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Research Article

Existence Principle for Advanced Integral Equations on Semiline

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The continuation principle for generalized contractions in gauge spaces is used to discuss nonlinear integral equations with advanced argument.

1. Introduction

This paper deals with an integral equation with advanced argument. The advanced argument makes necessary the use of two pseudometrics in the contraction condition. For this reason we will apply the continuation principle established in Chiş and Precup [1] involving contractions in Gheorghiu's sense, with respect to a family of pseudometrics rather than the existence principle from Frigon [2, 3].

In what follows we recall some notions and results from papers Chiş and Precup [1] and Chiş [4].

First recall the notion of a contraction on a gauge space introduced by Gheorghiu [5].

Definition 1.1 (Gheorghiu [5]). Let $(X,\mathcal{P})$ be a gauge space with the family of pseudometrics $\mathcal{P}=\{p_\alpha\}_{\alpha\in A}$, where A is a set of indices. A map $F:D\subset X\to X$ is a *contraction* if there exists a function $\varphi:A\to A$ and $a\in\mathbb{R}_+^A, a=\{a_\alpha\}_{\alpha\in A}$ such that

$$\begin{gathered} p_\alpha(F(x),F(y))\le a_\alpha p_{\varphi(\alpha)}(x,y),\quad \forall\alpha\in A,\ x,y\in D,\\ \sum_{n=1}^{\infty}a_\alpha a_{\varphi(\alpha)}a_{\varphi^2(\alpha)}\cdots a_{\varphi^{n-1}(\alpha)}p_{\varphi^n(\alpha)}(x,y)<\infty, \end{gathered}\tag{1.1}$$

for every $\alpha\in A$ and $x,y\in D$. Here, φ^n is the nth iteration of φ.

THEOREM 1.2 (Chiş [4]). *Let X be a set endowed with two separating gauge structures: $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$ and $\mathcal{Q} = \{q_\beta\}_{\beta\in B}$, let D_0 and D be two subsets of X with $D_0 \subset D$, and let $F : D \to X$ be a map. Assume that $F(D_0) \subset D_0$ and D is $\mathcal{P}$-closed. In addition, assume that the following conditions are satisfied:*

(i) *there is a function $\psi : A \to B$ and $c \in (0,\infty)^A$, $c = \{c_\alpha\}_{\alpha\in A}$ such that*

$$p_\alpha(x,y) \le c_\alpha q_{\psi(\alpha)}(x,y), \quad \forall\alpha \in A,\ x,y \in X; \tag{1.2}$$

(ii) *$(X,\mathcal{P})$ is a sequentially complete gauge space;*

(iii) *if $x_0 \in D$, $x_n = F(x_{n-1})$, for $n = 1,2,\dots$, and $\mathcal{P} - \lim_{n\to\infty} x_n = x$ for some $x \in D$, then $F(x) = x$;*

(iv) *F is a $\mathcal{Q}$-contraction on D_0.*

Then F has at least one fixed point which can be obtained by successive approximations starting from any element of D_0.

For a map $H : D \times [0,1] \to X$, where $D \subset X$, we will use the following notations:

$$\begin{aligned}\Sigma &= \{(x,\lambda) \in D \times [0,1] : H(x,\lambda) = x\},\\ S &= \{x \in D : H(x,\lambda) = x,\ \text{for some } \lambda \in [0,1]\},\\ \Lambda &= \{\lambda \in [0,1] : H(x,\lambda) = x,\ \text{for some } x \in D\}.\end{aligned} \tag{1.3}$$

THEOREM 1.3 (Chiş and Precup [1]). *Let X be a set endowed with the separating gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$ and $\mathcal{Q}^\lambda = \{q^\lambda_\beta\}_{\beta\in B}$, for $\lambda \in [0,1]$. Let $D \subset X$ be $\mathcal{P}$-sequentially closed, $H : D \times [0,1] \to X$ a map, and assume that the following conditions are satisfied:*

(i) *for each $\lambda \in [0,1]$, there exists a function $\varphi_\lambda : B \to B$ and $a^\lambda \in [0,1)^B$, $a^\lambda = \{a^\lambda_\beta\}_{\beta\in B}$ such that*

$$\begin{gathered}q^\lambda_\beta(H(x,\lambda),H(y,\lambda)) \le a^\lambda_\beta q^\lambda_{\varphi_\lambda(\beta)}(x,y),\\ \sum_{n=1}^{\infty} a^\lambda_\beta a^\lambda_{\varphi_\lambda(\beta)} a^\lambda_{\varphi^2_\lambda(\beta)} \cdots a^\lambda_{\varphi^{n-1}_\lambda(\beta)} < \infty,\end{gathered} \tag{1.4}$$

for every $\beta \in B$ and $x,y \in D$;

(ii) *there exists $\rho > 0$ such that for each $(x,\lambda) \in \Sigma$, there is a $\beta \in B$ with*

$$\inf\{q^\lambda_\beta(x,y) : y \in X\setminus D\} > \rho; \tag{1.5}$$

(iii) *for each $\lambda \in [0,1]$, there is a function $\psi : A \to B$ and $c \in (0,\infty)^A$, $c = \{c_\alpha\}_{\alpha\in A}$ such that*

$$p_\alpha(x,y) \le c_\alpha q^\lambda_{\psi(\alpha)}(x,y), \quad \forall\alpha \in A,\ x,y \in X; \tag{1.6}$$

(iv) *$(X,\mathcal{P})$ is a sequentially complete gauge space;*

(v) *if $\lambda \in [0,1]$, $x_0 \in D$, $x_n = H(x_{n-1},\lambda)$, for $n = 1,2,\dots$, and $\mathcal{P} - \lim_{n\to\infty} x_n = x$, then $H(x,\lambda) = x$;*

(vi) *for every* $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon) > 0$ *with*

$$q^{\lambda}_{\varphi^n_\lambda(\beta)}(x,H(x,\lambda)) \leq (1-a^{\lambda}_{\varphi^n_\lambda(\beta)})\varepsilon, \tag{1.7}$$

for $(x,\mu) \in \Sigma$, $|\lambda - \mu| \leq \delta$, *all* $\beta \in B$, *and* $n \in \mathbb{N}$.

In addition, assume that $H_0 := H(\cdot,0)$ *has a fixed point. Then, for each* $\lambda \in [0,1]$, *the map* $H_\lambda := H(\cdot,\lambda)$ *has at least a fixed point.*

2. The main result

We consider the integral equation inspired from biomathematics (see O'Regan and Precup [6]):

$$x(t) = \int_{t-1}^{t} f(s,x(s+2)), \quad t \in [0,\infty). \tag{2.1}$$

Let $I = [-1,\infty)$ and for a function $u \in L^1(a,b)$ we denote by $|u|_{L^1(a,b)}$ the norm in $L^1(a,b)$. We have the following existence principle for (2.1).

Theorem 2.1. *Let* $(E,\|\cdot\|)$ *be a Banach space, and let* $f : I \times E \to E$ *be a continuous function. Assume that the following conditions hold:*

(a) *there exists* $k : I \to (0,\infty)$, $k \in L^1_{\text{loc}}(I)$ *with* $|k|_{L^1_{\text{loc}}(I)} < 1$ *such that*

$$||f(t,x) - f(t,y)|| \leq k(t)|x-y| \tag{2.2}$$

for all $x, y \in E$, *and* $t \in I$;

(b) *for each* $n \in \mathbb{N}$ *there exists* $r_n > 0$ *such that, any continuous solution* x *of the equation*

$$x(t) = \lambda\left(\int_{t-1}^{t} f(s,x(s+1))\,ds\right), \quad t \in [0,\infty) \tag{2.3}$$

with $\lambda \in [0,1]$, *satisfies* $\|x(t)\| \leq r_n$ *for any* $t \in [n,2n+1]$;

(c) *there exists* $\alpha \in L^1_{\text{loc}}(I)$ *şi* $\beta : [0,\infty) \to (0,\infty)$ *nondecreasing such that*

$$||f(t,x)|| \leq \alpha(t)\beta(\|x\|) \tag{2.4}$$

for all $t \in I$ *and* $x \in E$;

(d) *there exists* $C > 0$ *such that* $\beta(r_{k+1})/(1-L_k) \leq C$ *for any* $k \in \mathbb{N}$, *where* $L_n = \int_{n-1}^{2n+1} k(s)\,ds$.

Then there exists at least one solution $x \in C(\mathbb{R}_+,E)$ *of the integral equation (2.1).*

Proof. For the proof we use Theorem 1.3. Let $X = C(R_+,E)$. For each $n \in \mathbb{N}$ we define the map $|\cdot|_n : X \to \mathbb{R}_+$ by $|x|_n = \max_{t\in[n,2n+1]} \|x(t)\|$. This map is a seminorm on X, and let $d_n : X \times X \to \mathbb{R}_+$ be given by

$$d_n(x,y) = |x-y|_n = \max_{t\in[n,2n+1]} ||x(t) - y(t)||. \tag{2.5}$$

It is easy to show that d_n is a pseudometric on X and the family $\{d_n\}_{n\in\mathbb{N}}$ defines on X a gauge structure, separated and complete by sequences.

Here $\mathcal{P} = \mathcal{Q}^\lambda = \{d_n\}_{n\in\mathbb{N}}$ for each $\lambda \in [0,1]$. Let D be the closure in X of the set

$$\{x \in X : \text{ there exists } n \in \mathbb{N} \text{ such that } d_n(x,0) \le r_n + \delta\}, \tag{2.6}$$

where $\delta > 0$ is a fixed number. We define $H : D \times [0,1] \to X$ by $H(x,\lambda) = \lambda A(x)$, where

$$A(x)(t) = \int_{t-1}^{t} f(s, x(s+2))\,ds. \tag{2.7}$$

First, we verify condition (i) from Theorem 1.3.

Let $t \in [n, 2n+1]$, where $n \ge 0$. We have

$$\begin{aligned}
||H(x,\lambda)(t) - H(y,\lambda)(t)|| &\le \lambda \int_{t-1}^{t} ||f(s,x(s+2)) - f(s,y(s+2))||\,ds \\
&\le \int_{n-1}^{2n+1} k(s)||x(s+2) - y(s+2)||\,ds \\
&\le \max_{s\in[n-1,2n+1]} ||x(s+2) - y(s+2)|| \int_{n-1}^{2n+1} k(s)\,ds \\
&\le \max_{\tau\in[n+1,2n+3]} ||x(\tau) - y(\tau)|| \int_{n-1}^{2n+1} k(s)\,ds \\
&= L_n d_{n+1}(x,y).
\end{aligned} \tag{2.8}$$

If we take the maximum with respect to t, we obtain

$$d_n(H(x,\lambda), H(y,\lambda)) \le L_n d_{n+1}(x,y) \tag{2.9}$$

for all $x, y \in D$ and all $n \in \mathbb{N}$. Hence, condition (i) in Theorem 1.3 holds with $\varphi_\lambda = \varphi$ where $\varphi : \mathbb{N} \to \mathbb{N}$ is defined by $\varphi(n) = n+1$. In addition, the series $\sum_{n=1}^{\infty} L_n L_{n+1} \cdots L_{2n}$ is finite since from assumption (a) we know that $|k|_{L^1_{\text{loc}}(I)} < 1$ so $L_n \le |k|_{L^1_{\text{loc}}(I)} < 1$.

Condition (ii) in our case becomes: there exists $\rho > 0$ such that for any solution $(x,\lambda) \in D \times [0,1]$, to $x = H(x,\lambda)$, there exists $n \in \mathbb{N}$ with

$$\inf\{d_n(x,y) : y \in X \backslash D\} > \rho. \tag{2.10}$$

If $y \in X \backslash D$, we have that $d_n(y,0) > r_n + \delta$ for each $n \in \mathbb{N}$. So there exists at least one $t \in [n, 2n+1]$ with

$$||x(t) - y(t)|| \ge ||y(t)|| - ||x(t)|| > r_n + \delta - r_n = \delta. \tag{2.11}$$

Hence $d_n(x,y) > \delta$ and (2.10) holds for any $\rho \in (0,\delta)$.

Condition (iii) in Theorem 1.3 is trivial since $\mathcal{P} = \mathcal{Q}^\lambda$ for any $\lambda \in [0,1]$.

Condition (iv) in Theorem 1.3 becomes: $(X, \{d_n\}_{n\in\mathbb{N}})$ is a gauge space sequatially complete because E is a Banach space.

Condition (v): Let $\lambda \in [0,1]$, $x_0 \in D$, $x_n = H(x_{n-1},\lambda)$ for $n = 1,2,\ldots$, and assume $\mathcal{P} - \lim_{n\to\infty} x_n = x$. We will prove that $H(x,\lambda) = x$.

Let $m \in \mathbb{N}$ and $t \in [m, 2m+1]$. We have

$$\begin{aligned}
||H(x,\lambda)(t) - x(t)|| &= ||H(x,\lambda)(t) - x_n(t) + x_n(t) - x(t)|| \\
&\le ||H(x,\lambda)(t) - x_n(t)|| + ||x_n(t) - x(t)|| \\
&= ||H(x,\lambda)(t) - H(x_{n-1},\lambda)(t)|| + ||x_n(t) - x(t)|| \\
&\le \int_{t-1}^{t} k(s)||x(s+2) - x_{n-1}(s+2)||ds + \max_{t\in[m,2m+1]} ||x_n(t) - x(t)|| \\
&\le L_m \max_{s\in[m-1,2m+1]} ||x(s+2) - x_{n-1}(s+2)|| + d_m(x_n,x) \\
&= L_m \max_{\tau\in[m+1,2m+3]} ||x(\tau) - x_{n-1}(\tau)|| + d_m(x_n,x) \\
&= L_m d_{m+1}(x,x_{n-1}) + d_m(x_n,x).
\end{aligned} \tag{2.12}$$

Consequently, passing to maximum after $t \in [m, 2m+1]$ we have

$$d_m(H(x,\lambda),x) \le L_m d_{m+1}(x,x_{n-1}) + d_m(x_n,x) \tag{2.13}$$

for all $m \in \mathbb{N}$. Letting $n \to \infty$, we deduce that $d_m(H(x,\lambda),x) = 0$ for each $m \in \mathbb{N}$ and since the family $\{d_m\}_{m\in\mathbb{N}}$ is separated, we have $H(x,\lambda) = x$.

Condition (vi) becomes: for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$d_{\varphi^n(m)}(x,H(x,\lambda)) \le (1 - L_{\varphi^n(m)})\varepsilon \tag{2.14}$$

for each $(x,\mu) \in D \times [0,1]$, $H(x,\mu) = x$, $|\lambda - \mu| \le \delta$, and $n,m \in \mathbb{N}$.

We have $\varphi^n(m) = n+m$. Let $t \in [n+m, 2(n+m)+1]$, and using conditions (c) and (d) we obtain

$$\begin{aligned}
||x(t) - H(x,\lambda)(t)|| &= ||H(x,\mu)(t) - H(x,\lambda)(t)|| \\
&= |\mu - \lambda| \left|\left| \int_{t-1}^{t} f(s,x(s+2))ds \right|\right| \\
&\le |\mu - \lambda| \int_{t-1}^{t} \alpha(s)\beta(||x(s+2)||)ds \\
&\le |\mu - \lambda|\beta(r_{m+n+1}) \int_{n+m-1}^{2(n+m)+1} \alpha(s)ds \\
&\le |\mu - \lambda|\,|\alpha|_{L^1_{\mathrm{loc}}(I)} C(1 - L_{m+n}).
\end{aligned} \tag{2.15}$$

So condition (vi) is true with $\delta(\varepsilon) = \varepsilon/C|\alpha|_{L^1_{\mathrm{loc}}(I)}$.

In addition, $H(\cdot,0) = 0$. So $H(\cdot,0)$ has a fixed point.

Therefore, all the assumptions of Theorem 1.3 are satisfied. Now the conclusion follows from Theorem 1.3. □

Other existence results for integral and differential equations established by the continuation method (see O'Regan and Precup [6]) are given in Chiş [4, 7].

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