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Research Article

Iteration Scheme with Perturbed Mapping for Common Fixed Points of a Finite Family of Nonexpansive Mappings

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We propose an iteration scheme with perturbed mapping for approximation of common fixed points of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$. We show that the proposed iteration scheme converges to the common fixed point $x^* \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ which solves some variational inequality.

1. Introduction and preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. A mapping T with domain $D(T)$ and range $R(T)$ in H is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D(T). \tag{1.1}$$

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-maps of H. Denote the common fixed points set of $\{T_i\}_{i=1}^N$ by $\bigcap_{i=1}^N \mathrm{Fix}(T_i)$. Let $F : H \to H$ be a mapping such that for some constants $k, \eta > 0$, F is k-Lipschitzian and η-strongly monotone. Let $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$, $\{\lambda_n\}_{n=1}^\infty \subset [0,1)$ and take a fixed number $\mu \in (0, 2\eta/k^2)$. The iterative schemes concerning nonlinear operators have been studied extensively by many authors, you may refer to [1–12]. Especially, in [13], Zeng and Yao introduced the following implicit iteration process with perturbed mapping F.

For an arbitrary initial point $x_0 \in H$, the sequence $\{x_n\}_{n=1}^\infty$ is generated as follows:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)\left[T_n x_n - \lambda_n \mu F(T_n x_n)\right], \quad n \geq 1, \tag{1.2}$$

where $T_n := T_{n \bmod N}$.

Using this iteration process, they proved the following weak and strong convergence theorems for nonexpansive mappings in Hilbert spaces.

Theorem 1.1 (see [13]). *Let H be a real Hilbert space and let $F : H \to H$ be a mapping such that for some constants $k, \eta > 0$, F is k-Lipschitzain vcommentand η-strongly monotone. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-mappings of H such that $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/k^2)$ and $x_0 \in H$. Let $\{\lambda_n\}_{n=1}^\infty \subset [0,1)$ and $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$ satisfying the conditions $\sum_{n=1}^\infty \lambda_n < \infty$ and $\alpha \le \alpha_n \le \beta$, $n \ge 1$, for some $\alpha, \beta \in (0,1)$. Then the sequence $\{x_n\}_{n=1}^\infty$ defined by (1.2) converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.*

Theorem 1.2 (see [13]). *Let H be a real Hilbert space and let $F : H \to H$ be a mapping such that for some constants $k, \eta > 0$, F is k-Lipschitzain and η-strongly monotone. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-mappings of H such that $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/k^2)$ and $x_0 \in H$. Let $\{\lambda_n\}_{n=1}^\infty \subset [0,1)$ and $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$ satisfying the conditions $\sum_{n=1}^\infty \lambda_n < \infty$ and $\alpha \le \alpha_n \le \beta$, $n \ge 1$, for some $\alpha, \beta \in (0,1)$. Then the sequence $\{x_n\}_{n=1}^\infty$ defined by (1.2) converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$ if and only if*

$$\liminf_{n\to\infty} d\left(x_n, \bigcap_{i=1}^N \mathrm{Fix}(T_i)\right) = 0. \tag{1.3}$$

Very recently, Wang [14] considered an explicit iterative scheme with perturbed mapping F and obtained the following result.

Theorem 1.3. *Let H be a Hilbert space, let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and let $F : H \to H$ be an η-strongly monotone and k-Lipschitzian mapping. For any given $x_0 \in H$, $\{x_n\}$ is defined by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^{\lambda_{n+1}} x_n, \quad n \ge 0, \tag{1.4}$$

where $T^{\lambda_{n+1}} x_n = T x_n - \lambda_{n+1} \mu F(T x_n)$, $\{\alpha_n\}$ and $\{\lambda_n\} \subset [0,1)$ satisfy the following conditions:

(1) *$\alpha \le \alpha_n \le \beta$ for some $\alpha, \beta \in (0,1)$;*
(2) *$\sum_{n=1}^\infty \lambda_n < \infty$;*
(3) *$0 < \mu < 2\eta/k^2$.*

Then

(1) *$\{x_n\}$ converges weakly to a fixed point of T,*
(2) *$\{x_n\}$ converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0. \tag{1.5}$$

This naturally brings us the following questions.

Questions 1.4. Let $T_i : H \to H$ $(i = 1, 2, \ldots, N)$ be a finite family of nonexpansive mappings and F is k-Lipschitzain and η-strongly monotone.

(i) Could we construct an explicit iterative algorithm to approximate the common fixed points of the mappings $\{T_i\}_{i=1}^N$?
(ii) Could we remove the assumption (2) imposed on the sequence $\{x_n\}$?

Motivated and inspired by the above research work of Zeng and Yao [13] and Wang [14], in this paper, we will propose a new explicit iteration scheme with perturbed mapping for approximation of common fixed points of a finite family of nonexpansive self-mappings of H. We will establish strong convergence theorem for this explicit iteration scheme. To be more specific, let $\alpha_{n1},\alpha_{n2},\dots,\alpha_{nN}\in(0,1]$, $n\in N$. Given the mappings $T_1,T_2,\dots,T_N$, following [15], one can define, for each n, mappings $U_{n1},U_{n2},\dots,U_{nN}$ by

$$\begin{aligned}
U_{n1}&=\alpha_{n1}T_1+(1-\alpha_{n1})I,\\
U_{n2}&=\alpha_{n2}T_2U_{n1}+(1-\alpha_{n2})I,\\
&\ \ \vdots\\
U_{n,N-1}&=\alpha_{n,N-1}T_{N-1}U_{n,N-2}+(1-\alpha_{n,N-1})I,\\
W_n:=U_{nN}&=\alpha_{nN}T_NU_{n,N-1}+(1-\alpha_{nN})I.
\end{aligned}\tag{1.6}$$

Such a mapping W_n is called the W-mapping generated by $T_1,\dots,T_N$ and $\alpha_{n1},\dots,\alpha_{nN}$.

First we introduce the following explicit iteration scheme with perturbed mapping F.

For an arbitrary initial point $x_0\in H$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated iteratively by

$$x_{n+1}=\beta x_n+(1-\beta)\big[W_nx_n-\lambda_n\mu F(W_nx_n)\big],\quad n\geq 0,\tag{1.7}$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$, β is a constant in $(0,1)$, F is k-Lipschitzian and η-strongly monotone, and W_n is the W-mapping defined by (1.6).

We have the following crucial conclusion concerning W_n.

Proposition 1.5 (see [15]). *Let C be a nonempty closed convex subset of a Banach space E. Let $T_1,T_2,\dots,T_N$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N\operatorname{Fix}(T_i)$ is nonempty, and let $\alpha_{n1},\alpha_{n2},\dots,\alpha_{nN}$ be real numbers such that $0<\alpha_{ni}\leq b<1$ for any $i\in N$. For any $n\in N$, let W_n be the W-mapping of C into itself generated by $T_N,T_{N-1},\dots,T_1$ and $\alpha_{nN},\alpha_{n,N-1},\dots,\alpha_{n1}$. Then W_n is nonexpansive. Further, if E is strictly convex, then $\operatorname{Fix}(W_n)=\bigcap_{i=1}^N\operatorname{Fix}(T_i)$.*

Now we recall some basic notations. Let $T:H\to H$ be nonexpansive mapping and $F:H\to H$ be a mapping such that for some constants $k,\eta>0$, F is k-Lipschitzian and η-strongly monotone; that is, F satisfies the following conditions:

$$\begin{aligned}
\|Fx-Fy\|&\leq k\|x-y\|,\quad \forall x,y\in H,\\
\langle Fx-Fy,x-y\rangle&\geq\eta\|x-y\|^2,\quad \forall x,y\in H,
\end{aligned}\tag{1.8}$$

respectively. We may assume, without loss of generality, that $\eta\in(0,1)$ and $k\in[1,\infty)$. Under these conditions, it is well known that the variational inequality problem—find $x^*\in\bigcap_{i=1}^N\operatorname{Fix}(T_i)$ such that

$$VI\Big(F,\bigcap_{i=1}^N\operatorname{Fix}(T_i)\Big):\langle F(x^*),x-x^*\rangle\geq 0,\quad \forall x\in\bigcap_{i=1}^N\operatorname{Fix}(T_i),\tag{1.9}$$

has a unique solution $x^* \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$. [Note: the unique existence of the solution $x^* \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$ is guaranteed automatically because F is k-Lipschitzian and η-strongly monotone over $\bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$.]

For any given numbers $\lambda \in [0,1)$ and $\mu \in (0, 2\eta/k^2)$, we define the mapping $T^\lambda : H \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in H. \tag{1.10}$$

Concerning the corresponding result of $T^\lambda x$, you can find it in [16].

Lemma 1.6 (see [16]). *If $0 \le \lambda < 1$ and $0 < \mu < 2\eta/k^2$, then there holds for $T^\lambda : H \to H$,*

$$||T^\lambda x - T^\lambda y|| \le (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in H, \tag{1.11}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \in (0,1)$.

Next, let us state four preliminary results which will be needed in the sequel. Lemma 1.7 is very interesting and important, you may find it in [17], the original prove can be found in [18]. Lemmas 1.8 and 1.9 well-known demiclosedness principle and subdifferential inequality, respectively. Lemma 1.10 is basic and important result, please consult it in [19].

Lemma 1.7 (see [17]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1. \tag{1.12}$$

Suppose

$$x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n, \tag{1.13}$$

for all integers $n \ge 0$ and

$$\limsup_{n\to\infty} (||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \le 0. \tag{1.14}$$

Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.8 (see [20]). *Assume that T is a nonexpansive self-mapping of a closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here, I is the identity operator of H.*

Lemma 1.9 (see [21]). $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$ *for all $x, y \in H$.*

Lemma 1.10 (see [19]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n) a_n + \delta_n, \tag{1.15}$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty}\gamma_n=\infty$,

(2) $\limsup_{n\to\infty}\delta_n/\gamma_n\leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n|<\infty$.

Then $\lim_{n\to\infty}a_n=0$.

2. Main result

Now we state and prove our main result.

THEOREM 2.1. *Let H be a real Hilbert space and let $F:H\to H$ be a k-Lipschitzian and η-strongly monotone mapping. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of H such that $\bigcap_{i=1}^N \operatorname{Fix}(T_i)\neq\emptyset$. Let $\mu\in(0,2\eta/k^2)$. Suppose the sequences $\{\alpha_{n,i}\}_{i=1}^N$ satisfy $\lim_{n\to\infty}(\alpha_{n,i}-\alpha_{n-1,i})=0$, for all $i=1,2,\dots,N$. If $\{\lambda_n\}_{n=1}^{\infty}\subset[0,1)$ satisfy the following conditions:*

(i) $\lim_{n\to\infty}\lambda_n=0$;

(ii) $\sum_{n=0}^{\infty}\lambda_n=\infty$,

then the sequence $\{x_n\}_{n=1}^{\infty}$ defined by (1.7) converges strongly to a common fixed point $x^*\in\bigcap_{i=1}^N\operatorname{Fix}(T_i)$ which solves the variational inequality (1.9).

Proof. Let x^* be an arbitrary element of $\bigcap_{i=1}^N\operatorname{Fix}(T_i)$. Observe that

$$\begin{aligned}||x_{n+1}-x^*|| &= ||\beta x_n+(1-\beta)W_n^{\lambda_n}x_n-x^*|| \\ &\leq \beta||x_n-x^*||+(1-\beta)||W_n^{\lambda_n}x_n-x^*||,\end{aligned}\tag{2.1}$$

where $W_n^{\lambda_n}x:=W_nx-\lambda_n\mu F(W_nx)$. Note that

$$W_n^{\lambda_n}x^*=x^*-\lambda_n\mu F(x^*).\tag{2.2}$$

Utilizing Lemma 1.6, we have

$$\begin{aligned}||W_n^{\lambda_n}x_n-x^*|| &= ||W_n^{\lambda_n}x_n-W_n^{\lambda_n}x^*+W_n^{\lambda_n}x^*-x^*|| \\ &\leq ||W_n^{\lambda_n}x_n-W_n^{\lambda_n}x^*||+||W_n^{\lambda_n}x^*-x^*|| \\ &\leq (1-\lambda_n\tau)||x_n-x^*||+\lambda_n\mu||F(x^*)||.\end{aligned}\tag{2.3}$$

From (2.1) and (2.3), we have

$$\begin{aligned}||x_{n+1}-x^*|| &\leq [\beta+(1-\beta)(1-\lambda_n\tau)]||x_n-x^*||+(1-\beta)\lambda_n\mu||F(x^*)|| \\ &= [1-(1-\beta)\lambda_n\tau]||x_n-x^*||+(1-\beta)\lambda_n\mu||F(x^*)|| \\ &\leq \max\Big\{||x_0-x^*||,\Big(\frac{\mu}{\tau}\Big)||F(x^*)||\Big\}.\end{aligned}\tag{2.4}$$

Hence, $\{x_n\}$ is bounded. We also can obtain that $\{W_nx_n\}$, $\{T_iU_{n,j}x_n\}(i=1,\dots,N;\ j=1,\dots,N)$, and $\{F(W_nx_n)\}$ are all bounded.

We will use M to denote the possible different constants appearing in the following reasoning.

We note that

$$
\begin{aligned}
&\|W_{n+1}^{\lambda_{n+1}}x_{n+1} - W_n^{\lambda_n}x_n\| \\
&\quad = \|W_{n+1}x_{n+1} - W_nx_n - \lambda_{n+1}\mu F(W_{n+1}x_{n+1}) + \lambda_n\mu F(W_nx_n)\| \\
&\quad \le \|W_{n+1}x_{n+1} - W_nx_n\| + \lambda_{n+1}\mu\|F(W_{n+1}x_{n+1})\| + \lambda_n\mu\|F(W_nx_n)\| \\
&\quad \le \|W_{n+1}x_{n+1} - W_{n+1}x_n\| + \|W_{n+1}x_n - W_nx_n\| + (\lambda_{n+1}+\lambda_n)M \\
&\quad \le \|x_{n+1} - x_n\| + \|W_{n+1}x_n - W_nx_n\| + (\lambda_{n+1}+\lambda_n)M.
\end{aligned}
\tag{2.5}
$$

From (1.6), since T_N and $U_{n,N}$ are nonexpansive,

$$
\begin{aligned}
&\|W_{n+1}x_n - W_nx_n\| \\
&\quad = \|\alpha_{n+1,N}T_NU_{n+1,N-1}x_n + (1-\alpha_{n+1,N})x_n - \alpha_{n,N}T_NU_{n,N-1}x_n - (1-\alpha_{n,N})x_n\| \\
&\quad \le \|\alpha_{n+1,N}T_NU_{n+1,N-1}x_n - \alpha_{n,N}T_NU_{n,N-1}x_n\| + |\alpha_{n+1,N} - \alpha_{n,N}|\,\|x_n\| \\
&\quad \le \|\alpha_{n+1,N}(T_NU_{n+1,N-1}x_n - T_NU_{n,N-1}x_n)\| + |\alpha_{n+1,N} - \alpha_{n,N}|\,\|T_NU_{n,N-1}x_n\| \\
&\qquad + |\alpha_{n+1,N} - \alpha_{n,N}|\,\|x_n\| \\
&\quad \le \alpha_{n+1,N}\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| + 2M|\alpha_{n+1,N} - \alpha_{n,N}|.
\end{aligned}
\tag{2.6}
$$

Again, from (1.6), we have

$$
\begin{aligned}
&\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad = \|\alpha_{n+1,N-1}T_{N-1}U_{n+1,N-2}x_n + (1-\alpha_{n+1,N-1})x_n \\
&\qquad - \alpha_{n,N-1}T_{N-1}U_{n,N-2}x_n - (1-\alpha_{n,N-1})x_n\| \\
&\quad \le \|\alpha_{n+1,N-1}T_{N-1}U_{n+1,N-2}x_n - \alpha_{n,N-1}T_{N-1}U_{n,N-2}x_n\| \\
&\qquad + |\alpha_{n+1,N-1} - \alpha_{n,N-1}|\,\|x_n\| \\
&\quad \le |\alpha_{n+1,N-1} - \alpha_{n,N-1}|\,\|x_n\| + |\alpha_{n+1,N-1} - \alpha_{n,N-1}|M \\
&\qquad + \alpha_{n+1,N-1}\|T_{N-1}U_{n+1,N-2}x_n - T_{N-1}U_{n,N-2}x_n\| \\
&\quad \le 2M|\alpha_{n+1,N-1} - \alpha_{n,N-1}| + \alpha_{n+1,N-1}\|U_{n+1,N-2}x_n - U_{n,N-2}x_n\| \\
&\quad \le 2M|\alpha_{n+1,N-1} - \alpha_{n,N-1}| + \|U_{n+1,N-2}x_n - U_{n,N-2}x_n\|.
\end{aligned}
\tag{2.7}
$$

Therefore, we have

$$\begin{aligned}
&\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad \le 2M\,|\alpha_{n+1,N-1}-\alpha_{n,N-1}| + 2M\,|\alpha_{n+1,N-2}-\alpha_{n,N-2}| \\
&\qquad + \|U_{n+1,N-3}x_n - U_{n,N-3}x_n\| \\
&\quad \le 2M\sum_{i=2}^{N-1}|\alpha_{n+1,i}-\alpha_{n,i}| + \|U_{n+1,1}x_n - U_{n,1}x_n\| \\
&\quad = \|\alpha_{n+1,1}T_1x_n + (1-\alpha_{n+1,1})x_n - \alpha_{n,1}T_1x_n - (1-\alpha_{n,1})x_n\| \\
&\qquad + 2M\sum_{i=2}^{N-1}|\alpha_{n+1,i}-\alpha_{n,i}|,
\end{aligned} \tag{2.8}$$

then

$$\begin{aligned}
&\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad \le |\alpha_{n+1,1}-\alpha_{n,1}|\,\|x_n\| + \|\alpha_{n+1,1}T_1x_n - \alpha_{n,1}T_1x_n\| \\
&\qquad + 2M\sum_{i=2}^{N-1}|\alpha_{n+1,i}-\alpha_{n,i}| \le 2M\sum_{i=1}^{N-1}|\alpha_{n+1,i}-\alpha_{n,i}|.
\end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.6), we have

$$\begin{aligned}
\|W_{n+1}x_n - W_nx_n\| &\le 2M\,|\alpha_{n+1,N}-\alpha_{n,N}| + 2\alpha_{n+1,N}M\sum_{i=1}^{N-1}|\alpha_{n+1,i}-\alpha_{n,i}| \\
&\le 2M\sum_{i=1}^{N}|\alpha_{n+1,i}-\alpha_{n,i}|.
\end{aligned} \tag{2.10}$$

Substituting (2.10) into (2.5), we have

$$\|W_{n+1}^{\lambda_{n+1}}x_{n+1} - W_n^{\lambda_n}x_n\| \le \|x_{n+1}-x_n\| + 2M\sum_{i=1}^{N}|\alpha_{n+1,i}-\alpha_{n,i}| + (\lambda_{n+1}+\lambda_n)M, \tag{2.11}$$

which implies that

$$\limsup_{n\to\infty}\left(\|W_{n+1}^{\lambda_{n+1}}x_{n+1} - W_n^{\lambda_n}x_n\| - \|x_{n+1}-x_n\|\right) \le 0. \tag{2.12}$$

We note that $x_{n+1} = \beta x_n + (1-\beta)W_n^{\lambda_n}x_n$ and $0<\beta<1$, then from Lemma 1.7 and (2.12), we have $\lim_{n\to\infty}\|W_n^{\lambda_n}x_n - x_n\| = 0$. It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta)\|W_n^{\lambda_n}x_n - x_n\| = 0. \tag{2.13}$$

On the other hand,

$$\begin{aligned}
\|x_n - W_nx_n\| &\le \|x_{n+1}-x_n\| + \|x_{n+1}-W_nx_n\| \\
&\le \|x_{n+1}-x_n\| + \beta\|x_n - W_nx_n\| + (1-\beta)\lambda_n\mu\|F(W_nx_n)\|,
\end{aligned} \tag{2.14}$$

that is,

$$\|x_n - W_n x_n\| \le \frac{1}{1-\beta}\|x_{n+1} - x_n\| + \lambda_n \mu \|F(W_n x_n)\|, \tag{2.15}$$

this together with (i) and (2.13) imply

$$\lim_{n\to\infty} \|x_n - W_n x_n\| = 0. \tag{2.16}$$

We next show that

$$\limsup_{n\to\infty} \langle -F(x^*), x_n - x^* \rangle \le 0. \tag{2.17}$$

To prove this, we pick a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle -F(x^*), x_n - x^* \rangle = \lim_{i\to\infty} \langle -F(x^*), x_{n_i} - x^* \rangle. \tag{2.18}$$

Without loss of generality, we may further assume that $x_{n_i} \to z$ weakly for some $z \in H$.

By Lemma 1.8 and (2.16), we have

$$z \in \operatorname{Fix}(W_n), \tag{2.19}$$

this together with Proposition 1.5 imply that

$$z \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i). \tag{2.20}$$

Since x^* solves the variational inequality (1.9), then we obtain

$$\limsup_{n\to\infty} \langle -F(x^*), x_n - x^* \rangle = \langle -F(x^*), z - x^* \rangle \le 0. \tag{2.21}$$

Finally, we show that $x_n \to x^*$. Indeed, from Lemma 1.9, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \|\beta(x_n - x^*) + (1-\beta)(W_n^{\lambda_n} x_n - W_n^{\lambda_n} x^*) + (1-\beta)(W_n^{\lambda_n} x^* - x^*)\|^2 \\
&\quad \le \|\beta(x_n - x^*) + (1-\beta)(W_n^{\lambda_n} x_n - W_n^{\lambda_n} x^*)\|^2 + 2(1-\beta)\langle W_n^{\lambda_n} x^* - x^*, x_{n+1} - x^* \rangle \\
&\quad \le [\beta\|x_n - x^*\| + (1-\beta)\|W_n^{\lambda_n} x_n - W_n^{\lambda_n} x^*\|]^2 + 2(1-\beta)\lambda_n\mu\langle -F(x^*), x_{n+1} - x^* \rangle \\
&\quad \le [\beta\|x_n - x^*\| + (1-\beta)(1-\lambda_n\tau)\|x_n - x^*\|]^2 + 2(1-\beta)\lambda_n\mu\langle -F(x^*), x_{n+1} - x^* \rangle \\
&\quad \le [1 - (1-\beta)\tau\lambda_n]\|x_n - x^*\|^2 + (1-\beta)\tau\lambda_n\left\{2\frac{\mu}{\tau}\langle -F(x^*), x_{n+1} - x^* \rangle\right\}.
\end{aligned} \tag{2.22}$$

Now applying Lemma 1.10 and (2.21) to (2.22) concludes that $x_n \to x^*$ $(n \to \infty)$. This completes the proof. □

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References

[1] L.-C. Zeng and J.-C. Yao, "Stability of iterative procedures with errors for approximating common fixed points of a couple of q-contractive-like mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 321, no. 2, pp. 661–674, 2006.

[2] Y.-C. Lin, N.-C. Wong, and J.-C. Yao, "Strong convergence theorems of Ishikawa iteration process with errors for fixed points of Lipschitz continuous mappings in Banach spaces," *Taiwanese Journal of Mathematics*, vol. 10, no. 2, pp. 543–552, 2006.

[3] L.-C. Zeng, N.-C. Wong, and J.-C. Yao, "Strong convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type," *Taiwanese Journal of Mathematics*, vol. 10, no. 4, pp. 837–849, 2006.

[4] Y.-C. Lin, "Three-step iterative convergence theorems with errors in Banach spaces," *Taiwanese Journal of Mathematics*, vol. 10, no. 1, pp. 75–86, 2006.

[5] L.-C. Zeng, G. M. Lee, and N.-C. Wong, "Ishikawa iteration with errors for approximating fixed points of strictly pseudocontractive mappings of Browder-Petryshyn type," *Taiwanese Journal of Mathematics*, vol. 10, no. 1, pp. 87–99, 2006.

[6] S. Schaible, J.-C. Yao, and L.-C. Zeng, "A proximal method for pseudomonotone type variational-like inequalities," *Taiwanese Journal of Mathematics*, vol. 10, no. 2, pp. 497–513, 2006.

[7] L.-C. Zeng, L. J. Lin, and J.-C. Yao, "Auxiliary problem method for mixed variational-like inequalities," *Taiwanese Journal of Mathematics*, vol. 10, no. 2, pp. 515–529, 2006.

[8] L.-C. Zeng, N.-C. Wong, and J.-C. Yao, "On the convergence analysis of modified hybrid steepest-descent methods with variable parameters for variational inequalities," to appear in *Journal of Optimization Theory and Applications*.

[9] L. C. Ceng, P. Cubiotti, and J.-C. Yao, "Approximation of common fixed points of families of nonexpansive mappings," to appear in *Taiwanese Journal of Mathematics*.

[10] L. C. Ceng, P. Cubiotti, and J.-C. Yao, "Strong convergence theorems for finitely many nonexpansive mappings and applications," to appear in *Nonlinear Analysis*.

[11] L.-C. Zeng, S. Y. Wu, and J.-C. Yao, "Generalized KKM theorem with applications to generalized minimax inequalities and generalized equilibrium problems," *Taiwanese Journal of Mathematics*, vol. 10, no. 6, pp. 1497–1514, 2006.

[12] L. C. Ceng, C. Lee, and J.-C. Yao, "Strong weak convergence theorems of implicit hybrid steepest-descent methods for variational inequalities," to appear in *Taiwanese Journal of Mathematics*.

[13] L.-C. Zeng and J.-C. Yao, "Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings," *Nonlinear Analysis*, vol. 64, no. 11, pp. 2507–2515, 2006.

[14] L. Wang, "An iteration method for nonexpansive mappings in Hilbert spaces," *Fixed Point Theory and Applications*, vol. 2007, Article ID 28619, 8 pages, 2007.

[15] W. Takahashi and K. Shimoji, "Convergence theorems for nonexpansive mappings and feasibility problems," *Mathematical and Computer Modelling*, vol. 32, no. 11–13, pp. 1463–1471, 2000.

[16] H. K. Xu and T. H. Kim, "Convergence of hybrid steepest-descent methods for variational inequalities," *Journal of Optimization Theory and Applications*, vol. 119, no. 1, pp. 185–201, 2003.

[17] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[18] T. Suzuki, "Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces," *Fixed Point Theory and Applications*, vol. 2005, no. 1, pp. 103–123, 2005.

[19] H.-K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[20] K. Geobel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, vol. 28 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.

[21] G. Marino and H.-K. Xu, "Convergence of generalized proximal point algorithms," *Communications on Pure and Applied Analysis*, vol. 3, no. 4, pp. 791–808, 2004.

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