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Research Article

An Extension of Gregus Fixed Point Theorem

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Let C be a closed convex subset of a complete metrizable topological vector space (X,d) and $T: C \to C$ a mapping that satisfies $d(Tx,Ty) \leq ad(x,y) + bd(x,Tx) + cd(y,Ty) + ed(y,Tx) + fd(x,Ty)$ for all $x,y \in C$, where $0 < a < 1$, $b \geq 0$, $c \geq 0$, $e \geq 0$, $f \geq 0$, and $a+b+c+e+f = 1$. Then T has a unique fixed point. The above theorem, which is a generalization and an extension of the results of several authors, is proved in this paper. In addition, we use the Mann iteration to approximate the fixed point of T.

1. Introduction

Gregus [1] proved the following theorem.

Theorem 1.1. *Let C be a closed convex subset of a Banach space X and $T: C \to C$ a mapping that satisfies $\|Tx - Ty\| \leq a\|x-y\| + b\|x - Tx\| + c\|y - Ty\|$ for all $x, y \in C$, where $0 < a < 1$, $b \geq 0$, $c \geq 0$, and $a+b+c = 1$. Then T has a unique fixed point.*

Several papers have been written on the Gregus fixed point theorem. For example, see [2, 3]. The theorem has been generalized to the condition when X is a complete metrizable toplogical vector space [4].

When $a = 1$, $b = 0$, $c = 0$, T becomes a nonexpansive map. In the past four decades, several papers have been written on the existence of a fixed point (which may not be unique) for a nonexpansive map defined on a closed bounded and convex subset C of a Banach space X. For example, see [5–7]. Recently, the existence of fixed points of T when the domain of T is unbounded was discussed in [6]. When $a = 0$, we have the Kannan maps. Similarly, several papers have been written on the existence of a fixed point for a

Kannan map defined on a Banach space, for example, see [8, 9]. The fixed point theorem of Gregus is interesting because it tells what happens if $0 < a < 1$.

Chatterjea [10] considered the existence of fixed point for T when T is defined on a metric space (X,d), such that for $0 < a < 1/2$,

$$d(Tx,Ty) \le a\{d(x,f(y)) + d(y,f(x))\}. \tag{1.1}$$

It is natural to combine this condition with that of Gregus to get the following condition:

$$d(Tx,Ty) \le ad(x,y) + bd(x,Tx) + cd(y,Ty) + ed(y,Tx) + fd(x,Ty) \tag{1.2}$$

for all $x, y \in C$, where $0 < a < 1$, $b \ge 0$, $c \ge 0$, $e \ge 0$, $f \ge 0$, and $a+b+c+e+f = 1$.

Observe that if T satisfies (1.2), then it also satisfies

$$d(Tx,Ty) \le ad(x,y) + pd(x,Tx) + pd(y,Ty) + pd(y,Tx) + pd(x,Ty) \tag{1.3}$$

for all $x, y \in C$, where $0 < a < 1$, $p \ge 0$, $a+4p = 1$, $(p = (1/4)b + (1/4)c + (1/4)e + (1/4)f)$. Thus b, c, e, and f will be used interchangeably as p in the proof of our main theorem.

As observed by Chidume [5, page 119], since the four points $\{x, y, Tx, Ty\}$ of (1.2) determine six distances in X, the inequality amounts to say that the image distance $d(Tx, Ty)$ never exceeds a fixed convex combination of the remaining five distances. Geometrically, this type of condition is quite natural.

In this paper, we extend Gregus result to the condition when T satisfies condition (1.2) and also generalize it to the condition when X is a complete metrizable topological vector space, thus answering the question posed in [4]. Complete metrizable topological vector spaces include uniformly convex Banach spaces, Banach spaces and complete metrizable locally convex spaces (see [11, 12]).

The following result will be needed for our result.

Theorem 1.2 [13, 14]. *A topological vector space X is metrizable if and only if it has a countable base of neighbourhoods of zero. The topology of a metrizable topological vector space can always be defined by a real-valued function $\|\cdot\| : X \to \Re$, called F-norm such that for all $x, y \in X$,*

(1) $\|x\| \ge 0$,
(2) $\|x\| = 0 \Rightarrow x = 0$,
(3) $\|x+y\| \le \|x\| + \|y\|$,
(4) $\|\lambda x\| \le \|x\|$ *for all* $\lambda \in K$ *with* $|\lambda| \le 1$,
(5) *if* $\lambda_n \to 0$, *and* $\lambda_n \in K$, *then* $\|\lambda_n x\| \to 0$.

For the same result see Kothe [15, Section 15.11]. Henceforth, unless otherwise indicated, F will denote an F-norm if it is characterizing a metrizable topological vector space. Observe that an F-norm will be a norm if it is defining a normed space.

We now prove our main theorem. We use the technique in [4] which is due to Gregus [1].

Theorem 1.3. *Let C be a closed convex subset of a complete metrizable space X and $T : C \to C$ a mapping that satisfies $F(Tx - Ty) \leq aF(x - y) + bF(x - Tx) + cF(y - Ty) + eF(y - Tx) + fF(x - Ty)$ for all $x, y \in C$, where $0 < a < 1$, $b \geq 0$, $c \geq 0$, $e \geq 0$, $f \geq 0$, and $a + b + c + e + f = 1$. Then T has a unique fixed point.*

Proof. Take any point $x \in C$ and consider the sequence $\{T_n(x)\}_{n=1}^{\infty}$,

$$\begin{aligned} F(T^n x - T^{n-1}x) &\leq aF(T^{n-1}x - T^{n-2}x) + bF(T^{n-1}x - T^n x) \\ &\quad + cF(T^{n-2}x - T^{n-1}x) + eF(T^{n-2}x - T^n x) \\ &\quad + fF(T^{n-1}x - T^{n-1}x) \\ &\leq \frac{a + c + e}{1 - b - e} F(T^{n-1}x - T^{n-2}x) \\ &\leq \frac{a + 2p}{1 - 2p} F(T^{n-1}x - T^{n-2}x) \leq F(Tx - x). \end{aligned} \tag{1.4}$$

Thus

$$F(T^n x - T^{n-1}x) \leq F(Tx - x). \tag{1.5}$$

In effect, it means that the distance between two consecutive elements of $\{T^n x\}$ is less or equal to the distance between the first and the second element. Now let us consider the distance between two consecutive elements with odd (resp., even) power of T. It is sufficient to consider only the distance between Tx and $T^3 x$,

$$\begin{aligned} F(T^3 x - Tx) &\leq aF(T^2 x - x) + bF(T^2 x - T^3 x) + cF(Tx - x) \\ &\quad + eF(x - T^3 x) + fF(T^2 x - Tx) \\ &\leq aF(T^2 x - Tx) + aF(Tx - x) + bF(T^2 x - T^3 x) \\ &\quad + cF(Tx - x) + eF(x - Tx) + eF(Tx - T^2 x) \\ &\quad + eF(T^2 x - T^3 x) + fF(T^2 x - Tx) \\ &\leq (2a + b + c + 3e + f)F(Tx - x) = (a + 2p + 1)F(Tx - x). \end{aligned} \tag{1.6}$$

Hence

$$F(T^3 x - Tx) \leq (a + 2p + 1)F(Tx - x) \quad \forall x \in C. \tag{1.7}$$

Since C is convex, therefore $z = (1/2)T^2 x + (1/2)T^3 x$ is in C, and from the properties of the F-norm, we have

$$\begin{aligned} F(Tz - z) &\leq \frac{1}{2}F(Tz - T^2 x) + \frac{1}{2}F(Tz - T^3 x) \\ &\leq \frac{1}{2}\{aF(z - Tx) + bF(Tz - z) + cF(Tx - T^2 x) \\ &\qquad + eF(Tx - Tz) + fF(z - T^2 x)\} \\ &\quad + \frac{1}{2}\{aF(z - T^2 x) + bF(Tz - z) + cF(T^3 x - T^2 x) \\ &\qquad + eF(T^2 x - Tz) + fF(z - T^3 x)\}, \end{aligned}$$

$$
\begin{aligned}
F(z-Tx) &\le \frac{1}{2}F(T^2x-Tx)+\frac{1}{2}F(T^3x-Tx)\\
&\le \frac{1}{2}F(Tx-x)+\frac{1}{2}(a+2p+1)F(Tx-x)=\left(1+p+\frac{1}{2}a\right)F(Tx-x),\\
F(z-T^2x) &\le \frac{1}{2}F(T^3x-T^2x)\le \frac{1}{2}F(Tx-x).
\end{aligned} \tag{1.8}
$$

Similarly,

$$
\begin{aligned}
F(z-T^3x) &\le \frac{1}{2}F(Tx-x),\\
F(Tx-Tz) &\le \frac{1}{2}F(Tx-T^3x)+\frac{1}{2}F(Tx-T^4x)\\
&\le \frac{1}{2}(a+2p+1)F(Tx-x)+\frac{1}{2}\{F(Tx-T^2x)+F(T^2x-T^4x)\}\\
&\le \frac{1}{2}(a+2p+1)F(Tx-x)+\frac{1}{2}\{F(Tx-x)+(a+2p+1)F(Tx-x)\}\\
&\le \left(a+2p+\frac{3}{2}\right)F(Tx-x),\\
F(T^2x-Tz) &\le \frac{1}{2}F(T^2x-T^3x)+\frac{1}{2}F(T^2x-T^4x)\le\left(\frac{1}{2}a+p+1\right)F(Tx-x).
\end{aligned} \tag{1.9}
$$

Thus

$$
\begin{aligned}
(1-b)F(Tz-z) \le \frac{1}{2}\Big\{&a\left(1+p+\frac{1}{2}a\right)F(Tx-x)+cF(Tx-x)\\
&+e\left(a+2p+\frac{3}{2}\right)F(Tx-x)+\frac{1}{2}fF(Tx-x)\Big\}\\
&+\frac{1}{2}\Big\{\frac{1}{2}aF(Tx-x)+cF(Tx-x)+\frac{1}{2}e(a+2p+1)F(Tx-x)\\
&+\frac{1}{2}fF(Tx-x)\Big\}=\left(\frac{3}{4}a+\frac{1}{4}a^2+\frac{5}{4}ap+\frac{5}{2}p+\frac{3}{2}p^2\right)F(Tx-x).
\end{aligned} \tag{1.10}
$$

Thus

$$
\begin{aligned}
4(1-p)F(z-Tz) &\le (3a+a^2+5ap+10p+6p^2)F(Tx-x)\\
&\le (2p^2-5p+4)F(Tx-x).
\end{aligned} \tag{1.11}
$$

Hence

$$
\begin{aligned}
F(z-Tz) &\le \frac{26-22a-a^2}{8(a+3)}F(Tx-x),\\
F(Tz-z) &\le \lambda F(Tx-x),
\end{aligned} \tag{1.12}
$$

where $\lambda=(26-22a-a^2)/8(a+3)$. It is clear that $0<\lambda<1$.

Now let $i = \inf\{F(Tx - x) : x \in C\}$. Then there exists a point $x \in C$ such that $F(Tx - x) < i + \epsilon$ for $\epsilon > 0$.

Suppose $i > 0$. Then for $0 < \epsilon < (1-\lambda)i/\lambda$ and $F(Tx - x) < i + \epsilon$, we have

$$F(Tz - z) \le \lambda F(Tx - x) \le \lambda(i + \epsilon) < i, \tag{1.13}$$

that is, $F(Tz - z) < i$, which is a contradiction with the definition of i. Hence $\inf\{F(Tx - x) : x \in C\} = 0$.

To prove that the infimum is attained is the easy part of the proof. Take the following system of sets: $K_n = \{x : F(x - Tx) \le 1/2n(q+1)\}$; $T(K_n)$ and $\overline{T(K_n)}$, where $n \in N$, $q = (a+p)/(1-a)$, and $\overline{T(K_n)}$ is the closure of $T(K_n)$. Then for any $x, y \in K_n$,

$$\begin{gathered} F(Tx - Ty) \le qF(Tx - x) + qF(Ty - y) \le \frac{1}{n}, \\ F(x - y) \le (q+1)F(Tx - x) + (q+1)F(Ty - y) \le \frac{1}{n}, \end{gathered} \tag{1.14}$$

that is, $\operatorname{diam}(K_n) \le 1/n$, $\operatorname{diam}(T(K_n)) \le 1/n$ and therefore, since $\operatorname{diam}(T(K_n)) = \operatorname{diam}(\overline{T(K_n)})$, we have $\operatorname{diam}(\overline{T(K_n)}) \le 1/n$. It is clear that $\{K_n\}$ and $\{\overline{T(K_n)}\}$ form monotone sequences of sets and from (1.5) we have $T(K_n) \subset K_n$. Suppose $y \in \overline{T(K_n)}$, then there exists $y' \in K_n$ such that $F(y - Ty') < \epsilon$ for $\epsilon > 0$ and

$$\begin{aligned} F(y - Ty) &\le F(y - Ty') + F(Ty' - Ty) \\ &\le F(y - Ty') + aF(y - y') + bF(y' - Ty') \\ &\quad + cF(Ty - y) + eF(y - Ty') + fF(y' - Ty). \end{aligned} \tag{1.15}$$

Hence

$$(1 - c))F(y - Ty) \le (1 + a + e + f)\epsilon + (a+b)F(Ty' - y'). \tag{1.16}$$

Since $F(y' - Ty') \le 1/2n(q+1)$, then

$$F(y - Ty) \le \frac{1+a+e+f}{1-c}\epsilon + \frac{a+b}{1-c}\frac{1}{2n(q+1)}. \tag{1.17}$$

Since $\epsilon > 0$ is arbitrary and $a + b + c \le 1$, then $F(y - Ty) \le 1/2n(q+1)$ and we have $y \in K_n$. Hence $\overline{T(K_n)} \subset K_n$, too.

$\{\overline{T(K_n)}\}$ is a decreasing sequence of closed nonempty sets with $\operatorname{diam}(\overline{T(K_n)}) \to 0$ as $n \to \infty$. Hence they have a nonempty intersection $\{x*\}$ and T has a unique fixed point $Tx* = x*$. □

Corollary 1.4. *Let C be a closed convex subset of a Banach space X and $T : C \to C$ a mapping that satisfies $\|Tx - Ty\| \le a\|x - y\| + b\|Tx - x\| + c\|Ty - y\| + e\|Tx - y\| + f\|Ty - x\|$ for all $x, y \in C$ where $0 < a < 1$, $b \ge 0$, $c \ge 0$, $e \ge 0$, $f \ge 0$, and $a + b + c + e + f = 1$. Then T has a unique fixed point.*

Corollary 1.5 [1]. *Let C be a closed convex subset of a Banach space X and $T : C \to C$ a mapping that satisfies $\|Tx - Ty\| \le a\|x - y\| + b\|Tx - x\| + c\|Ty - y\|$ for all $x, y \in C$, where $0 < a < 1$, $b \ge 0$, $c \ge 0$, and $a + b + c = 1$. Then T has a unique fixed point.*

Corollary 1.6. *Let C be a closed convex subset of a complete metrizable topological vector space X and $T : C \to C$ a mapping that satisfies $\|Tx - Ty\| \leq a\|x - y\| + b\|Tx - y\| + c\|Ty - x\|$ for all $x, y \in C$, where $0 < a < 1$, $b \geq 0$, $c \geq 0$, and $a + b + c = 1$. Then T has a unique fixed point.*

We now proceed to use the Mann iteration scheme [16] to approximate the fixed point of our mapping under consideration.

Theorem 1.7. *Let C be a nonempty closed convex subset of a complete metrizable topological vector space X and let $T : C \to C$ be a mapping that satisfies $F(Tx - Ty) \leq aF(x - y) + bF(Tx - x) + cF(Ty - y) + eF(Tx - y) + fF(Ty - x)$ for all $x, y \in C$, where $0 < a < 1$, $b \geq 0$, $c \geq 0$, $e \geq 0$, $f \geq 0$, and $a + b + c + e + f = 1$. Suppose $\{x_n\}$ is a Mann iteration sequence defined by $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n$, $x_0 \in C$, $n \geq 0$, where $\{\alpha_n\}$ satisfy $0 < \alpha_n \leq 1$ for all n, $\sum_0^\infty \alpha_n = \infty$. Assume $2c < c + b$, then $\{x_n\}$ converges to the unique fixed point of T.*

Proof. The fact that T has a unique fixed point is already shown in Theorem 1.3.

If $F(Tx - Ty) \leq aF(x - y) + bF(Tx - x) + cF(Ty - y) + eF(Tx - y) + fF(Ty - x)$, then

$$\begin{aligned} F(Tx - Ty) \leq\ & aF(x - y) + bF(Tx - x) + c\{F(Ty - Tx) + F(Tx - x) + F(x - y)\} \\ & + e\{F(Tx - x) + F(x - y)\} + f\{F(Ty - Tx) + F(Tx - x)\}. \end{aligned} \tag{1.18}$$

After computation, we have $F(Tx - Ty) \leq ((a + c + e)/(1 - (c + f)))F(x - y) + ((b + c + e + f)/(1 - (c + f)))F(Tx - x)$. If $\delta = (a + c + e)/(1 - (c + f))$, then

$$F(Tx - Ty) \leq \delta F(x - y) + \frac{b + c + e + f}{1 - (c + f)} F(Tx - x)\}. \tag{1.19}$$

Since by assumption $2c < b + c$, it is clear that $\delta < 1$.

Suppose p is a fixed point of T, then if $x = p$ and $y = x_n$, from (1.19), we obtain

$$\begin{aligned} F(Tx_n - p) &\leq \delta F(x_n - p), \\ F(x_{n+1} - p) &= F((1 - \alpha_n)x_n + \alpha_n Tx_n - (1 - \alpha_n + \alpha_n)p) \\ &= F((1 - \alpha_n)(x_n - p) + \alpha_n(Tx_n - p)) \\ &\leq (1 - \alpha_n)F(x_n - p) + \alpha_n F(Tx_n - p) \\ &\leq (1 - \alpha_n(1 - \delta))F(x_n - p). \end{aligned} \tag{1.20}$$

Since $1 - \alpha_n(1 - \delta) < 1$ by the choice of α_n in the theorem, then $\{x_n\}$ converges to p. □

Remarks 1.8. (1) Gregus [1] gave an example in which $a = 1$, C is closed convex and bounded but yet T does not have a fixed point. If $a = 1$, some form of boundedness must be assumed on C for T to have a fixed point, for example, see [7, 6]. The same is true if $a = 0$ (see [8, 9]).

(2) If (X, d) is a complete metric space and $a + b + c + e + f < 1$, it was shown in [17] that T as defined in (1.2) has a unique fixed point. However, if $a + b + c + e + f = 1$, Hardy

and Rogers [17] assumed that T is continuous and X is compact in order to prove the existence of fixed point for T as defined in (1.2). Goebel et al. [18] obtained the existence of fixed point for T as defined by (1.2) when $a+b+c+e+f=1$. In which case, it was assumed that X is a uniformly convex Banach space, T is continuous and C is bounded, closed, and convex. In our result, T is not assumed to be continuous, X is assumed to be neither a compact nor a uniformly convex Banach space, and there is no boundedness assumption on C.

(3) Berinde [14] showed that the Ishikawa iteration sequence [16] of a class of quasi-contractive operators, called Zamfirescu operators, defined on a closed convex subset C of a Banach space X converges to the fixed point of T. The first author [19] showed that if X is a complete metrizable locally convex space, and C is closed and convex, then the Mann iteration sequence of the Zamfirescu operator T defined on C converges to the fixed point of T. In both cases, the sum of the constants is less than 1 while in Theorem 1.7, the sum is 1. In addition, X is generalized to a complete metrizable topological vector spaces. Can Theorem 1.7 still be proved without the assumption that $2c<a+b$?

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