Research Article

An Extragradient Approximation Method for Equilibrium Problems and Fixed Point Problems of a Countable Family of Nonexpansive Mappings

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We introduce a new iterative scheme for finding the common element of the set of common fixed points of nonexpansive mappings, the set of solutions of an equilibrium problem, and the set of solutions of the variational inequality. We show that the sequence converges strongly to a common element of the above three sets under some parameters controlling conditions. Moreover, we apply our result to the problem of finding a common fixed point of a countable family of nonexpansive mappings, and the problem of finding a zero of a monotone operator. This main theorem extends a recent result of Yao et al. (2007) and many others.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $\phi : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$\phi(x, y) \geq 0 \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{EP}(\phi)$. Given a mapping $T : C \to H$, let $\phi(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then $z \in \mathrm{EP}(\phi)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in C$, that is, z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). In 1997, Flåm and Antipin [1] introduced an iterative scheme of finding the best approximation to initial data when $\mathrm{EP}(\phi)$ is nonempty and proved a strong convergence theorem.

Let $A : C \to H$ be a mapping. The classical variational inequality, denoted by $\mathrm{VI}(A, C)$, is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^* \rangle \geq 0 \tag{1.2}$$

for all $v \in C$. The variational inequality has been extensively studied in the literature. See, for example, [2, 3] and the references therein. A mapping A of C into H is called α-inverse-strongly monotone [4, 5] if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2 \tag{1.3}$$

for all $u, v \in C$. It is obvious that any α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous. A mapping S of C into itself is called nonexpansive if

$$\|Su - Sv\| \leq \|u - v\| \tag{1.4}$$

for all $u, v \in C$. We denote by $F(S)$ the set of fixed points of S. For finding an element of $F(S) \cap VI(A, C)$, under the assumption that a set $C \subseteq H$ is nonempty, closed, and convex, a mapping $S : C \to C$ is nonexpansive and a mapping $A : C \to H$ is α-inverse-strongly monotone, Takahashi and Toyoda [6] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n A x_n) \tag{1.5}$$

for every $n = 0, 1, 2, \ldots$, where $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that if $F(S) \cap VI(A, C) \neq \varnothing$, then the sequence $\{x_n\}$ generated by (1.5) converges weakly to some $z \in F(S) \cap VI(A, C)$. Recently, motivated by the idea of Korpelevič's extragradient method [7], Nadezhkina and Takahashi [8] introduced an iterative scheme for finding an element of $F(S) \cap VI(A, C)$ and the weak convergence theorem is presented. Moreover, Zeng and Yao [9] proposed some new iterative schemes for finding elements in $F(S) \cap VI(A, C)$ and obtained the weak convergence theorem for such schemes. Very recently, Yao et al. [10] introduced the following iterative scheme for finding an element of $F(S) \cap VI(A, C)$ under some mild conditions. Let C be a closed convex subset of a real Hilbert space H, $A : C \to H$ a monotone, L-Lipschitz continuous mapping, and S a nonexpansive mapping of C into itself such that $F(S) \cap VI(A, C) \neq \varnothing$. Suppose that $x_1 = u \in C$ and $\{x_n\}$, $\{y_n\}$ are given by

$$y_n = P_C (x_n - \lambda_n A x_n),$$
$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C (x_n - \lambda_n A y_n) \quad \forall n \in \mathbb{N}, \tag{1.6}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$ and $\{\lambda_n\} \subseteq (0, 1)$ satisfy some parameters controlling conditions. They proved that the sequence $\{x_n\}$ defined by (1.6) converges strongly to a common element of $F(S) \cap VI(A, C)$.

On the other hand, S. Takahashi and W. Takahashi [11] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solution (1.1) and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Let $S : C \to C$ be a nonexpansive mapping. Starting with arbitrary initial $x_1 \in C$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \quad \forall y \in C,$$
$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S u_n \quad \forall n \in \mathbb{N}. \tag{1.7}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(\phi)$, where $z = P_{F(S) \cap EP(\phi)} f(z)$.

Moreover, Aoyama et al. [12] introduced an iterative scheme for finding a common fixed point of a countable family of nonexpansive mappings in Banach spaces and obtained the strong convergence theorem for such scheme.

In this paper, motivated by Yao et al. [10], S. Takahashi and W. Takahashi [11] and Aoyama et al. [12], we introduce a new extragradient method (4.2) which is mixed the iterative schemes considered in [10–12] for finding a common element of the set of common fixed points of nonexpansive mappings, the set of solutions of an equilibrium problem, and the solution set of the classical variational inequality problem for a monotone L-Lipschitz continuous mapping in a real Hilbert space. Then, the strong convergence theorem is proved under some parameters controlling conditions. Further, we apply our result to the problem of finding a common fixed point of a countable family of nonexpansive mappings, and the problem of finding a zero of a monotone operator. The results obtained in this paper improve and extend the recent ones announced by Yao et al. results [10] and many others.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\| \quad \forall y \in C. \tag{2.1}$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \ge \|P_C x - P_C y\|^2 \tag{2.2}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \le 0, \tag{2.3}$$

$$\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.4}$$

for all $x \in H$, $y \in C$. For more details, see [13]. It is easy to see that the following is true:

$$u \in \mathrm{VI}(A, C) \Longleftrightarrow u = P_C(u - \lambda A u), \quad \lambda > 0. \tag{2.5}$$

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$, and $g \in Ty$ imply $\langle x - y, f - g \rangle \ge 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \ge 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of C into H, L-Lipschitz continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle u - v, w \rangle \ge 0 \text{ for all } u \in C\}$. Define

$$Tv = \begin{cases} Bv + N_C v, & v \in C; \\ \varnothing, & v \notin C. \end{cases} \tag{2.6}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, B)$; see [14].

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1 (see [15]). *Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, one has*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta \|x - y\|^2 - \alpha\gamma \|x - z\|^2 - \beta\gamma \|y - z\|^2. \qquad (2.7)$$

Lemma 2.2 (see [16]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|z_n - x_n\| = 0$.*

Lemma 2.3 (see [17]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 1, \qquad (2.8)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and*

(ii) $\limsup_{n\to\infty}(\delta_n/\alpha_n) \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty}a_n = 0$.

Lemma 2.4 (see [12, Lemma 3.2]). *Let C be a nonempty closed subset of a Banach space and let $\{S_n\}$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty}\sup\{\|S_{n+1}z - S_nz\| : z \in C\} < \infty$. Then, for each $y \in C$, $\{S_ny\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by*

$$Sy = \lim_{n\to\infty}S_ny \quad \forall y \in C. \qquad (2.9)$$

Then $\lim_{n\to\infty}\sup\{\|Sz - S_nz\| : z \in C\} = 0$.

For solving the equilibrium problem for a bifunction $\phi : C \times C \to \mathbb{R}$, let us assume that ϕ satisfies the following conditions:

(A1) $\phi(x, x) = 0$ for all $x \in C$;

(A2) ϕ is monotone, that is, $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\to 0}\phi(tz + (1 - t)x, y) \leq \phi(x, y)$;

(A4) for each $x \in C$, $y \mapsto \phi(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [18].

Lemma 2.5 (see [18]). *Let C be a nonempty closed convex subset of H and let ϕ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0 \quad \forall y \in C. \qquad (2.10)$$

The following lemma was also given in [1].

Lemma 2.6 (see [1]). *Assume that $\phi : C \times C \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : \phi(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0 \ \forall y \in C \right\} \tag{2.11}$$

for all $z \in H$. Then, the following hold:

    (i) *T_r is single-valued;*

    (ii) *T_r is firmly nonexpansive, that is, for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$;*

    (iii) *$F(T_r) = EP(\phi)$;*

    (iv) *$EP(\phi)$ is closed and convex.*

3. Main results

In this section, we prove a strong convergence theorem.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let ϕ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), $A : C \to H$ a monotone L-Lipschitz continuous mapping and let $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(S_n) \cap VI(A, C) \cap EP(\phi) \neq \varnothing$. Let the sequences $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be generated by*

$$x_1 = x \in C \text{ chosen arbitrarily,}$$

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0 \quad \forall y \in C,$$

$$y_n = P_C(u_n - \lambda_n A u_n), \tag{3.1}$$

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n P_C(u_n - \lambda_n A y_n) \quad \forall n \geq 1,$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$, $\{\lambda_n\} \subseteq (0, 1)$, and $\{r_n\} \subseteq (0, \infty)$ satisfy the following conditions:

    (C1) $\alpha_n + \beta_n + \gamma_n = 1$,

    (C2) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

    (C3) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$,

    (C4) $\lim_{n \to \infty} \lambda_n = 0$,

    (C5) $\liminf_{n \to \infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C. Let S be a mapping of C into itself defined by $Sy = \lim_{n \to \infty} S_n y$ for all $y \in C$ and suppose that $F(S) = \cap_{n=1}^{\infty} F(S_n)$. Then the sequences $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ converge strongly to the same point $q \in \cap_{n=1}^{\infty} F(S_n) \cap VI(A, C) \cap EP(\phi)$, where $q = P_{\cap_{n=1}^{\infty} F(S_n) \cap VI(A,C) \cap EP(\phi)} f(q)$.

Proof. Let $Q = P_{\cap_{n=1}^{\infty} F(S_n) \cap VI(A,C) \cap EP(\phi)}$. Since f is a contraction with $\alpha \in [0, 1)$, we obtain

$$\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq \alpha\|x - y\| \quad \forall x, y \in C. \tag{3.2}$$

Therefore, Qf is a contraction of C into itself, which implies that there exists a unique element $q \in C$ such that $q = Qf(q)$. Then we divide the proof into several steps.

Step 1 ($\{x_n\}$ is bounded). Indeed, put $t_n = P_C(u_n - \lambda_n Ay_n)$ for all $n \geq 1$. Let $x^* \in \cap_{n=1}^{\infty} F(S_n) \cap$ VI$(A,C) \cap$ EP(ϕ). From (2.5) we have $x^* = P_C(x^* - \lambda_n Ax^*)$. Also it follows from (2.4) that

$$
\begin{aligned}
\|t_n - x^*\|^2 &\leq \|u_n - \lambda_n Ay_n - x^*\|^2 - \|u_n - \lambda_n Ay_n - t_n\|^2 \\
&= \|u_n - x^*\|^2 - 2\lambda_n \langle Ay_n, u_n - x^* \rangle + \lambda_n^2 \|Ay_n\|^2 - \|u_n - t_n\|^2 \\
&\quad + 2\lambda_n \langle Ay_n, u_n - t_n \rangle - \lambda_n^2 \|Ay_n\|^2 \\
&= \|u_n - x^*\|^2 + 2\lambda_n \langle Ay_n, x^* - t_n \rangle - \|u_n - t_n\|^2 \\
&= \|u_n - x^*\|^2 - \|u_n - t_n\|^2 + 2\lambda_n \langle Ay_n - Ax^*, x^* - y_n \rangle \\
&\quad + 2\lambda_n \langle Ax^*, x^* - y_n \rangle + 2\lambda_n \langle Ay_n, y_n - t_n \rangle.
\end{aligned}
\tag{3.3}
$$

Since A is monotone and x^* is a solution of the variational inequality problem VI(A,C), we have

$$
\langle Ay_n - Ax^*, x^* - y_n \rangle \leq 0, \qquad \langle Ax^*, x^* - y_n \rangle \leq 0.
\tag{3.4}
$$

This together with (3.3) implies that

$$
\begin{aligned}
\|t_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - t_n\|^2 + 2\lambda_n \langle Ay_n, y_n - t_n \rangle \\
&= \|u_n - x^*\|^2 - \|(u_n - y_n) + (y_n - t_n)\|^2 + 2\lambda_n \langle Ay_n, y_n - t_n \rangle \\
&= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - t_n \rangle - \|y_n - t_n\|^2 + 2\lambda_n \langle Ay_n, y_n - t_n \rangle \\
&= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n \rangle.
\end{aligned}
\tag{3.5}
$$

From (2.3), we have

$$
\langle u_n - \lambda_n Au_n - y_n, t_n - y_n \rangle \leq 0,
\tag{3.6}
$$

so that

$$
\begin{aligned}
\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n \rangle &= \langle u_n - \lambda_n Au_n - y_n, t_n - y_n \rangle + \langle \lambda_n Au_n - \lambda_n Ay_n, t_n - y_n \rangle \\
&\leq \langle \lambda_n Au_n - \lambda_n Ay_n, t_n - y_n \rangle \\
&\leq \lambda_n \|Au_n - Ay_n\| \|t_n - y_n\| \\
&\leq \lambda_n L \|u_n - y_n\| \|t_n - y_n\|.
\end{aligned}
\tag{3.7}
$$

Hence it follows from (3.5) and (3.7) that

$$
\begin{aligned}
\|t_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n L \|u_n - y_n\| \|t_n - y_n\| \\
&\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n L \left(\|u_n - y_n\|^2 + \|y_n - t_n\|^2 \right) \\
&= \|u_n - x^*\|^2 + (\lambda_n L - 1) \|u_n - y_n\|^2 + (\lambda_n L - 1) \|y_n - t_n\|^2.
\end{aligned}
\tag{3.8}
$$

Since $\lambda_n \to 0$ as $n \to \infty$, there exists a positive integer N_0 such that $\lambda_n L - 1 \leq -1/3$, when $n \geq N_0$. Hence it follows from (3.8) that

$$\|t_n - x^*\| \leq \|u_n - x^*\|. \tag{3.9}$$

Observe that

$$\|u_n - x^*\| = \|T_{r_n} x_n - T_{r_n} x^*\| \leq \|x_n - x^*\|, \tag{3.10}$$

and hence

$$\|t_n - x^*\| \leq \|x_n - x^*\|. \tag{3.11}$$

Thus, we can calculate

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n t_n - x^*\| \\
&\leq \alpha_n \|f(x_n) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|t_n - x^*\| \\
&\leq \alpha_n \|f(x_n) - f(x^*)\| + \alpha_n \|f(x^*) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|x_n - x^*\| \\
&\leq (1 - \alpha_n(1 - \alpha)) \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| \\
&= (1 - \alpha_n(1 - \alpha)) \|x_n - x^*\| + \alpha_n(1 - \alpha) \frac{\|f(x^*) - x^*\|}{(1 - \alpha)}.
\end{aligned} \tag{3.12}$$

It follows from induction that

$$\|x_n - x^*\| \leq \max \left\{ \|x_1 - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \alpha} \right\}, \quad n \geq N_0. \tag{3.13}$$

Therefore, $\{x_n\}$ is bounded. Hence, so are $\{t_n\}$, $\{S_n t_n\}$, $\{Au_n\}$, $\{Ay_n\}$, and $\{f(x_n)\}$.

Step 2 ($\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$). Indeed, we observe that for any $x, y \in C$,

$$\begin{aligned}
\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 &= \|(x - y) - \lambda_n(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2\lambda_n \langle x - y, Ax - Ay \rangle + \lambda_n^2 \|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 + \lambda_n^2 L^2 \|x - y\|^2 \\
&= (1 + \lambda_n^2 L^2) \|x - y\|^2,
\end{aligned} \tag{3.14}$$

which implies that

$$\|(I - \lambda_n A)x - (I - \lambda_n A)y\| \leq (1 + \lambda_n L) \|x - y\|. \tag{3.15}$$

Thus

$$\begin{aligned}
\|t_{n+1} - t_n\| &\leq \|P_C(u_{n+1} - \lambda_{n+1} Ay_{n+1}) - P_C(u_n - \lambda_n Ay_n)\| \\
&\leq \|u_{n+1} - \lambda_{n+1} Ay_{n+1} - (u_n - \lambda_n Ay_n)\| \\
&= \|(u_{n+1} - \lambda_{n+1} Au_{n+1}) - (u_n - \lambda_{n+1} Au_n) + \lambda_{n+1}(Au_{n+1} - Ay_{n+1} - Au_n) + \lambda_n Ay_n\| \\
&\leq \|(u_{n+1} - \lambda_{n+1} Au_{n+1}) - (u_n - \lambda_{n+1} Au_n)\| \\
&\quad + \lambda_{n+1}(\|Au_{n+1}\| + \|Ay_{n+1}\| + \|Au_n\|) + \lambda_n \|Ay_n\| \\
&\leq (1 + \lambda_{n+1} L) \|u_{n+1} - u_n\| + \lambda_{n+1}(\|Au_{n+1}\| + \|Ay_{n+1}\| + \|Au_n\|) + \lambda_n \|Ay_n\|.
\end{aligned} \tag{3.16}$$

On the other hand, from $u_n = T_{r_n} x_n$ and $u_{n+1} = T_{r_{n+1}} x_{n+1}$, we note that

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \forall y \in C, \tag{3.17}$$

$$\phi(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 \quad \forall y \in C. \tag{3.18}$$

Putting $y = u_{n+1}$ in (3.17) and $y = u_n$ in (3.18), we have

$$\phi(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0,$$

$$\phi(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0. \tag{3.19}$$

So, from (A2), we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0 \tag{3.20}$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle \geq 0. \tag{3.21}$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbb{N}$. Then, we have

$$\|u_{n+1} - u_n\|^2 \leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1})\right\rangle$$

$$\leq \|u_{n+1} - u_n\|\left\{\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\|\right\} \tag{3.22}$$

and hence

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\|$$

$$\leq \|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n|M, \tag{3.23}$$

where $M = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. It follows from (3.16) and the last inequality that

$$\|t_{n+1} - t_n\| \leq (1 + \lambda_{n+1}L)\|x_{n+1} - x_n\| + (1 + \lambda_{n+1}L)\frac{1}{c}|r_{n+1} - r_n|M$$

$$+ \lambda_{n+1}(\|Au_{n+1}\| + \|Ay_{n+1}\| + \|Au_n\|) + \lambda_n\|Ay_n\|. \tag{3.24}$$

Setting $z_n = (\alpha_n f(x_n) + \gamma_n S_n t_n)/(1 - \beta_n)$, we obtain $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ for all $n \in \mathbb{N}$. Thus, we have

$$
\begin{aligned}
\|z_{n+1} - z_n\| &= \left\| \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} S_{n+1} t_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n S_n t_n}{1 - \beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} f(x_{n+1}) + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(S_{n+1} t_{n+1} - S_n t_n) - \frac{\alpha_n}{1 - \beta_n} f(x_n) \right. \\
&\qquad \left. - \left(1 - \frac{\alpha_n}{1 - \beta_n}\right) S_n t_n + \left(1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} S_n t_n\right) \right\| \\
&\le \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|f(x_{n+1}) - S_n t_n\| + \frac{\alpha_n}{1 - \beta_n} \|S_n t_n - f(x_n)\| \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|S_{n+1} t_{n+1} - S_n t_n\|.
\end{aligned}
\tag{3.25}
$$

It follows from (3.24) that

$$
\begin{aligned}
\|S_{n+1} t_{n+1} - S_n t_n\| &\le \|S_{n+1} t_{n+1} - S_{n+1} t_n\| + \|S_{n+1} t_n - S_n t_n\| \\
&\le \|t_{n+1} - t_n\| + \|S_{n+1} t_n - S_n t_n\| \\
&\le (1 + \lambda_{n+1} L) \|x_{n+1} - x_n\| + (1 + \lambda_{n+1} L)\frac{1}{c}|r_{n+1} - r_n|M \\
&\quad + \lambda_{n+1}(\|Au_{n+1}\| + \|Ay_{n+1}\| + \|Au_n\|) + \lambda_n \|Ay_n\| + \|S_{n+1} t_n - S_n t_n\|.
\end{aligned}
\tag{3.26}
$$

Combining (3.25) and (3.26), we have

$$
\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\le \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|f(x_{n+1}) - S_n t_n\| + \frac{\alpha_n}{1 - \beta_n} \|S_n t_n - f(x_n)\| \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(1 + \lambda_{n+1} L)\|x_{n+1} - x_n\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(1 + \lambda_{n+1} L)\frac{1}{c}|r_{n+1} - r_n|M \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \lambda_{n+1}(\|Au_{n+1}\| + \|Ay_{n+1}\| + \|Au_n\|) + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \lambda_n \|Ay_n\| \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|S_{n+1} t_n - S_n t_n\| - \|x_{n+1} - x_n\| \\
&\le \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|f(x_{n+1}) - S_n t_n\| + \frac{\alpha_n}{1 - \beta_n} \|S_n t_n - f(x_n)\| \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \lambda_{n+1} L \|x_{n+1} - x_n\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(1 + \lambda_{n+1} L)\frac{1}{c}|r_{n+1} - r_n|M \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \lambda_{n+1}(\|Au_{n+1}\| + \|Ay_{n+1}\| + \|Au_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \lambda_n \|Ay_n\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \sup\{\|S_{n+1} t - S_n t\| : t \in \{t_n\}\}.
\end{aligned}
\tag{3.27}
$$

This together with (C1)–(C5) and $\lim_{n\to\infty} \sup\{\|S_{n+1}t - S_n t\| : t \in \{t_n\}\} = 0$ implies that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{3.28}$$

Hence, by Lemma 2.2, we obtain $\|z_n - x_n\| \to 0$ as $n \to \infty$. It then follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \beta_n)\|z_n - x_n\| = 0. \tag{3.29}$$

By (3.23) and (3.24), we also have

$$\lim_{n\to\infty}\|t_{n+1} - t_n\| = \lim_{n\to\infty}\|u_{n+1} - u_n\| = 0. \tag{3.30}$$

Step 3 ($\lim_{n\to\infty}\|St_n - t_n\| = 0$). Indeed, pick any $x^* \in \cap_{n=1}^{\infty}F(S_n) \cap \mathrm{VI}(A,C) \cap \mathrm{EP}(\phi)$, to obtain

$$
\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}x_n - T_{r_n}x^*\|^2 \leq \langle T_{r_n}x_n - T_{r_n}x^*, x_n - x^*\rangle = \langle u_n - x^*, x_n - x^*\rangle \\
&= \frac{1}{2}\left(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2\right).
\end{aligned}
\tag{3.31}
$$

Therefore, $\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2$. From Lemma 2.1 and (3.9), we obtain, when $n \geq N_0$, that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n t_n - x^*\|^2 \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|S_n t_n - x^*\|^2 \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|t_n - x^*\|^2 \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2 \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\left(\|x_n - x^*\|^2 - \|x_n - u_n\|^2\right) \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 - \gamma_n\|x_n - u_n\|^2
\end{aligned}
\tag{3.32}
$$

and hence

$$
\begin{aligned}
\gamma_n\|x_n - u_n\|^2 &\leq \alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|).
\end{aligned}
\tag{3.33}
$$

It now follows from the last inequality, (C1), (C2), (C3) and (3.29), that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.34}$$

Noting that

$$
\begin{aligned}
\|y_n - x_n\| &= \|P_C(u_n - \lambda_n A u_n) - x_n\| \leq \|u_n - x_n\| + \lambda_n\|A u_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\
\|y_n - t_n\| &= \|P_C(u_n - \lambda_n A u_n) - P_C(u_n - \lambda_n A y_n)\| \leq \lambda_n\|A u_n - A y_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned}
\tag{3.35}
$$

Thus

$$\|t_n - x_n\| \le \|t_n - y_n\| + \|y_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.36}$$

We note that

$$\begin{aligned}
\|S_n y_n - x_{n+1}\| &\le \|S_n y_n - S_n t_n\| + \|S_n t_n - x_{n+1}\| \\
&\le \|y_n - t_n\| + \alpha_n \|S_n t_n - f(x_n)\| + \beta_n \|S_n t_n - x_n\| \\
&\le \|y_n - t_n\| + \alpha_n \|S_n t_n - f(x_n)\| + \beta_n \|S_n t_n - S_n x_n\| + \beta_n \|S_n x_n - x_n\| \\
&\le \|y_n - t_n\| + \alpha_n \|S_n t_n - f(x_n)\| + \beta_n \|t_n - x_n\| + \beta_n \|S_n x_n - x_n\|.
\end{aligned} \tag{3.37}$$

Using (3.37), we have

$$\begin{aligned}
\|S_n x_n - x_n\| &\le \|S_n x_n - S_n y_n\| + \|S_n y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&\le \|x_n - y_n\| + \|y_n - t_n\| + \alpha_n \|S_n t_n - f(x_n)\| + \beta_n \|t_n - x_n\| \\
&\quad + \beta_n \|S_n x_n - x_n\| + \|x_{n+1} - x_n\|,
\end{aligned} \tag{3.38}$$

so that

$$(1-\beta_n)\|S_n x_n - x_n\| \le \|x_n - y_n\| + \|y_n - t_n\| + \alpha_n \|S_n t_n - f(x_n)\| + \beta_n \|t_n - x_n\| + \|x_{n+1} - x_n\|. \tag{3.39}$$

This implies that

$$\lim_{n\to\infty} \|S_n x_n - x_n\| = 0. \tag{3.40}$$

It now follows from (3.36) and (3.40) that

$$\begin{aligned}
\|S_n t_n - t_n\| &\le \|S_n t_n - S_n x_n\| + \|S_n x_n - x_n\| + \|x_n - t_n\| \\
&\le 2\|t_n - x_n\| + \|S_n x_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.41}$$

Applying Lemma 2.4 and (3.41), we have

$$\begin{aligned}
\|S t_n - t_n\| &\le \|S t_n - S_n t_n\| + \|S_n t_n - t_n\| \\
&\le \sup\left\{ \|S t - S_n t\| : t \in \{t_n\} \right\} + \|S_n t_n - t_n\| \longrightarrow 0.
\end{aligned} \tag{3.42}$$

It follows from the last inequality and (3.36) that

$$\|x_n - S t_n\| \le \|x_n - t_n\| + \|t_n - S t_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.43}$$

Step 4 ($\limsup_{n\to\infty} \langle f(q) - q, x_n - q \rangle \le 0$). Indeed, we choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$\limsup_{n\to\infty} \langle f(q) - q, S t_n - q \rangle = \lim_{i\to\infty} \langle f(q) - q, S t_{n_i} - q \rangle. \tag{3.44}$$

Without loss of generality, we may assume that $\{t_{n_i}\}$ converges weakly to $z \in C$. From $\|St_n - t_n\| \to 0$, we obtain $St_{n_i} \rightharpoonup z$. Now, we will show that $z \in F(S) \cap VI(A, C) \cap EP(\phi)$. Firstly, we will show $z \in EP(\phi)$. Indeed, we observe that $u_n = T_{r_n} x_n$, and

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \forall y \in C. \tag{3.45}$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \phi(y, u_n), \tag{3.46}$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq \phi(y, u_{n_i}). \tag{3.47}$$

From $\|u_n - x_n\| \to 0$, $\|x_n - St_n\| \to 0$, and $\|St_n - t_n\| \to 0$, we get $u_{n_i} \rightharpoonup z$. Since $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$, it follows by (A4) that $0 \geq \phi(y, z)$ for all $y \in C$. For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $\phi(y_t, z) \leq 0$. So, from (A1) and (A4), we have

$$0 = \phi(y_t, y_t) \leq t\phi(y_t, y) + (1-t)\phi(y_t, z) \leq t\phi(y_t, y) \tag{3.48}$$

and hence $0 \leq \phi(y_t, y)$. From (A3), we have $0 \leq \phi(z, y)$ for all $y \in C$, and hence $z \in EP(\phi)$. By the Opial's condition, we can obtain that $z \in F(S)$. Next we will show that $z \in VI(A, C)$. Let

$$Tv = \begin{cases} Av + N_C v, & v \in C; \\ \varnothing, & v \notin C. \end{cases} \tag{3.49}$$

Then T is maximal monotone (see [14]). Let $(v, w) \in G(T)$. Since $w - Av \in N_C(v)$ and $t_n \in C$, we have $\langle v - t_n, w - Av\rangle \geq 0$. On the other hand, from $t_n = P_C(u_n - \lambda_n Ay_n)$, we have

$$\langle v - t_n, t_n - (u_n - \lambda_n Ay_n)\rangle \geq 0, \tag{3.50}$$

that is,

$$\left\langle v - t_n, \frac{t_n - u_n}{\lambda_n} + Ay_n\right\rangle \geq 0. \tag{3.51}$$

Therefore, we obtain

$$\begin{aligned}
\langle v - t_{n_i}, w\rangle &\geq \langle v - t_{n_i}, Av\rangle \geq \langle v - t_{n_i}, Av\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i}\right\rangle \\
&= \left\langle v - t_{n_i}, Av - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle \\
&= \langle v - t_{n_i}, Av - At_{n_i}\rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle \\
&\geq \langle v - t_{n_i}, At_{n_i}\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i}\right\rangle \\
&= \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle.
\end{aligned} \tag{3.52}$$

Noting that $\|t_{n_i} - y_{n_i}\| \to 0$, $\|t_{n_i} - u_{n_i}\| \to 0$ as $n \to \infty$, A is Lipschitz continuous and (3.52), we obtain

$$\langle v - z, w \rangle \geq 0. \tag{3.53}$$

Since T is maximal monotone, we have $z \in T^{-1}0$, and hence $z \in \text{VI}(A, C)$. Hence $z \in F(S) \cap \text{VI}(A, C) \cap \text{EP}(\phi)$. The property of the metric projection implies that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(q) - q, x_n - q \rangle &= \limsup_{n\to\infty}\langle f(q) - q, St_n - q \rangle \\
&= \lim_{i\to\infty}\langle f(q) - q, St_{n_i} - q \rangle = \langle f(q) - q, z - q \rangle \leq 0.
\end{aligned} \tag{3.54}$$

Step 5 ($\lim_{n\to\infty}\|x_n - q\| = 0$). Indeed, we observe that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n t_n, x_{n+1} - q \rangle \\
&= \alpha_n \langle f(x_n) - q, x_{n+1} - q \rangle + \beta_n \langle x_n - q, x_{n+1} - q \rangle + \gamma_n \langle S_n t_n - q, x_{n+1} - q \rangle \\
&\leq \frac{1}{2}\beta_n\left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\right) + \frac{1}{2}\gamma_n\left(\|t_n - q\|^2 + \|x_{n+1} - q\|^2\right) \\
&\quad + \alpha_n \langle f(x_n) - f(q), x_{n+1} - q \rangle + \alpha_n \langle f(q) - q, x_{n+1} - q \rangle \\
&\leq \frac{1}{2}(1 - \alpha_n)\left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\right) + \frac{1}{2}\alpha_n\left(\|f(x_n) - f(q)\|^2 + \|x_{n+1} - q\|^2\right) \\
&\quad + \alpha_n \langle f(q) - q, x_{n+1} - q \rangle \\
&\leq \frac{1}{2}(1 - \alpha_n(1 - \alpha^2))\|x_n - q\|^2 + \frac{1}{2}(1 - \alpha_n)\|x_{n+1} - q\|^2 \\
&\quad + \frac{1}{2}\alpha_n\|x_{n+1} - q\|^2 + \alpha_n \langle f(q) - q, x_{n+1} - q \rangle,
\end{aligned} \tag{3.55}$$

which implies that

$$\|x_{n+1} - q\|^2 \leq (1 - \alpha_n(1 - \alpha^2))\|x_n - q\|^2 + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle. \tag{3.56}$$

Setting $\delta_n = 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle$, we have $\limsup_{n\to\infty}(\delta_n/\alpha_n(1 - \alpha^2)) \leq 0$. Applying Lemma 2.3 to (3.56), we conclude that $\{x_n\}$ converges strongly to q. Consequently, $\{u_n\}$ and $\{y_n\}$ converge strongly to q. This completes the proof. $\qquad\square$

As in [12, Theorem 4.1], we can generate a sequence $\{S_n\}$ of nonexpansive mappings satisfying condition $\sum_{n=1}^{\infty}\sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C by using convex combination of a general sequence $\{T_k\}$ of nonexpansive mappings with a common fixed point.

Corollary 3.2. *Let C be a closed convex subset of a real Hilbert space H. Let ϕ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), $A : C \to H$ a monotone, L-Lipschitz continuous mapping and let*

$\{\beta_n^k\}$ *be a family of nonnegative numbers with indices* $n, k \in N$ *with* $k \leq n$ *such that*

(i) $\sum_{k=1}^n \beta_n^k = 1$ *for all* $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty} \beta_n^k > 0$ *for every* $k \in \mathbb{N}$;

(iii) $\sum_{n=1}^\infty \sum_{k=1}^n |\beta_{n+1}^k - \beta_n^k| < \infty$.

Let $\{T_k\}$ *be a sequence of nonexpansive mappings of* C *into itself with* $\cap_{k=1}^\infty F(T_k) \cap \mathrm{VI}(A, C) \cap$
$\mathrm{EP}(\phi) \neq \emptyset$. *Let* $x_1 = x \in C$ *and* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *be the sequences generated by*

$$x_1 = x \in C \text{ chosen arbitrary,}$$

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0 \quad \forall y \in C,$$

$$y_n = P_C(u_n - \lambda_n A u_n), \tag{3.57}$$

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{k=1}^n \beta_n^k T_k P_C(u_n - \lambda_n A y_n) \quad \forall n \geq 1,$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$, $\{\lambda_n\} \subseteq (0,1)$, *and* $\{r_n\} \subseteq (0, \infty)$ *satisfy the following conditions:*

(C1) $\alpha_n + \beta_n + \gamma_n = 1$,

(C2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,

(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(C4) $\lim_{n\to\infty} \lambda_n = 0$,

(C5) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^\infty |r_{n+1} - r_n| < \infty$.

Then the sequences $\{x_n\}, \{u_n\}$, *and* $\{y_n\}$ *converge strongly to the same point* $q \in \cap_{k=1}^\infty F(T_k) \cap$
$\mathrm{VI}(A, C) \cap \mathrm{EP}(\phi)$, *where* $q = P_{\cap_{k=1}^\infty F(T_k) \cap \mathrm{VI}(A,C) \cap \mathrm{EP}(\phi)} f(q)$.

Setting $S_n \equiv S$ and $f := u$ in Theorem 3.1, we have the following result.

Corollary 3.3 (see [10, Theorem 3.1]). *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $A : C \to H$ *be a monotone, L-Lipschitz continuous mapping, and let* S *be a nonexpansive mapping of* C *into itself such that* $F(S) \cap \mathrm{VI}(A, C) \neq \emptyset$. *Suppose* $x_1 = u \in C$ *and* $\{x_n\}, \{y_n\}$ *are given by*

$$y_n = P_C(x_n - \lambda_n A x_n),$$
$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(x_n - \lambda_n A y_n), \tag{3.58}$$

where $\{\lambda_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(iv) $\lim_{n\to\infty} \lambda_n = 0$.

Then $\{x_n\}$ *converges strongly to* $P_{F(S) \cap \mathrm{VI}(A,C)} u$.

Proof. Put $\phi(x, y) = 0$ for all $x, y \in C$ and $\{r_n\} = 1$ in Theorem 3.1. Thus, we have $u_n = x_n$. Then the sequence $\{x_n\}$ generated in Corallary 3.3 converges strongly to $P_{F(S) \cap \mathrm{VI}(A,C)} u$. $\square$

4. Applications

In this section, we consider the problem of finding a zero of a monotone operator. A multivalued operator $S : H \rightarrow 2^H$ with domain $D(S) = \{z \in H : Sz \neq \varnothing\}$ and range $R(S) = \{Sz : z \in D(T)\}$ is said to be monotone if for each $x_i \in D(S)$ and $y_i \in Sx_i$, $i = 1, 2$, we have $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$. A monotone operator S is said to be maximal if its graph $G(S) = \{(x, y) : y \in Sx\}$ is not properly contained in the graph of any other monotone operator. Let I denote the identity operator on H and let $S : H \rightarrow 2^H$ be a maximal monotone operator. Then we can define, for each $r > 0$, a nonexpansive single-valued mapping $J_r : H \rightarrow H$ by $J_r = (I + rS)^{-1}$. It is called the resolvent (or the proximal mapping) of S. We also define the Yosida approximation A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in SJ_r x$ and $\|A_r x\| \leq \inf\{\|y\| : y \in Sx\}$ for all $x \in H$. We also know that $S^{-1}0 = F(J_r)$ for all $r > 0$; see, for instance, Rockafellar [19] or Takahashi [20].

Lemma 4.1 (the resolvent identity). *For $\lambda, \mu > 0$, there holds the identity*

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} + \left(1 - \frac{\mu}{\lambda} \right) J_\lambda x \right), \quad x \in H. \tag{4.1}$$

By using Theorem 3.1 and Lemma 4.1, we may obtain the following improvement.

Theorem 4.2. *Let $S : H \rightarrow 2^H$ be a maximal monotone operator. Let ϕ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), $A : C \rightarrow H$ a monotone L-Lipschitz continuous mapping of C into H such that $S^{-1}(0) \cap \mathrm{VI}(A, C) \cap \mathrm{EP}(\phi) \neq \varnothing$ and f a contraction of C into itself with coefficient $\alpha \in (0, 1)$. Let the sequences $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be generated by*

$$x_1 = x \in C \text{ chosen arbitrary,}$$

$$\phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \quad \forall y \in C,$$

$$y_n = P_C(u_n - \lambda_n A u_n), \tag{4.2}$$

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{r_n} P_C(u_n - \lambda_n A y_n) \quad \forall n \geq 1,$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$, $\{\lambda_n\} \subseteq (0, 1)$, and $\{r_n\} \subseteq (0, \infty)$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$,

(C2) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C3) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$,

(C4) $\lim_{n \to \infty} \lambda_n = 0$,

(C5) $\liminf_{n \to \infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ converges strongly to $q = P_{S^{-1}(0) \cap \mathrm{VI}(A,C) \cap \mathrm{EP}(\phi)} f(q)$.

Proof. We first verify that $\sum_{n=1}^{\infty} \sup\{\|J_{r_{n+1}} z - J_{r_n} z\| : z \in B\} < \infty$ for any bounded subset B of C. Let B be a bounded subset of C. Since $S^{-1}(0) = F(J_{r_n})$ for each $n \in \mathbb{N}$, $\{J_{r_n} z : z \in B, n \in \mathbb{N}\}$ is bounded. It follows from Lemma 4.1 that

$$J_{r_{n+1}} z = J_{r_n} \left(\frac{r_n}{r_{n+1}} z + \left(1 - \frac{r_n}{r_{n+1}} \right) J_{r_{n+1}} z \right), \quad z \in H. \tag{4.3}$$

Thus

$$\|J_{r_{n+1}}z - J_{r_n}z\| \leq \frac{|r_{n+1} - r_n|}{r_{n+1}} \|J_{r_{n+1}}z - z\|$$

$$\leq M|r_{n+1} - r_n|$$

$$(4.4)$$

for each $z \in B$ and $n \in \mathbb{N}$, where $M = \sup\{\|J_{r_{n+1}}z - z\| : z \in B, n \in \mathbb{N}\} / \inf\{r_n : n \in \mathbb{N}\}$. Hence we get

$$\sum_{n=1}^{\infty} \sup\{\|J_{r_{n+1}}z - J_{r_n}z\| : z \in B\} \leq M \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty. \qquad (4.5)$$

By the assumption that $\sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$, we obtain $r_n \to r$ for some $r > 0$. Since $\|J_r z - J_{r_n}z\| \leq (|r - r_n|/r)\|z - J_r z\|$, we obtain that $\lim_{n\to\infty} J_{r_n}z = J_r z$ for all $z \in C$. Since $F(J_\mu) = S^{-1}(0)$ for all $\mu > 0$, we have $F(J_r) = \cap_{n=1}^{\infty} F(J_{r_n}) = S^{-1}(0) \neq \varnothing$. Therefore, by Theorem 3.1, $\{x_n\}$ converges strongly to $q = P_{S^{-1}(0) \cap \mathrm{VI}(A,C) \cap \mathrm{EP}(\phi)} f(q)$. □

Acknowledgments

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