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Research Article

Strong Convergence of Cesàro Mean Iterations for Nonexpansive Nonself-Mappings in Banach Spaces

Rabian Wangkeeree

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Let E be a real uniformly convex Banach space which admits a weakly sequentially continuous duality mapping from E to E^*, C a nonempty closed convex subset of E which is also a sunny nonexpansive retract of E, and $T : C \to E$ a non-expansive nonself-mapping with $F(T) \neq \emptyset$. In this paper, we study the strong convergence of two sequences generated by $x_{n+1} = \alpha_n x + (1-\alpha_n)(1/n+1)\sum_{j=0}^{n}(PT)^j x_n$ and $y_{n+1} = (1/n+1)\sum_{j=0}^{n} P(\alpha_n y + (1-\alpha_n)(TP)^j y_n)$ for all $n \geq 0$, where $x, x_0, y, y_0 \in C$, $\{\alpha_n\}$ is a real sequence in an interval $[0,1]$, and P is a sunny non-expansive retraction of E onto C. We prove that $\{x_n\}$ and $\{y_n\}$ converge strongly to Qx and Qy, respectively, as $n \to \infty$, where Q is a sunny non-expansive retraction of C onto $F(T)$. The results presented in this paper generalize, extend, and improve the corresponding results of Matsushita and Kuroiwa (2001) and many others.

1. Introduction

Let C be a nonempty closed convex subset of a Hilbert space E and let T be a nonexpansive mapping from C into itself, that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. In 1997, Shimizu and Takahashi [1] originally studied the convergence of an iteration process $\{x_n\}$ for a family of nonexpansive mappings in the framework of a Hilbert space. We restate the sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n \quad \text{for } n = 0,1,2,\ldots, \tag{1.1}$$

where x_0, x are all elements of C, and $\{\alpha_n\}$ is an appropriate sequence in $[0,1]$. They proved that $\{x_n\}$ converges strongly to an element of fixed point of T which is the nearest to x. Shioji and Takahashi [2] extended the result of Shimizu and Takahashi [1] to a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T which is the nearest to x. Very recently, Song and Chen [3] also extended the result of Shimizu and Takahashi [1] to a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping. But this approximation method is not suitable for some nonexpansive nonself-mappings. In 2004, Matsushita and Kuroiwa [4] studied the strong convergence of the sequences $\{x_n\}$ and $\{y_n\}$ for nonexpansive nonself-mappings in the framework of a real Hilbert space. We can restate the sequences $\{x_n\}$ and $\{y_n\}$ as follows:

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j x_n \quad \text{for } n = 0,1,2,\dots, \tag{1.2}$$

$$y_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n}P(\alpha_n y + (1-\alpha_n)(TP)^j y_n) \quad \text{for } n = 0,1,2,\dots, \tag{1.3}$$

where x_0, x, y_0, y are all elements of C, P is the metric projection from H onto C, and T is a nonexpansive nonself-mapping from C into H. By using the nowhere normal outward condition for such a mapping T and appropriate conditions on $\{\alpha_n\}$, they proved that $\{x_n\}$ generated by (1.2) converges strongly to a fixed point of T which is the nearest to x; further they proved that $\{y_n\}$ generated by (1.3) converges strongly to a fixed point of T which is the nearest to y when $F(T)$ is nonempty.

In this paper, our purpose is to establish two strong convergence theorems of the iterative processes $\{x_n\}$ and $\{y_n\}$ defined by (1.2) and (1.3), respectively, for nonexpansive nonself-mappings in a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping from E to E^*. Our results extend and improve the results of Matsushita and Kuroiwa [4] to a Banach space setting.

2. Preliminaries

Throughout this paper, it is assumed that E is a real Banach space with norm $\|\cdot\|$; let J denote the normalized duality mapping from E into E^* given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\} \tag{2.1}$$

for each $x \in E$, where E^* denotes the dual space of E, $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing, and $\mathbb{N}$ denotes the set of all positive integers. In the sequel, we will denote the single-valued duality mapping by j, and denote $F(T) = \{x \in C : Tx = x\}$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x, x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp., weak, weak*) convergence of the sequence $\{x_n\}$ to x. In a Banach space E, the following result (*the subdifferential inequality*) is well known [5, Theorem 4.2.1]: for all $x, y \in E$, for all $j(x+y) \in J(x+y)$, for all $j(x) \in J(x)$,

$$\|x\|^2 + 2\langle y, j(x)\rangle \le \|x+y\|^2 \le \|x\|^2 + \langle y, j(x+y)\rangle. \tag{2.2}$$

Let E be a real Banach space and T a mapping with domain $D(T)$ and range $R(T)$ in E. T is called *nonexpansive* (resp., *contractive*) if for any $x, y \in D(T)$,

$$\|Tx - Ty\| \leq \|x - y\| \tag{2.3}$$

(resp., $\|Tx - Ty\| \leq \beta\|x - y\|$ for some $0 \leq \beta < 1$). A Banach space E is said to be *strictly convex* if

$$\|x\| = \|y\| = 1, \quad x \neq y \text{ imply } \frac{\|x + y\|}{2} < 1. \tag{2.4}$$

A Banach space E is said to be *uniformly convex* if for all $\epsilon \in (0,2]$, there exits $\delta_\epsilon > 0$ such that

$$\|x\| = \|y\| = 1 \quad \text{ with } \|x - y\| \geq \epsilon \text{ imply } \frac{\|x + y\|}{2} < 1 - \delta_\epsilon. \tag{2.5}$$

Recall that the norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.6}$$

exists for each x, y on the unit sphere $S(E)$ of E. The following results are well known and can be found in [5].

(i) A uniformly convex Banach space E is reflexive and strictly convex [5, Theorems 4.1.2 and 4.1.6].

(ii) If C is a nonempty convex subset of a strictly convex Banach space E and $T : C \to C$ is a nonexpansive mapping, then fixed point set $F(T)$ of T is a closed convex subset of C [5, Theorem 4.5.3].

If a Banach space E admits a weakly sequentially continuous duality mapping J from weak topology to weak star topology, from [6, Lemma 1], it follows that the duality mapping J is single-valued and also E is smooth. In this case, duality mapping J is also said to be *weakly sequentially continuous*, that is, for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, then $J(x_n) \overset{*}{\rightharpoonup} J(x)$ (see [6, 7]).

In the sequel, we also need the following lemma which can be found in [8].

Lemma 2.1 (Browder's demiclosed principle [8]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E, and suppose that $T : C \to E$ is nonexpansive. Then, the mapping I-T is demiclosed at zero, that is, $x_n \rightharpoonup x$, $x_n - Tx_n \to 0$ imply $x = Tx$.*

If C is a nonempty closed convex subset of a Banach space E and D is a nonempty subset of C, then a mapping $P : C \to D$ is called a *retraction* if $Px = x$ for all $x \in D$. A mapping $P : C \to D$ is called *sunny* if

$$P(Px + t(x - Px)) = Px, \quad \forall x \in C, \tag{2.7}$$

whenever $Px + t(x - Px) \in C$ and $t > 0$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D. For more details, see [5, 6]. The following lemma can be found in [5].

LEMMA 2.2. *Let C be a nonempty closed convex subset of a smooth Banach space E, $D \subset C$, $J : E \to E^*$ the normalized duality mapping of E, and $P : C \to D$ a retraction. Then, the following are equivalent:*

(i) $\langle x - Px, j(y - Px)\rangle \leq 0$, *for all* $x \in C$, *for all* $y \in D$;

(ii) *P is both sunny and nonexpansive.*

Let E be a smooth Banach space and let C be a nonempty closed convex subset of E. Let P be a sunny nonexpansive retraction from E onto C. Then, P is unique. For more details, see [9]. For a nonself-mapping T from C into E, Matsushita and Takahashi [9] studied the following condition:

$$Tx \in S_x^c \tag{2.8}$$

for all $x \in C$, where $S_x = \{y \in E : y \neq x, Py = x\}$ and P is a sunny nonexpansive retraction from E onto C.

Remark 2.3 [9, Remark 2.1]. If C is a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E, then for any $x \in E$, there exists a unique point $x_0 \in C$ such that

$$||x_0 - x|| = \min_{y \in C} \|y - x\|. \tag{2.9}$$

The mapping Q from E onto C defined by $Qx = x_0$ is called the *metric projection*. Using the metric projection Q, Halpern and Bergman [10] studied the following condition:

$$Tx \in \{y \in E : y \neq x, Qy = x\}^c \tag{2.10}$$

for all $x \in C$. Such a condition is called the *nowhere-normal outward condition*. Note that if E is a Hilbert space, then the condition (2.8) and the nowhere-normal outward condition are equivalent.

In the sequel, we also need the following lemmas which can be found in [9].

LEMMA 2.4 [9, Lemma 3.1]. *Let C be a closed convex subset of a smooth Banach space E and let T be a mapping form C into E. Suppose that C is a sunny nonexpansive retract of E. If T satisfies the condition (2.8), then $F(T) = F(PT)$, where P is a sunny nonexpansive retraction from E onto C.*

LEMMA 2.5 [9, Lemma 3.3]. *Let C be a closed convex subset of a strictly convex Banach space E and let T be a nonexpansive mapping from C into E. Suppose that C is a sunny nonexpansive retract of E. If $F(T) \neq \emptyset$, then T satisfies the condition (2.8).*

The following theorem was proved by Bruck [11].

THEOREM 2.6. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let $T : C \to C$ be nonexpansive. For each $x \in C$ and the Cesàro means $T_n x = 1/n \sum_{j=0}^{n-1} T^j x$, then $\lim_{n\to\infty} \sup_{x \in C} \|T_n x - T(T_n x)\| = 0$.*

3. Main results

In this section, we prove two strong convergence theorems for a nonexpansive nonself-mapping in a uniformly convex Banach space.

Theorem 3.1. *Let E be a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J from E to E^* and C a nonempty closed convex subset of E. Suppose that C is a sunny nonexpansive retract of E. Let P be the sunny nonexpansive retraction of E onto C, T a nonexpansive nonself-mapping from C into E with $F(T) \neq \emptyset$, and $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let the sequence $\{x_n\}$ be defined by (1.2). Then, $\{x_n\}$ converges strongly to $Qx \in F(T)$, where Q is the sunny nonexpansive retraction from C onto $F(T)$.*

Proof. Let $x \in C$, $z \in F(T)$, and $M = \max\{\|x - z\|, \|x_0 - z\|\}$. Then, we have

$$||x_1 - z|| = ||\alpha_0 x + (1-\alpha_0)x_0 - z|| \leq \alpha_0\|x - z\| + (1-\alpha_0)||x_0 - z|| \leq M. \tag{3.1}$$

If $\|x_n - z\| \leq M$ for some $n \in \mathbb{N}$, then we can show that $\|x_{n+1} - z\| \leq M$ similarly. Therefore, by induction on n, we obtain $\|x_n - z\| \leq M$ for all $n \in \mathbb{N}$, and hence $\{x_n\}$ is bounded, so is $\{(1/n+1)\sum_{j=0}^{n}(PT)^j x_n\}$. We define $T_n := (1/n+1)\sum_{j=0}^{n}(PT)^j$ for all $n \in \mathbb{N}$. Then, for any $p \in F(T)$, we get $\|T_n x_n - p\| \leq (1/n+1)\sum_{j=0}^{n}\|(PT)^j x_n - (PT)^j p\| \leq \|x_n - p\|$. Therefore, $\{T_n x_n\}$ is also bounded. We observe that

$$\begin{aligned} ||x_{n+1} - T_n x_n|| &= \left|\left|x_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j x_n\right|\right| \\ &= \left|\left|\alpha_n x + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j x_n - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j x_n\right|\right| \\ &= \alpha_n\left|\left|x - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j x_n\right|\right| = \alpha_n||x - T_n x_n||. \end{aligned} \tag{3.2}$$

It follows from (3.2) and $\lim_{n\to\infty}\alpha_n = 0$ that

$$\lim_{n\to\infty}||x_{n+1} - T_n x_n|| = 0. \tag{3.3}$$

Next, we prove that $\lim_{n\to\infty}\|x_n - PTx_n\| = 0$. Take $w \in F(T)$ and define a subset D of C by $D = \{x \in C : \|x - w\| \leq M\}$. Then, D is a nonempty closed bounded convex subset of C, $PT(D) \subset D$, and $\{x_n\} \subset D$. Hence, Theorem 2.6 implies that

$$\lim_{n\to\infty}\sup_{x\in D}||T_n x - PT(T_n x)|| = 0. \tag{3.4}$$

Furthermore,

$$\lim_{n\to\infty}||T_n x_n - PT(T_n x_n)|| \leq \lim_{n\to\infty}\sup_{x\in D}||T_n x - PT(T_n x)|| = 0. \tag{3.5}$$

Hence,

$$\lim_{n\to\infty}||T_n x_n - PT(T_n x_n)|| = 0. \tag{3.6}$$

It follows from (3.3) and (3.6) that

$$\begin{aligned} \|x_{n+1} - PTx_{n+1}\| &\le \|x_{n+1} - T_n x_n\| + \|T_n x_n - PT(T_n x_n)\| + \|PT(T_n x_n) - PTx_{n+1}\| \\ &\le 2\|x_{n+1} - T_n x_n\| + \|T_n x_n - PT(T_n x_n)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.7}$$

That is,

$$\lim_{n\to\infty} \|x_n - PTx_n\| = 0. \tag{3.8}$$

Next, we will show that

$$\limsup_{n\to\infty} \langle Qx - x, j(Qx - x_n)\rangle \le 0. \tag{3.9}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{n\to\infty} \langle Qx - x, j(Qx - x_{n_k})\rangle = \limsup_{n\to\infty} \langle Qx - x, j(Qx - x_n)\rangle. \tag{3.10}$$

It follows from reflexivity of E and boundedness of the sequence $\{x_{n_k}\}$ that there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ converging weakly to $w \in C$ as $i \to \infty$. It follows from (3.8) and the nonexpansivity of PT that we have $w \in F(PT)$ by Lemma 2.1. Since $F(T)$ is nonempty, it follows from Lemma 2.5 that T satisfies condition (2.8). Applying Lemma 2.4, we obtain that $w \in F(T)$. Since the duality map j is single-valued and weakly sequentially continuous from E to E^*, we get that

$$\begin{aligned} \limsup_{n\to\infty} \langle Qx - x, j(Qx - x_n)\rangle &= \lim_{k\to\infty} \langle Qx - x, j(Qx - x_{n_k})\rangle \\ &= \lim_{i\to\infty} \langle Qx - x, j(Qx - x_{n_{k_i}})\rangle \\ &= \langle Qx - x, j(Qx - w)\rangle \le 0 \end{aligned} \tag{3.11}$$

by Lemma 2.2 as required. Then, for any $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$\langle Qx - x, j(Qx - x_n)\rangle \le \epsilon \tag{3.12}$$

for all $n \ge m$. On the other hand, from

$$x_{n+1} - Qx + \alpha_n(Qx - x) = \alpha_n x + (1 - \alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j x_n - (\alpha_n x + (1 - \alpha_n)Qx) \tag{3.13}$$

and the inequality (2.2), we have

$$\begin{aligned}
&\|x_{n+1}-Qx\|^2\\
&\quad=\|x_{n+1}-Qx+\alpha_n(Qx-x)-\alpha_n(Qx-x)\|^2\\
&\quad\le\|x_{n+1}-Qx+\alpha_n(Qx-x)\|^2-2\alpha_n\langle Qx-x,j(x_{n+1}-Qx)\rangle\\
&\quad=\left\{\left\|(1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}\left((PT)^jx_n-Qx\right)\right\|\right\}^2-2\alpha_n\langle Qx-x,j(x_{n+1}-Qx)\rangle\\
&\quad\le\left\{(1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}\|(PT)^jx_n-Qx\|\right\}^2-2\alpha_n\langle Qx-x,j(x_{n+1}-Qx)\rangle\\
&\quad\le(1-\alpha_n)^2\|x_n-Qx\|^2+2\alpha_n\langle x-Qx,j(x_{n+1}-Qx)\rangle\\
&\quad\le(1-\alpha_n)\|x_n-Qx\|^2+2\alpha_n\epsilon\\
&\quad=2\epsilon(1-(1-\alpha_n))+(1-\alpha_n)\|x_n-Qx\|^2\\
&\quad\le2\epsilon(1-(1-\alpha_n))+(1-\alpha_n)(2\epsilon(1-(1-\alpha_{n-1}))+(1-\alpha_{n-1})\|x_{n-1}-Qx\|^2\\
&\quad=2\epsilon(1-(1-\alpha_n)(1-\alpha_{n-1}))+(1-\alpha_n)(1-\alpha_{n-1})\|x_{n-1}-Qx\|^2
\end{aligned}\tag{3.14}$$

for all $n\ge m$. By induction, we obtain

$$\|x_{n+1}-Qx\|^2\le2\epsilon\left(1-\prod_{k=m}^{n}(1-\alpha_k)\right)+\prod_{k=m}^{n}(1-\alpha_k)\|x_m-Qx\|^2.\tag{3.15}$$

Therefore, from $\sum_{n=0}^{\infty}\alpha_n=\infty$, we have

$$\limsup_{n\to\infty}\|x_{n+1}-Qx\|\le2\epsilon.\tag{3.16}$$

By arbitrarity of ϵ, we conclude that $\{x_n\}$ converges strongly to Qx in $F(T)$. This completes the proof. □

If in Theorem 3.1, T is self-mapping and $\{\alpha_n\}\subset(0,1)$, then the requirement that C is a sunny nonexpansive retract of E is not necessary. Furthermore, we have $PT=T$, then the iteration (1.2) reduces to the iteration (1.1). In fact, the following corollary can be obtained from Theorem 3.1 immediately.

Corollary 3.2 [3, Corollary 4.2]. *Let E be a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J from E to E^* and C a nonempty closed convex subset of E. Suppose that $T:C\to C$ is a nonexpansive mapping with $F(T)\neq\emptyset$, and $\{x_n\}$ is defined by (1.1), where $\{\alpha_n\}$ is a sequence of real numbers in $(0,1)$ satisfying $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=0}^{\infty}\alpha_n=\infty$. Then, as $n\to\infty$, $\{x_n\}$ converges strongly to $Qx\in F(T)$, where Q is the sunny nonexpansive retraction from C onto $F(T)$.*

If in Theorem 3.1 $E = H$ is a real Hilbert space, then the requirement that C is a sunny nonexpansive retract of E is not necessary. In fact, we have the following corollary due to Matsushita and Kuroiwa [4].

Corollary 3.3 [4, Theorem 1]. *Let H be a real Hilbert space, C a closed convex subset of H, P the metric projection of H onto C, T a nonexpansive nonself-mapping from C into H such that $F(T)$ is nonempty, and $\{\alpha_n\}$ a sequence of real numbers in $[0,1]$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then, $\{x_n\}$ defined by (1.2) converges strongly to Qx, where Q is the metric projection from C onto $F(T)$.*

Theorem 3.4. *Let E be a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J from E to E^* and C a nonempty closed convex subset of E. Suppose that C is a sunny nonexpansive retract of E. Let P be the sunny nonexpansive retraction of E onto C, T a nonexpansive nonself-mapping from C into E with $F(T) \neq \emptyset$, and $\{\alpha_n\}$ a sequence of real numbers such that $0 \le \alpha_n \le 1$, $\lim_{n\to\infty}\alpha_n = 0$, and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Let the sequence $\{y_n\}$ be defined by (1.3). Then, $\{y_n\}$ converges strongly to $Qy \in F(T)$, where Q is the sunny nonexpansive retraction from C onto $F(T)$.*

Proof. Let $y \in C$, $z \in F(T)$, and $M = \max\{\|y - z\|, \|y_0 - z\|\}$. Then, we have

$$||y_1 - z|| = ||P(\alpha_0 y + (1-\alpha_0)y_0) - z|| \le \alpha_0\|y - z\| + (1-\alpha_0)||y_0 - z|| \le M. \tag{3.17}$$

If $\|y_n - z\| \le M$ for some $n \in \mathbb{N}$, then we can show that $\|y_{n+1} - z\| \le M$ similarly. Therefore, by induction, we obtain $\|y_n - z\| \le M$ for all $n \in \mathbb{N}$ and hence $\{y_n\}$ is bounded, so is $\{(1/n+1)\sum_{j=0}^{n}(PT)^j y_n\}$. We observe that

$$\begin{aligned}\left\|y_{n+1} - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\right\| &= \left\|\frac{1}{n+1}\sum_{j=0}^{n}P(\alpha_n y + (1-\alpha_n)(TP)^j y_n) - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\right\| \\ &\le \frac{1}{n+1}\sum_{j=0}^{n}||P(\alpha_n y + (1-\alpha_n)(TP)^j y_n) - (PT)^j y_n|| \\ &\le \frac{1}{n+1}\sum_{j=0}^{n}||\alpha_n y + (1-\alpha_n)(TP)^j y_n - (TP)^j y_n|| \\ &= \alpha_n\frac{1}{n+1}\sum_{j=0}^{n}||y - (PT)^j y_n||.\end{aligned} \tag{3.18}$$

We define $T_n := (1/n+1)\sum_{j=0}^{n}(PT)^j$ for all $n \in \mathbb{N}$. It follows from $\lim_{n\to\infty}\alpha_n = 0$ and (3.18) that

$$\lim_{n\to\infty}||y_{n+1} - T_n y_n|| = 0. \tag{3.19}$$

Next, we prove that $\lim_{n\to\infty}\|y_n - PTy_n\| = 0$. Take $w \in F(T)$ and define a subset D of C by $D = \{y \in C : \|y - w\| \le M\}$. Then, clearly D is a nonempty closed bounded convex subset of C and $TP(D) \subset D$ and $\{y_n\} \subset D$. Since $PT(D) \subset D$, Theorem 2.6 implies that

$$\lim_{n\to\infty} \sup_{y\in D} ||T_n y - PT(T_n y)|| = 0. \tag{3.20}$$

Furthermore,

$$\lim_{n\to\infty} ||T_n y_n - PT(T_n y)|| \le \lim_{n\to\infty} \sup_{y\in D} ||T_n y - PT(T_n y)|| = 0. \tag{3.21}$$

Hence, using $\lim_{n\to\infty} \|T_n y_n - PT(T_n y)\| = 0$ along with (3.19), we obtain that

$$\begin{aligned} ||y_{n+1} - PTy_{n+1}|| &\le ||y_{n+1} - T_n y_n|| + ||T_n y_n - PT(T_n y_n)|| + ||PT(T_n y_n) - PTy_{n+1}|| \\ &\le 2||y_{n+1} - T_n y_n|| + ||T_n y_n - PT(T_n y_n)|| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.22}$$

That is,

$$\lim_{n\to\infty} ||y_n - PTy_n|| = 0. \tag{3.23}$$

Next, we will show that

$$\limsup_{n\to\infty} \langle Qy - y, j(Qy - y_n)\rangle \le 0. \tag{3.24}$$

Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ such that

$$\lim_{n\to\infty} \langle Qy - y, j(Qy - y_{n_k})\rangle = \limsup_{n\to\infty} \langle Qy - y, j(Qy - y_n)\rangle. \tag{3.25}$$

If follows from reflexivity of E and boundedness of sequence $\{y_{n_k}\}$ that there exists a subsequence $\{y_{n_{k_i}}\}$ of $\{y_{n_k}\}$ converging weakly to $w \in C$ as $i \to \infty$. Then, from (3.23) and the nonexpansivity of PT, we obtain that $w \in F(PT)$ by Lemma 2.1. Since $F(T)$ is nonempty, it follows from Lemma 2.5 that T satisfies condition (2.8). Applying Lemma 2.4, we obtain that $w \in F(T)$. By the assumption that the duality map J is single-valued and weakly sequentially continuous from E to E^*, Lemma 2.2 gives that

$$\begin{aligned} \limsup_{n\to\infty} \langle Qy - y, j(Qy - y_n)\rangle &= \lim_{k\to\infty} \langle Qy - y, j(Qy - y_{n_k})\rangle \\ &= \lim_{i\to\infty} \langle Qy - y, j(Qy - y_{n_{k_i}})\rangle \\ &= \langle Qy - y, j(Qy - w)\rangle \le 0 \end{aligned} \tag{3.26}$$

as required. Then for any $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$\langle Qy - y, j(Qy - y_n)\rangle \le \epsilon \tag{3.27}$$

for all $n \ge m$. On the other hand, from

$$y_{n+1} - Qy + \alpha_n(Qy - y) = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n y + (1-\alpha_n)(TP)^j y_n) - P(\alpha_n y + (1-\alpha_n)Qy) \tag{3.28}$$

and the inequality (2.2), we have

$$\begin{aligned}
&||y_{n+1}-Qy||^2\\
&\quad= ||y_{n+1}-Qy+\alpha_n(Qy-y)-\alpha_n(Qy-y)||^2\\
&\quad\le ||y_{n+1}-Qy+\alpha_n(Qy-y)||^2-2\alpha_n\langle Qy-y,j(y_{n+1}-Qy)\rangle\\
&\quad\le \left\|\frac{1}{n+1}\sum_{j=0}^{n}P(\alpha_n y+(1-\alpha_n)(TP)^j y_n)-P(\alpha_n y+(1-\alpha_n)Qy)\right\|^2\\
&\qquad -2\alpha_n\langle Qy-y,j(y_{n+1}-Qy)\rangle\\
&\quad= \left\{\frac{1}{n+1}\sum_{j=0}^{n}||P(\alpha_n y+(1-\alpha_n)(TP)^j y_n)-P(\alpha_n y+(1-\alpha_n)Qy)||\right\}^2\\
&\qquad -2\alpha_n\langle Qy-y,j(y_{n+1}-Qy)\rangle\\
&\quad\le \left\{(1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}||(TP)^j y_n-Qy||\right\}^2-2\alpha_n\langle Qy-y,j(y_{n+1}-Qy)\rangle\\
&\quad\le (1-\alpha_n)^2||y_n-Qy||^2+2\alpha_n\langle y-Qy,j(y_{n+1}-Qy)\rangle\\
&\quad\le (1-\alpha_n)||y_n-Qy||^2+2\alpha_n\epsilon\\
&\quad= 2\epsilon(1-(1-\alpha_n))+(1-\alpha_n)||y_n-Qy||^2\\
&\quad\le 2\epsilon(1-(1-\alpha_n))+(1-\alpha_n)(2\epsilon(1-(1-\alpha_{n-1}))+(1-\alpha_{n-1})||y_{n-1}-Qy||^2\\
&\quad= 2\epsilon(1-(1-\alpha_n)(1-\alpha_{n-1}))+(1-\alpha_n)(1-\alpha_{n-1})||y_{n-1}-Qy||^2
\end{aligned} \tag{3.29}$$

for all $n \ge m$. By induction, we obtain

$$||y_{n+1}-Qy||^2 \le 2\epsilon\left(1-\prod_{k=m}^{n}(1-\alpha_k)\right)+\prod_{k=m}^{n}(1-\alpha_k)||y_m-Qy||^2. \tag{3.30}$$

It follows from $\sum_{n=0}^{\infty}\alpha_n=\infty$ that

$$\limsup_{n\to\infty}||y_{n+1}-Qy|| \le 2\epsilon. \tag{3.31}$$

By arbitrarity of ϵ, we conclude that $\{y_n\}$ converges strongly to Qy in $F(T)$. This completes the proof. □

If in Theorem 3.4, $E = H$ is a real Hilbert space, then the requirement that C is a sunny nonexpansive retract of E is not necessary. In fact, we have the following corollary due to Matsushita and Kuroiwa [4].

Corollary 3.5 [4, Theorem 2]. *Let H be a real Hilbert space, C a closed convex subset of H, P the metric projection of H onto C, T a nonexpansive nonself-mapping from C into H such that $F(T)$ is nonempty, and $\{\alpha_n\}$ a sequence of real numbers in* $[0,1]$ *satisfying*

$\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$. *Then,* $\{y_n\}$ *defined by (1.3) converges strongly to* Qy, *where* Q *is the metric projection from* C *onto* $F(T)$.

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Rabian Wangkeeree: Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand
Email address: rabianw@nu.ac.th