]) An S_b-metric space (X, S_b) is called complete if every S_b-Cauchy sequence is S_b-convergent in X.

Lemma 2.10 ([20]) *If (X, S_b) is an S_b-metric space with $b \geq 1$, and suppose that $\{x_n\}$ is S_b-convergent to x, then we have*

$$\text{(i)}\quad \frac{1}{2b} S_b(y,x,x) \leq \lim_{n\to\infty} \inf S_b(y,y,x_n) \leq \lim_{n\to\infty} \sup S_b(y,y,x_n) \leq 2bS_b(y,y,x)$$

and

$$\text{(ii)}\quad \frac{1}{b^2} S_b(x,x,y) \leq \lim_{n\to\infty} \inf S_b(x_n,x_n,y) \leq \lim_{n\to\infty} \sup S_b(x_n,x_n,y) \leq b^2 S_b(x,x,y)$$

for all $y \in X$.

In particular, if $x = y$, then we have $\lim_{n\to\infty} S_b(x_n,x_n,y) = 0$.

Now we prove our main results.

3 Results and discussions

Definition 3.1 Let $(X, S_b, \preceq)$ be a partially ordered complete S_b-metric space which is said to be regular if every two elements of X are comparable,

i.e., if $x, y \in X \Rightarrow$ either $x \preceq y$ or $y \preceq x$.

Definition 3.2 Let $(X, S_b, \preceq)$ be a partially ordered complete S_b-metric space which is also regular; let $f : X \to X$ be a mapping. We say that f satisfies (ψ, ϕ)-contraction if there exist $\psi, \phi : [0, \infty) \to [0, \infty)$ such that

(3.2.1) f is non-decreasing,
(3.2.2) ψ is continuous, monotonically non-decreasing and ϕ is lower semi-continuous,
(3.2.3) $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$,
(3.2.4) $\psi(4b^4 S_b(fx, fx, fy)) \leq \psi(M_f^i(x,y)) - \phi(M_f^i(x,y))$, $\forall x, y \in X$, $x \preceq y$, $i = 3, 4, 5$ and

$$M_f^5(x,y) = \max\left\{S_b(x,x,y), S_b(x,x,fx), S_b(y,y,fy), S_b(x,x,fy), S_b(y,y,fx)\right\},$$

$$M_f^4(x,y) = \max\left\{S_b(x,x,y), S_b(x,x,fx), S_b(y,y,fy), \frac{1}{4b^4}\left[S_b(x,x,fy) + S_b(y,y,fx)\right]\right\},$$

$$M_f^3(x,y) = \max\left\{S_b(x,x,y), \frac{1}{4b^4}\left[S_b(x,x,fx) + S_b(y,y,fy)\right], \frac{1}{4b^4}\left[S_b(x,x,fy) + S_b(y,y,fx)\right]\right\}.$$

Definition 3.3 Suppose that $(X, \preceq)$ is a partially ordered set and f is a mapping of X into itself. We say that f is non-decreasing if for every $x, y \in X$,

$$x \preceq y \text{ implies that } fx \preceq fy. \tag{1}$$

Theorem 3.4 *Let $(X, S_b, \preceq)$ be an ordered complete S_b metric space, which is also regular, and let $f : X \to X$ satisfy (ψ, ϕ)-contraction with $i = 5$. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a unique fixed point in X.*

Proof Since f is a mapping from X into X, there exists a sequence $\{x_n\}$ in X such that

$$x_{n+1} = fx_n, \quad n = 0, 1, 2, 3, \ldots.$$

Case (i): If $x_n = x_{n+1}$, then x_n is a fixed point of f.
Case (ii): Suppose $x_n \neq x_{n+1}$ $\forall n$.
Since $x_0 \preceq fx_0 = x_1$ and f is non-decreasing, it follows that

$$x_0 \preceq fx_0 \preceq f^2x_0 \preceq f^3x_0 \preceq \cdots \preceq f^nx_0 \preceq f^{n+1}x_0 \preceq \cdots.$$

Now

$$\begin{aligned}\psi\left(4b^4S_b\left(fx_0, fx_0, f^2x_0\right)\right) &= \psi\left(4b^4S_b(fx_0, fx_0, fx_1)\right)\\ &\leq \psi\left(M_f^5(x_0, x_1)\right) - \phi\left(M_f^5(x_0, x_1)\right),\end{aligned}$$

where

$$\begin{aligned}M_f^5(x_0, x_1) &= \max\left\{\begin{matrix} S_b(x_0, x_0, x_1), S_b(x_0, x_0, fx_0), S_b(x_1, x_1, fx_1) \\ S_b(x_0, x_0, f^2x_0), S_b(fx_0, fx_0, fx_0)\end{matrix}\right\}\\ &= \max\left\{S_b(x_0, x_0, fx_0), S_b\left(fx_0, fx_0, f^2x_0\right), S_b\left(x_0, x_0, f^2x_0\right)\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\psi\left(4b^4S_b\left(fx_0, fx_0, f^2x_0\right)\right)\\ &\quad\leq \psi\left(\max\left\{S_b(x_0, x_0, fx_0), S_b\left(fx_0, fx_0, f^2x_0\right), S_b\left(x_0, x_0, f^2x_0\right)\right\}\right)\\ &\qquad - \phi\left(\max\left\{S_b(x_0, x_0, fx_0), S_b\left(fx_0, fx_0, f^2x_0\right), S_b\left(x_0, x_0, f^2x_0\right)\right\}\right)\\ &\quad\leq \psi\left(\max\left\{S_b(x_0, x_0, fx_0), S_b\left(fx_0, fx_0, f^2x_0\right), S_b\left(x_0, x_0, f^2x_0\right)\right\}\right).\end{aligned}$$

By the definition of ψ, we have that

$$S_b\left(fx_0, fx_0, f^2x_0\right) \leq \max\left\{\begin{matrix} \frac{1}{4b^4}S_b(x_0, x_0, fx_0) \\ \frac{1}{4b^4}S_b(fx_0, fx_0, f^2x_0) \\ \frac{1}{4b^4}S_b(x_0, x_0, f^2x_0)\end{matrix}\right\}. \tag{2}$$

But

$$\begin{aligned}\frac{1}{4b^4}S_b\left(x_0, x_0, f^2x_0\right) &\leq \frac{1}{4b^4}\left[2bS_b(x_0, x_0, fx_0) + b^2S_b\left(fx_0, fx_0, f^2x_0\right)\right]\\ &\leq \max\left\{\frac{1}{b^3}S_b(x_0, x_0, fx_0), \frac{1}{2b^2}S_b\left(fx_0, fx_0, f^2x_0\right)\right\}.\end{aligned}$$

From (2) we have that

$$S_b\left(fx_0, fx_0, f^2x_0\right) \leq \max\left\{\frac{1}{b^3}S_b(x_0, x_0, fx_0), \frac{1}{2b^2}S_b\left(fx_0, fx_0, f^2x_0\right)\right\}.$$

If $\frac{1}{2b^2} S_b(fx_0, fx_0, f^2x_0)$ is maximum, we get a contradiction. Hence

$$S_b\bigl(fx_0, fx_0, f^2x_0\bigr) \le \frac{1}{b^3} S_b(x_0, x_0, fx_0). \tag{3}$$

Also

$$\begin{aligned}\psi\bigl(4b^4 S_b\bigl(f^2x_0, f^2x_0, f^3x_0\bigr)\bigr) &= \psi\bigl(4b^4 S_b(fx_1, fx_1, fx_2)\bigr)\\ &\le \psi\bigl(M_f^5(x_1, x_2)\bigr) - \phi\bigl(M_f^4(x_1, x_2)\bigr),\end{aligned}$$

where

$$\begin{aligned}M_f^5(x_1, x_2) &= \max\left\{\begin{matrix} S_b(fx_0, fx_0, f^2x_0), S_b(fx_0, fx_0, f^2x_0), S_b(f^2x_0, f^2x_0, f^3x_0) \\ S_b(fx_0, fx_0, f^3x_0), S_b(f^2x_0, f^2x_0, f^2x_0) \end{matrix}\right\}\\ &= \max\bigl\{S_b\bigl(fx_0, fx_0, f^2x_0\bigr), S_b\bigl(f^2x_0, f^2x_0, f^3x_0\bigr), S_b\bigl(fx_0, fx_0, f^3x_0\bigr)\bigr\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\psi\bigl(4b^4 S_b\bigl(f^2x_0, f^2x_0, f^3x_0\bigr)\bigr)\\ &\quad\le \psi\left(\max\left\{\begin{matrix} S_b(fx_0, fx_0, f^2x_0), S_b(f^2x_0, f^2x_0, f^3x_0) \\ S_b(fx_0, fx_0, f^3x_0) \end{matrix}\right\}\right)\\ &\qquad - \phi\left(\max\left\{\begin{matrix} S_b(fx_0, fx_0, f^2x_0), S_b(f^2x_0, f^2x_0, f^3x_0) \\ S_b(fx_0, fx_0, f^3x_0) \end{matrix}\right\}\right)\\ &\quad\le \psi\left(\max\left\{\begin{matrix} S_b(fx_0, fx_0, f^2x_0), S_b(f^2x_0, f^2x_0, f^3x_0) \\ S_b(fx_0, fx_0, f^3x_0) \end{matrix}\right\}\right).\end{aligned}$$

By the definition of ψ, we have that

$$S_b\bigl(f^2x_0, f^2x_0, f^3x_0\bigr) \le \max\left\{\begin{matrix} \frac{1}{4b^4} S_b(fx_0, fx_0, f^2x_0) \\ \frac{1}{4b^4} S_b(f^2x_0, f^2x_0, f^3x_0) \\ \frac{1}{4b^4} S_b(fx_0, fx_0, f^3x_0) \end{matrix}\right\}. \tag{4}$$

But

$$\begin{aligned}&\frac{1}{4b^4} S_b\bigl(fx_0, fx_0, f^3x_0\bigr)\\ &\quad\le \frac{1}{4b^4}\bigl[2bS_b\bigl(fx_0, fx_0, f^2x_0\bigr) + b^2 S_b\bigl(f^2x_0, f^2x_0, f^3x_0\bigr)\bigr]\\ &\quad\le \max\left\{\frac{1}{b^3} S_b\bigl(fx_0, fx_0, f^2x_0\bigr), \frac{1}{2b^2} S_b\bigl(f^2x_0, f^2x_0, f^3x_0\bigr)\right\}.\end{aligned}$$

From (4) we have that

$$S_b\bigl(f^2x_0, f^2x_0, f^3x_0\bigr) \le \max\left\{\frac{1}{b^3} S_b\bigl(fx_0, fx_0, f^2x_0\bigr), \frac{1}{2b^2} S_b\bigl(f^2x_0, f^2x_0, f^3x_0\bigr)\right\}.$$

If $\frac{1}{2b^2}S_b(f^2x_0,f^2x_0,f^3x_0)$ is maximum, we get a contradiction. Hence

$$\begin{aligned} S_b(f^2x_0,f^2x_0,f^3x_0) &\le \frac{1}{b^3}S_b(fx_0,fx_0,f^2x_0) \\ &\le \frac{1}{(b^3)^2}S_b(x_0,x_0,fx_0). \end{aligned}$$

Continuing this process, we can conclude that

$$\begin{aligned} S_b(f^nx_0,f^nx_0,f^{n+1}x_0) &\le \frac{1}{(b^3)^n}S_b(x_0,x_0,fx_0) \\ &\to 0 \quad \text{as } n\to\infty. \end{aligned}$$

That is,

$$\lim_{n\to\infty} S_b(f^nx_0,f^nx_0,f^{n+1}x_0) = 0. \tag{5}$$

Now we prove that $\{f^nx_0\}$ is an S_b-Cauchy sequence in (X,S_b). On the contrary, we suppose that $\{f^nx_0\}$ is not S_b-Cauchy. Then there exist $\epsilon > 0$ and monotonically increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$.

$$S_b(f^{m_k}x_0,f^{m_k}x_0,f^{n_k}x_0) \ge \epsilon \tag{6}$$

and

$$S_b(f^{m_k}x_0,f^{m_k}x_0,f^{n_k-1}x_0) < \epsilon. \tag{7}$$

From (6) and (7), we have

$$\begin{aligned} \epsilon &\le S_b(f^{m_k}x_0,f^{m_k}x_0,f^{n_k}x_0) \\ &\le 2bS_b(f^{m_k}x_0,f^{m_k}x_0,f^{m_k+1}x_0) \\ &\quad + b^2S_b(f^{m_k+1}x_0,f^{m_k+1}x_0,f^{n_k}x_0). \end{aligned}$$

So that

$$\begin{aligned} 4b^2\epsilon &\le 8b^3S_b(f^{m_k}x_0,f^{m_k}x_0,f^{m_k+1}x_0) \\ &\quad + 4b^4S_b(f^{m_k+1}x_0,f^{m_k+1}x_0,f^{n_k}x_0). \end{aligned}$$

Letting $k\to\infty$ and applying ψ on both sides, we have that

$$\begin{aligned} \psi(4b^2\epsilon) &\le \lim_{k\to\infty} \psi(4b^4S_b(f^{m_k+1}x_0,f^{m_k+1}x_0,f^{n_k}x_0)) \\ &= \lim_{k\to\infty} \psi(4b^4S_b(fx_{m_k},fx_{m_k},fx_{n_k-1})) \\ &\le \lim_{k\to\infty} \psi(M_f^5(x_{m_k},x_{n_k-1})) - \lim_{k\to\infty} \phi(M_f^5(x_{m_k},x_{n_k-1})), \end{aligned} \tag{8}$$

where

$$\begin{aligned}
&\lim_{k\to\infty} M_f^5(x_{m_k}, x_{n_k-1}) \\
&= \lim_{k\to\infty} \max \left\{ \begin{array}{c} S_b(f^{m_k}x_0, f^{m_k}x_0, f^{n_k-1}x_0), S_b(f^{m_k}x_0, f^{m_k}x_0, f^{m_k+1}x_0) \\ S_b(f^{n_k-1}x_0, f^{n_k-1}x_0, f^{n_k}x_0), S_b(f^{m_k}x_0, f^{m_k}x_0, f^{n_k}x_0) \\ S_b(f^{n_k-1}x_0, f^{n_k-1}x_0, f^{m_k+1}x_0) \end{array} \right\} \\
&< \lim_{k\to\infty} \max \left\{ \epsilon, 0, 0, S_b\big(f^{m_k}x_0, f^{m_k}x_0, f^{n_k}x_0\big), S_b\big(f^{n_k-1}x_0, f^{n_k-1}x_0, f^{m_k+1}x_0\big) \right\}.
\end{aligned}$$

But

$$\lim_{k\to\infty} S_b\big(f^{m_k}x_0, f^{m_k}x_0, f^{n_k}x_0\big) \le \lim_{k\to\infty} \left[\begin{array}{c} 2bS_b(f^{m_k}x_0, f^{m_k}x_0, f^{n_k-1}x_0) \\ + b^2 S_b(f^{n_k-1}x_0, f^{n_k-1}x_0, f^{n_k}x_0) \end{array} \right] < 2b\epsilon.$$

Also

$$\lim_{k\to\infty} S_b\big(f^{n_k-1}x_0, f^{n_k-1}x_0, f^{m_k+1}x_0\big) \le \lim_{k\to\infty} \left[\begin{array}{c} 2bS_b(f^{n_k-1}x_0, f^{n_k-1}x_0, f^{m_k}x_0) \\ + b^2 S_b(f^{m_k}x_0, f^{m_k}x_0, f^{m_k+1}x_0) \end{array} \right] < 2b^2\epsilon.$$

Therefore

$$\begin{aligned}
\lim_{k\to\infty} M_f^5(x_{m_k}, x_{n_k-1}) &\le \max\left\{\epsilon, 2b\epsilon, 2b^2\epsilon\right\} \\
&= 2b^2\epsilon.
\end{aligned}$$

From (8), by the definition of ψ, we have that

$$4b^2\epsilon \le 2b^2\epsilon,$$

which is a contradiction. Hence $\{f^n x_0\}$ is an S_b-Cauchy sequence in complete regular S_b-metric spaces $(X, S_b, \preceq)$. By the completeness of (X, S_b), it follows that the sequence $\{f^n x_0\}$ converges to α in (X, S_b). Thus

$$\lim_{k\to\infty} f^n x_0 = \alpha = \lim_{k\to\infty} f^{n+1} x_0.$$

Since $x_n, \alpha \in X$ and X is regular, it follows that either $x_n \preceq \alpha$ or $\alpha \preceq x_n$.

Now we have to prove that α is a fixed point of f.

Suppose $f\alpha \neq \alpha$, by Lemma (2.10), we have that

$$\frac{1}{2b} S_b(f\alpha, f\alpha, \alpha) \le \lim_{n\to\infty} \inf S_b(f\alpha, f\alpha, f^{n+1}x_0).$$

Now from (3.2.4) and applying ψ on both sides, we have that

$$\begin{aligned}
\psi\big(2b^3 S_b(f\alpha, f\alpha, \alpha)\big) &\le \lim_{n\to\infty} \inf \psi\big(4b^4 S_b\big(f\alpha, f\alpha, f^{n+1}x_0\big)\big) \\
&\le \lim_{n\to\infty} \inf \psi\big(M_f^5(\alpha, x_n)\big) - \lim_{n\to\infty} \inf \phi\big(M_f^5(\alpha, x_n)\big). \qquad (9)
\end{aligned}$$

Here

$$\begin{aligned}\lim_{n\to\infty}\inf M_f^5(\alpha,x_n) &= \lim_{n\to\infty}\inf\max\left\{\begin{matrix}S_b(\alpha,\alpha,x_n),S_b(\alpha,\alpha,f\alpha),S_b(x_n,x_n,fx_n)\\ S_b(\alpha,\alpha,fx_n),S_b(x_n,x_n,f\alpha)\end{matrix}\right\}\\ &\leq \lim_{n\to\infty}\sup\max\left\{0,S_b(\alpha,\alpha,f\alpha),0,0,S_b(x_n,x_n,f\alpha)\right\}\\ &\leq \max\left\{S_b(\alpha,\alpha,f\alpha),b^2S_b(\alpha,\alpha,f\alpha)\right\}\\ &\leq b^3S_b(f\alpha,f\alpha,\alpha).\end{aligned}$$

Hence from (9) we have that

$$\begin{aligned}\psi\left(2b^3S_b(f\alpha,f\alpha,\alpha)\right) &\leq \psi\left(b^3S_b(\alpha,\alpha,f\alpha)\right) - \lim_{n\to\infty}\inf\phi\left(M_f^5(\alpha,x_n)\right)\\ &\leq \psi\left(b^3S_b(f\alpha,f\alpha,\alpha)\right),\end{aligned}$$

which is a contradiction. So that α is a fixed point of f.

Suppose that α^* is another fixed point of f such that $\alpha \neq \alpha^*$.

Consider

$$\begin{aligned}\psi\left(4b^4S_b\left(\alpha,\alpha,\alpha^*\right)\right) &\leq \psi\left(M_f^5\left(\alpha,\alpha^*\right)\right) - \phi\left(M_f^5\left(\alpha,\alpha^*\right)\right)\\ &= \psi\left(\max\left\{S_b\left(\alpha,\alpha,\alpha^*\right),S_b\left(\alpha^*,\alpha^*,\alpha\right)\right\}\right)\\ &\quad - \phi\left(\max\left\{S_b\left(\alpha,\alpha,\alpha^*\right),S_b\left(\alpha^*,\alpha^*,\alpha\right)\right\}\right)\\ &\leq \psi\left(bS_b\left(\alpha,\alpha,\alpha^*\right)\right),\end{aligned}$$

which is a contradiction.

Hence α is a unique fixed point of f in (X,S_b). □

Example 3.5 Let $X = [0,1]$ and $S : X\times X\times X \to \mathbb{R}^+$ by $S_b(x,y,z) = (|y+z-2x| + |y-z|)^2$ and $\preceq$ by $a \preceq b \iff a \leq b$, then $(X,S_b,\preceq)$ is a complete ordered S_b-metric space with $b = 4$. Define $f : X \to X$ by $f(x) = \frac{x}{32\sqrt{2}}$. Also define $\psi,\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = t$ and $\phi(t) = \frac{t}{2}$.

$$\begin{aligned}\psi\left(4b^4S_b(fx,fx,fy)\right) &= 4b^4\left(|fx+fy-2fx| + |fx-fy|\right)^2\\ &= 4b^4\left(2\left|\frac{x}{32\sqrt{2}} - \frac{y}{32\sqrt{2}}\right|\right)^2\\ &= \frac{4b^4}{8b^4}S_b(x,x,y)\\ &\leq \frac{1}{2}M_f^5(x,y)\\ &\leq \psi\left(M_f^5(x,y)\right) - \phi\left(M_f^5(x,y)\right),\end{aligned}$$

where

$$M_f^5(x,y) = \max\left\{S_b(x,x,y),S_b(x,x,fx),S_b(y,y,fy),S_b(x,x,fy),S_b(y,y,fx)\right\}.$$

Hence, all the conditions of Theorem 3.4 are satisfied and 0 is a unique fixed point of f.

Theorem 3.6 *Let $(X, S_b, \preceq)$ be an ordered complete S_b metric space, and let $f : X \to X$ satisfy (ψ, ϕ)-contraction with $i = 3$ or 4. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a unique fixed point in X.*

Proof Follows along similar lines of Theorem 3.4 if we take $M_f^3(x,y)$ or $M_f^4(x,y)$ in place of $M_f^5(x,y)$ in Theorem 3.4. □

Theorem 3.7 *Let $(X, S_b, \preceq)$ be an ordered complete S_b metric space, and let $f : X \to X$ satisfy*

$$4b^4 S_b(fx,fx,fy) \leq M_f^i(x,y) - \varphi\big(M_f^i(x,y)\big),$$

where $\varphi : [0,\infty) \to [0,\infty)$ and $i = 3$ or 4 or 5. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a unique fixed point in X.

Proof The proof follows from Theorems 3.4 and 3.6 by taking $\psi(t) = t$ and $\phi(t) = \varphi(t)$. □

Theorem 3.8 *Let $(X, S_b, \preceq)$ be an ordered complete S_b metric space, and let $f : X \to X$ satisfy*

$$S_b(fx,fx,fy) \leq \lambda M_f^i(x,y),$$

where $\lambda \in [0, \frac{1}{4b^4})$ and $i = 3,4,5$. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a unique fixed point in X.

3.1 Application to integral equations

In this section, we study the existence of a unique solution to an initial value problem as an application to Theorem 3.4.

Theorem 3.9 *Consider the initial value problem*

$$x^1(t) = T\big(t, x(t)\big), \quad t \in I = [0,1], x(0) = x_0, \tag{10}$$

where $T : I \times [\frac{x_0}{4}, \infty) \to [\frac{x_0}{4}, \infty)$ and $x_0 \in \mathbb{R}$. Then there exists a unique solution in $C(I, [\frac{x_0}{4}, \infty))$ for initial value problem (10).

Proof The integral equation corresponding to initial value problem (10) is

$$x(t) = x_0 + 3b^2 \int_0^t T\big(s, x(s)\big)\, ds.$$

Let $X = C(I, [\frac{x_0}{4}, \infty))$ and $S_b(x,y,z) = (|y+z-2x| + |y-z|)^2 for x, y \in X$. Define $\psi, \phi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t$, $\phi(t) = \frac{5t}{9}$. Define $f : X \to X$ by

$$f(x)(t) = \frac{x_0}{3b^2} + \int_0^t T\big(s, x(s)\big)\, ds. \tag{11}$$

Now

$$\begin{aligned}
&\psi\big(4b^4 S_b\big(fx(t),fx(t),fy(t)\big)\big)\\
&\quad = 4b^4\big\{\big|fx(t)+fy(t)-2fx(t)\big| + \big|fx(t)-fy(t)\big|\big\}^2\\
&\quad = 16b^4\big|fx(t)-fy(t)\big|^2\\
&\quad = \frac{16b^4}{9b^4}\left|x_0 + 3b^2\int_0^t T\big(s,x(s)\big)\,ds - y_0 - 3b^2\int_0^t T\big(s,y(s)\big)\,ds\right|^2\\
&\quad = \frac{16}{9}\big|x(t)-y(t)\big|^2\\
&\quad = \frac{4}{9}S(x,x,y)\\
&\quad \le \frac{4}{9}M_f^5(x,y)\\
&\quad = \psi\big(M_f^5(x,y)\big) - \phi\big(M_f^5(x,y)\big),
\end{aligned}$$

where

$$M_f^5(x,y) = \max\big\{S_b(x,x,y), S_b(x,x,fx), S_b(y,y,fy), S_b(x,x,fy), S_b(y,y,fx)\big\}.$$

It follows from Theorem 3.4 that f has a unique fixed point in X. □

3.2 Application to homotopy

In this section, we study the existence of a unique solution to homotopy theory.

Theorem 3.10 *Let (X,S_b) be a complete S_b-metric space, U be an open subset of X and $\overline{U}$ be a closed subset of X such that $U \subseteq \overline{U}$. Suppose that $H : \overline{U} \times [0,1] \to X$ is an operator such that the following conditions are satisfied:*

(i) $x \neq H(x,\lambda)$ *for each* $x \in \partial U$ *and* $\lambda \in [0,1]$ (*here* ∂U *denotes the boundary of* U *in* X),

(ii) $\psi(4b^4 S_b(H(x,\lambda),H(x,\lambda),H(y,\lambda))) \le \psi(S_b(x,x,y)) - \phi(S_b(x,x,y))$ $\forall x,y \in \overline{U}$ *and* $\lambda \in [0,1]$, *where* $\psi : [0,\infty) \to [0,\infty)$ *is continuous, non-decreasing and* $\phi : [0,\infty) \to [0,\infty)$ *is lower semi-continuous with* $\phi(t) > 0$ *for* $t > 0$,

(iii) *there exists* $M \ge 0$ *such that*

$$S_b\big(H(x,\lambda),H(x,\lambda),H(x,\mu)\big) \le M|\lambda-\mu|$$

for every $x \in \overline{U}$ *and* $\lambda,\mu \in [0,1]$.

Then $H(\cdot,0)$ *has a fixed point if and only if* $H(\cdot,1)$ *has a fixed point.*

Proof Consider the set

$$A = \big\{\lambda \in [0,1] : x = H(x,\lambda) \text{ for some } x \in U\big\}.$$

Since $H(\cdot,0)$ has a fixed point in U, we have that $0 \in A$. So that A is a non-empty set.

We will show that A is both open and closed in $[0,1]$, and so, by the connectedness, we have that $A = [0,1]$. As a result, $H(\cdot,1)$ has a fixed point in U. First we show that A is closed in $[0,1]$. To see this, let $\{\lambda_n\}_{n=1}^{\infty} \subseteq A$ with $\lambda_n \to \lambda \in [0,1]$ as $n \to \infty$.

We must show that $\lambda \in A$. Since $\lambda_n \in A$ for $n = 1, 2, 3, \ldots$, there exists $x_n \in U$ with $x_n = H(x_n, \lambda_n)$.

Consider

$$\begin{aligned} S_b(x_n, x_n, x_{n+1}) &= S_b\big(H(x_n, \lambda_n), H(x_n, \lambda_n), H(x_{n+1}, \lambda_{n+1})\big) \\ &\leq 2bS_b\big(H(x_n, \lambda_n), H(x_n, \lambda_n), H(x_{n+1}, \lambda_n)\big) \\ &\quad + b^2 S_b\big(H(x_{n+1}, \lambda_n), H(x_{n+1}, \lambda_n), H(x_{n+1}, \lambda_{n+1})\big) \\ &\leq S_b\big(H(x_n, \lambda_n), H(x_n, \lambda_n), H(x_{n+1}, \lambda_n)\big) + M|\lambda_n - \lambda_{n+1}|. \end{aligned}$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} S_b(x_n, x_n, x_{n+1}) \leq \lim_{n\to\infty} S_b\big(H(x_n, \lambda_n), H(x_n, \lambda_n), H(x_{n+1}, \lambda_n)\big) + 0.$$

Since ψ is continuous and non-decreasing, we obtain

$$\begin{aligned} \lim_{n\to\infty} \psi\big(4b^4 S_b(x_n, x_n, x_{n+1})\big) &\leq \lim_{n\to\infty} \psi\big(4b^4 S_b\big(H(x_n, \lambda_n), H(x_n, \lambda_n), H(x_{n+1}, \lambda_n)\big)\big) \\ &\leq \lim_{n\to\infty} \big[\psi\big(S_b(x_n, x_n, x_{n+1})\big) - \phi\big(S_b(x_n, x_n, x_{n+1})\big)\big]. \end{aligned}$$

By the definition of ψ, it follows that

$$\lim_{n\to\infty} \big(4b^4 - 1\big) S_b(x_n, x_n, x_{n+1}) \leq 0.$$

So that

$$\lim_{n\to\infty} S_b(x_n, x_n, x_{n+1}) = 0. \tag{12}$$

Now we prove that $\{x_n\}$ is an S_b-Cauchy sequence in (X, d_p). On the contrary, suppose that $\{x_n\}$ is not S_b-Cauchy.

There exists $\epsilon > 0$ and monotone increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$,

$$S_b(x_{m_k}, x_{m_k}, x_{n_k}) \geq \epsilon \tag{13}$$

and

$$S_b(x_{m_k}, x_{m_k}, x_{n_k-1}) < \epsilon. \tag{14}$$

From (13) and (14), we obtain

$$\begin{aligned} \epsilon &\leq S_b(x_{m_k}, x_{m_k}, x_{n_k}) \\ &\leq 2bS_b(x_{m_k}, x_{m_k}, x_{m_k+1}) + b^2 S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k}). \end{aligned}$$

Letting $k \to \infty$ and applying ψ on both sides, we have that

$$\psi\big(2b^2\epsilon\big) \leq \lim_{n\to\infty} \psi\big(4b^4 S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k})\big). \tag{15}$$

But

$$\begin{aligned}
&\lim_{n\to\infty} \psi\big(4b^4 S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k})\big) \\
&\quad = \lim_{n\to\infty} \psi\big(S_b\big(4b^4 H(x_{m_k+1}, \lambda_{m_k+1}), H(x_{m_k+1}, \lambda_{m_k+1}), H(x_{n_k}, \lambda_{n_k})\big)\big) \\
&\quad \leq \lim_{n\to\infty}\big[\psi\big(S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k})\big) - \phi\big(S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k})\big)\big].
\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\big(4b^4 - 1\big) S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k}) \leq 0.$$

Thus

$$\lim_{n\to\infty} S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k}) = 0.$$

Hence from (15) and the definition of ψ, we have that

$$\epsilon \leq 0,$$

which is a contradiction.

Hence $\{x_n\}$ is an S_b-Cauchy sequence in (X, S_b) and, by the completeness of (X, S_b), there exists $\alpha \in U$ with

$$\lim_{n\to\infty} x_n = \alpha = \lim_{n\to\infty} x_{n+1}, \tag{16}$$

$$\begin{aligned}
\psi\big(2b^3 S_b\big(H(\alpha,\lambda), H(\alpha,\lambda), \alpha\big)\big) &\leq \lim_{n\to\infty} \inf \psi\big(4b^4 S_b\big(H(\alpha,\lambda), H(\alpha,\lambda), H(x_n,\lambda)\big)\big) \\
&\leq \lim_{n\to\infty} \inf\big[\psi\big(S_b(\alpha,\alpha,x_n)\big) - \phi\big(S_b(\alpha,\alpha,x_n)\big)\big] \\
&= 0.
\end{aligned}$$

It follows that $\alpha = H(\alpha, \lambda)$.

Thus $\lambda \in A$. Hence A is closed in $[0,1]$.

Let $\lambda_0 \in A$. Then there exists $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$.

Since U is open, there exists $r > 0$ such that $B_{S_b}(x_0, r) \subseteq U$.

Choose $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ such that $|\lambda - \lambda_0| \leq \frac{1}{M^n} < \epsilon$.

Then, for $x \in \overline{B_p(x_0, r)} = \{x \in X / S_b(x, x, x_0) \leq r + b^2 S_b(x_0, x_0, x_0)\}$,

$$\begin{aligned}
&S_b\big(H(x,\lambda), H(x,\lambda), x_0\big) \\
&\quad = S_b\big(H(x,\lambda), H(x,\lambda), H(x_0,\lambda_0)\big) \\
&\quad \leq 2b S_b\big(H(x,\lambda), H(x,\lambda), H(x,\lambda_0)\big) + b^2 S_b\big(H(x,\lambda_0), H(x,\lambda_0), H(x_0,\lambda_0)\big) \\
&\quad \leq 2bM|\lambda - \lambda_0| + b^2 S_b\big(H(x,\lambda_0), H(x,\lambda_0), H(x_0,\lambda_0)\big) \\
&\quad \leq \frac{2b}{M^{n-1}} + b^2 S_b\big(H(x,\lambda_0), H(x,\lambda_0), H(x_0,\lambda_0)\big).
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$S_b\big(H(x,\lambda),H(x,\lambda),x_0\big) \le b^2 S_b\big(H(x,\lambda_0),H(x,\lambda_0),H(x_0,\lambda_0)\big).$$

Since ψ is continuous and non-decreasing, we have

$$\begin{aligned}\psi\big(S_b\big(H(x,\lambda),H(x,\lambda),x_0\big)\big) &\le \psi\big(4b^2 S_b\big(H(x,\lambda),H(x,\lambda),x_0\big)\big)\\ &\le \psi\big(4b^4 S_b\big(H(x,\lambda_0),H(x,\lambda_0),H(x_0,\lambda_0)\big)\big)\\ &\le \psi\big(S_b(x,x,x_0)\big) - \phi\big(S_b(x,x,x_0)\big)\\ &\le \psi\big(S_b(x,x,x_0)\big).\end{aligned}$$

Since ψ is non-decreasing, we have

$$\begin{aligned}S_b\big(H(x,\lambda),H(x,\lambda),x_0\big) &\le S_b(x,x,x_0)\\ &\le r + b^2 S_b(x_0,x_0,x_0).\end{aligned}$$

Thus, for each fixed $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, $H(\cdot,\lambda) : \overline{B_p(x_0,r)} \to \overline{B_p(x_0,r)}$.

Since also (ii) holds and ψ is continuous and non-decreasing and ϕ is continuous with $\phi(t) > 0$ for $t > 0$, then all the conditions of Theorem (3.10) are satisfied.

Thus we deduce that $H(\cdot,\lambda)$ has a fixed point in $\overline{U}$. But this fixed point must be in U since (i) holds.

Thus $\lambda \in A$ for any $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$.

Hence $(\lambda_0 - \epsilon, \lambda_0 + \epsilon) \subseteq A$ and therefore A is open in $[0,1]$.

For the reverse implication, we use the same strategy. □

Corollary 3.11 *Let (X,p) be a complete partial metric space, U be an open subset of X and $H : \overline{U} \times [0,1] \to X$ with the following properties*:

(1) $x \ne H(x,t)$ *for each* $x \in \partial U$ *and each* $\lambda \in [0,1]$ (*here* ∂U *denotes the boundary of* U *in* X),

(2) *there exist* $x,y \in \overline{U}$ *and* $\lambda \in [0,1], L \in [0,\frac{1}{4b^4})$ *such that*

$$S_b\big(H(x,\lambda),H(x,\lambda),H(y,\mu)\big) \le L S_b(x,x,y),$$

(3) *there exists* $M \ge 0$ *such that*

$$S_b\big(H(x,\lambda),H(x,\lambda),H(x,\mu)\big) \le M|\lambda - \mu|$$

for all $x \in \overline{U}$ *and* $\lambda,\mu \in [0,1]$.

If $H(\cdot,0)$ *has a fixed point in* U, *then* $H(\cdot,1)$ *has a fixed point in* U.

Proof Proof follows by taking $\psi(x) = x, \phi(x) = x - Lx$ with $L \in [0,\frac{1}{4b^4})$ in Theorem (3.10). □

4 Conclusions

In this paper we conclude some applications to homotopy theory and integral equations by using fixed point theorems in partially ordered S_b-metric spaces.

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