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Research Article

Fixed Points and Stability in Neutral Stochastic Differential Equations with Variable Delays

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We consider the mean square asymptotic stability of a generalized linear neutral stochastic differential equation with variable delays by using the fixed point theory. An asymptotic mean square stability theorem with a necessary and sufficient condition is proved, which improves and generalizes some results due to Burton, Zhang and Luo. Two examples are also given to illustrate our results.

1. Introduction

Liapunov's direct method has been successfully used to investigate stability properties of a wide variety of differential equations. However, there are many difficulties encountered in the study of stability by means of Liapunov's direct method. Recently, Burton [1–4], Jung [5], Luo [6], and Zhang [7] studied the stability by using the fixed point theory which solved the difficulties encountered in the study of stability by means of Liapunov's direct method.

Up till now, the fixed point theory is almost used to deal with the stability for deterministic differential equations, not for stochastic differential equations. Very recently, Luo [6] studied the mean square asymptotic stability for a class of linear scalar neutral stochastic differential equations. For more details of the stability concerned with the stochastic differential equations, we refer to [8, 9] and the references therein.

Motivated by previous papers, in this paper, we consider the mean square asymptotic stability of a generalized linear neutral stochastic differential equation with variable delays by using the fixed point theory. An asymptotic mean square stability theorem with a necessary and sufficient condition is proved. Two examples is also given to illustrate our results. The results presented in this paper improve and generalize the main results in [1, 6, 7].

2. Main results

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space and let $W(t)$ denote a one-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of $W(t)$. Let $a(t), b(t), \overline{b}(t), c(t), e(t), q(t) \in C(R^+, R)$, and $\tau(t), \delta(t) \in C(R^+, R^+)$ with $t - \tau(t) \to \infty$ and $t - \delta(t) \to \infty$ as $t \to \infty$. Here $C(S_1, S_2)$ denotes the set of all continuous functions $\phi : S_1 \to S_2$ with the supremum norm $\|\cdot\|$.

In 2003, Burton [1] studied the equation

$$x'(t) = -\overline{b}(t)x(t - \tau(t)) \tag{2.1}$$

and proved the following theorem.

Theorem A (Burton [1]). *Suppose that $\tau(t) = r$ and there exists a constant $\alpha < 1$ such that*

$$\int_{t-r}^{t} |\overline{b}(s+r)| ds + \int_0^t |\overline{b}(s+r)| e^{-\int_s^t \overline{b}(u+r)du} \int_{s-r}^{s} |\overline{b}(u+r)| du\, ds \leq \alpha \tag{2.2}$$

for all $t \geq 0$ and $\int_0^\infty \overline{b}(s)ds = \infty$. Then, for every continuous initial function $\phi : [-r, 0] \to \mathbb{R}$, the solution $x(t) = x(t, 0, \phi)$ of (2.1) *is bounded and tends to zero as $t \to \infty$.*

Recently, Zhang [7] studied the generalization of (2.1) as follows:

$$x'(t) = -\sum_{j=1}^{n} \overline{b}_j(t)x(t - \tau_j(t)) \tag{2.3}$$

and obtained the following theorem.

Theorem B (Zhang [7]). *Suppose that τ_j is differential, the inverse function $g_j(t)$ of $t - \tau_j(t)$ exists, and there exists a constant $\alpha \in (0, 1)$ such that for $t \geq 0$, $\liminf_{t\to\infty} \int_0^t Q(s)ds > -\infty$ and*

$$\sum_{j=1}^{n} \left[\int_{t-\tau_j(t)}^{t} |\overline{b}_j(g_j(s))| ds + \int_0^t e^{-\int_s^t Q(u)du} |\overline{b}_j(s)| |\tau_j'(s)| ds + \int_0^t e^{-\int_s^t Q(u)du} |Q(s)| \int_{s-\tau_j(s)}^{s} |\overline{b}_j(g_j(v))| dv\, ds \right] \leq \alpha, \tag{2.4}$$

where $Q(t) = \sum_{j=1}^{n} \overline{b}_j(g_j(t))$. Then the zero solution of (2.3) *is asymptotically stable if and only if $\int_0^t Q(s)ds \to \infty$, as $t \to \infty$.*

Very recently, Luo [6] considered the following neutral stochastic differential equation:

$$d[x(t) - q(t)x(t - \tau(t))] = [a(t)x(t) + b(t)x(t - \tau(t))]dt + [c(t)x(t) + e(t)x(t - \delta(t))]dW(t) \tag{2.5}$$

and obtained the following theorem.

Theorem C (Luo [6]). *Let $\tau(t)$ be derivable. Assume that there exists a constant $\alpha \in (0,1)$ and a continuous function $h(t) : [0,\infty) \to \mathbb{R}$ such that for $t \geq 0$, $\liminf_{t\to\infty}\int_0^t h(s)ds > -\infty$ and*

$$|q(t)|+\int_{t-\tau(t)}^{t}|a(s)+h(s)|ds+\int_0^t e^{-\int_s^t h(u)du}\left|\left(a(s-\tau(s))+h(s-\tau(s))\right)(1-\tau'(s))+b(s)-q(s)h(s)\right|ds$$
$$+\int_0^t e^{-\int_s^t h(u)du}|h(s)|\int_{s-\tau(s)}^{s}|a(u)+h(u)|du\,ds+\left(\int_0^t e^{-2\int_s^t h(u)du}\left(|c(s)|+|e(s)|\right)^2 ds\right)^{1/2} \leq \alpha. \tag{2.6}$$

Then the zero solution of (2.5) *is mean square asymptotically stable if and only if $\int_0^t h(s)ds \to \infty$, as $t \to \infty$.*

Now, we consider the generalization of (2.5):

$$d\left[x(t)-\sum_{j=1}^{n}q_j(t)x(t-\tau_j(t))\right]=\sum_{j=1}^{n}b_j(t)x(t-\tau_j(t))dt+\sum_{j=1}^{n}c_j(t)x(t-\delta_j(t))dW(t), \tag{2.7}$$

with the initial condition

$$x(s)=\phi(s) \quad \text{for } s\in[m(t_0),t_0], \tag{2.8}$$

where $\phi \in C([m(t_0),t_0],R)$, $b_j(t), c_j(t), q_j(t) \in C(R^+,R)$, $\tau_j(t), \delta_j(t) \in C(R^+,R^+)$, $t-\tau_j(t)\to\infty$, and $t-\delta_j(t)\to\infty$ as $t\to\infty$ and for each $t_0 \geq 0$,

$$\begin{gathered}m_j(t_0)=\min\left\{\inf\{s-\tau_j(s),\ s\geq t_0\},\ \inf\{s-\delta_j(s),\ s\geq t_0\}\right\},\\ m(t_0)=\min\{m_j(t_0),\ 1\leq j\leq n\}.\end{gathered} \tag{2.9}$$

Note that (2.7) becomes (2.5) for $n=2$, $\tau_1(t)=0, \tau_2(t)=\tau(t)$, $b_1(t)=a(t)$, $b_2(t)=b(t)$, $q_1(t)=0$, $q_2(t)=q(t)$, $\delta_1(t)=0$, $\delta_2(t)=\delta(t)$, $c_1(t)=c(t)$, and $c_2(t)=e(t)$. Thus, we know that (2.7) includes (2.1), (2.3), and (2.5) as special cases.

Our aim here is to generalize Theorems B and C to (2.7).

Theorem 2.1. *Suppose that τ_j is differential, and there exist continuous functions $h_j(t) : [0,\infty)\to\mathbb{R}$ for $j=1\cdots n$ and a constant $\alpha\in(0,1)$ such that for $t\geq 0$*

(i) $\liminf_{t\to\infty}\int_0^t H(s)ds > -\infty$,

(ii)

$$\sum_{j=1}^{n}|q_j(t)|+\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}|h_j(s)|ds+\sum_{j=1}^{n}\int_0^t e^{-\int_s^t H(u)du}\left|(h_j(s-\tau_j(s))(1-\tau_j'(s))+b_j(s)-q_j(s)H(s)\right|ds$$
$$+\sum_{j=1}^{n}\int_0^t e^{-\int_s^t H(u)du}|H(s)|\int_{s-\tau_j(s)}^{s}|h_j(u)|du\,ds+2\left(\int_0^t e^{-2\int_s^t H(u)du}\left(\sum_{j=1}^{n}|c_j(s)|\right)^2 ds\right)^{1/2}\leq\alpha<1, \tag{2.10}$$

where $H(t)=\sum_{j=1}^{n}h_j(t)$.

Then the zero solution of (2.7) *is mean square asymptotically stable if and only if*

$$\int_0^t H(s)ds \longrightarrow \infty \quad \text{as } t \longrightarrow \infty. \tag{2.11}$$

Proof. For each t_0, denote by S the Banach space of all $\mathcal{F}$-adapted processes $\varphi(t,\omega) : [m(t_0),\infty)\times\Omega \to \mathbb{R}$ which are almost surely continuous in t with norm

$$\|\varphi\|_S = \left\{ E\left(\sup_{s\geq m(t_0)} |\varphi(s,\omega)|^2 \right) \right\}^{1/2}. \tag{2.12}$$

Moreover, we set $\varphi(t,\omega) = \phi(t)$ for $t \in [m(t_0), t_0]$ and $E|\varphi(t,\omega)|^2 \to 0$, as $t \to \infty$.

At first, we suppose that (2.11) holds. Define an operator $P : S \to S$ by $(Px)(t) = \phi(t)$ for $t \in [m(t_0), t_0]$ and for $t \geq t_0$,

$$\begin{aligned}
(Px)(t) = {} & \left(\phi(t_0) - \sum_{j=1}^n q_j(t_0)\phi(t_0 - \tau_j(t_0)) - \sum_{j=1}^n \int_{t_0-\tau_j(t_0)}^{t_0} h_j(s)\phi(s)ds \right) e^{-\int_{t_0}^t H(u)du} \\
& + \sum_{j=1}^n q_j(t)x(t-\tau_j(t)) + \sum_{j=1}^n \int_{t-\tau_j(t)}^t h_j(s)x(s)ds \\
& + \int_{t_0}^t e^{-\int_s^t H(u)du} \sum_{j=1}^n \left(h_j(s-\tau_j(s))(1-\tau_j'(s)) + b_j(s) - q_j(s)H(s)\right)x(s-\tau_j(s))ds \\
& - \int_{t_0}^t e^{-\int_s^t H(u)du} H(s) \left(\sum_{j=1}^n \int_{s-\tau_j(s)}^s h_j(u)x(u)du \right) ds \\
& + \int_{t_0}^t e^{-\int_s^t H(u)du} \left(\sum_{j=1}^n c_j(s)x(s-\delta_j(s)) \right) dW(s) := \sum_{i=1}^5 I_i(t).
\end{aligned} \tag{2.13}$$

Now, we show the mean square continuity of P on $[t_0,\infty)$. Let $x \in S$, $T_1 > 0$, and let $|r|$ be sufficiently small. Then

$$E\left|(Px)(T_1+r) - (Px)(T_1)\right|^2 \leq 5\sum_{i=1}^5 E\left|I_i(T_1+r) - I_i(T_1)\right|^2. \tag{2.14}$$

It is easy to verify that

$$E\left|I_i(T_1+r) - I_i(T_1)\right|^2 \longrightarrow 0, \quad \text{as } r \longrightarrow 0,\ i = 1,2,3,4. \tag{2.15}$$

It follows from the last term I_5 in (2.13) that

$$E\left|I_5(T_1+r)-I_5(T_1)\right|^2 = E\left|\int_{t_0}^{T_1} e^{-\int_s^{T_1} H(u)du}\left(e^{-\int_{T_1}^{T_1+r} H(u)du}-1\right)\sum_{j=1}^{n} c_j(s)x(s-\delta_j(s))dW(s)\right.$$
$$\left.+\int_{T_1}^{T_1+r} e^{-\int_s^{T_1+r} H(u)du}\sum_{j=1}^{n} c_j(s)x(s-\delta_j(s))dW(s)\right|^2$$
$$\le 2E\int_{t_0}^{T_1} e^{-2\int_s^{T_1} H(u)du}\left(e^{-\int_{T_1}^{T_1+r} H(u)du}-1\right)^2\left(\sum_{j=1}^{n}\left|c_j(s)\right|\cdot\left|x(s-\delta_j(s))\right|\right)^2 ds$$
$$+2E\int_{T_1}^{T_1+r} e^{-2\int_s^{T_1+r} H(u)du}\left(\sum_{j=1}^{n}\left|c_j(s)\right|\cdot\left|x(s-\delta_j(s))\right|\right)^2 ds \longrightarrow 0, \quad \text{as } r\longrightarrow 0. \tag{2.16}$$

Therefore, P is mean square continuous on $[t_0,\infty)$.

Next, we verify that $Px\in S$. Since $E|x(t)|\to 0$, $t-\delta_j(t)\to\infty$ as $t\to\infty$, for each $\epsilon>0$, there exists a $T_1>t_0$ such that $s\ge T_1$ implies $E|x(s)|^2<\epsilon$ and $E|x(s-\delta_j(s))|^2<\epsilon$. Thus, for $t\ge T_1$, the last term I_5 in (2.13) satisfies

$$E\left|I_5(t)\right|^2$$
$$\le E\int_{t_0}^{T_1} e^{-2\int_s^{t}H(u)du}\left(\sum_{j=1}^{n} c_j(s)x(s-\delta_j(s))\right)^2 ds + E\int_{T_1}^{t} e^{-2\int_s^{t}H(u)du}\left(\sum_{j=1}^{n} c_j(s)x(s-\delta_j(s))\right)^2 ds$$
$$\le E\left(\sup_{s\ge m(t_0)}\left|x(s)\right|^2\right)\int_{t_0}^{T_1} e^{-2\int_s^{t}H(u)du}\left(\sum_{j=1}^{n}\left|c_j(s)\right|\right)^2 ds + \epsilon\int_{T_1}^{t} e^{-2\int_s^{t}H(u)du}\left(\sum_{j=1}^{n}\left|c_j(s)\right|\right)^2 ds. \tag{2.17}$$

By condition (ii) and (2.11), there exists $T_2>T_1$ such that $t\ge T_2$ implies

$$E|I_5(t)|^2<\epsilon+\alpha\epsilon. \tag{2.18}$$

Thus, $E|I_5(t)|^2\to 0$, as $t\to\infty$. Similarly, we can show that $E|I_i(t)|^2\to 0$, $i=1,2,3,4$, as $t\to\infty$. Thus, $E|(Px)(t)|^2\to 0$ as $t\to\infty$. This yields $Px\in S$.

Now we show that $P: S\to S$ is a contraction mapping. From (ii), we can choose $\varepsilon>0$ such that $\alpha^2+\varepsilon<1$. Thus, for each $t_0\ge 0$, we can find a constant $L>0$ such that

$$\left(1+\frac{1}{L}\right)\left(\sum_{j=1}^{n}\left|q_j(t)\right| + \sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_s^{t}H(u)du}\left|H(s)\right|\int_{s-\tau_j(s)}^{s}\left|h_j(u)\right|du\,ds\right.$$
$$\left.+\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}\left|h_j(s)\right|ds+\sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_s^{t}H(u)du}\left|(h_j(s-\tau_j(s))(1-\tau_j'(s))+b_j(s)-q_j(s)H(s)\right|ds\right)^2$$
$$+4(1+L)\int_{t_0}^{t} e^{-2\int_s^{t}H(u)du}\left(\sum_{j=1}^{n}\left|c_j(s)\right|\right)^2 ds \le \alpha^2+\varepsilon<1. \tag{2.19}$$

For any $x, y \in S$, it follows from (2.13), conditions (i) and (ii), and Doob's L^p-inequality (see [10]) that

$$
\begin{aligned}
& e \sup_{s\geq m(t_0)} \left|(px)(s)-(py)(s)\right|^2 \\
&\quad = e\sup_{s\geq t_0}\Bigg|\sum_{j=1}^{n} q_j(s)\left(x\left(s-\tau_j(s)\right)-y\left(s-\tau_j(s)\right)\right)+\sum_{j=1}^{n}\int_{s-\tau_j(s)}^{s} h_j(v)\left(x(v)-y(v)\right)dv \\
&\qquad\qquad +\int_{t_0}^{s} e^{-\int_v^s h(u)du}\sum_{j=1}^{n}\left(h_j\left(v-\tau_j(v)\right)\left(1-\tau_j'(v)\right)+b_j(v)-q_j(v)h(v)\right) \\
&\qquad\qquad\qquad \times\left(x\left(v-\tau_j(v)\right)-y\left(v-\tau_j(v)\right)\right)dv \\
&\qquad\qquad -\int_{t_0}^{s} e^{-\int_v^s h(u)du}h(v)\left(\sum_{j=1}^{n}\int_{v-\tau_j(v)}^{v} h_j(u)\left(x(u)-y(u)\right)du\right)dv \\
&\qquad\qquad +\int_{t_0}^{s} e^{-\int_v^s h(u)du}\left(\sum_{j=1}^{n}c_j(v)\left(x\left(v-\delta_j(v)\right)-y\left(v-\delta_j(v)\right)\right)\right)dw(v)\Bigg|^2 \\
&\quad \leq \left(1+\frac{1}{l}\right)e\sup_{s\geq t_0}\Bigg(\sum_{j=1}^{n}\left|q_j(s)\right|\cdot\left|x\left(s-\tau_j(s)\right)-y\left(s-\tau_j(s)\right)\right| \\
&\qquad\qquad +\sum_{j=1}^{n}\int_{s-\tau_j(s)}^{s}\left|h_j(v)\right|\cdot\left|x(v)-y(v)\right|dv \\
&\qquad\qquad +\int_{t_0}^{s} e^{-\int_v^s h(u)du}\sum_{j=1}^{n}\left|h_j\left(v-\tau_j(v)\right)\left(1-\tau_j'(v)\right)+b_j(v)-q_j(v)h(v)\right| \\
&\qquad\qquad\qquad \cdot\left|x\left(v-\tau_j(v)\right)-y\left(v-\tau_j(v)\right)\right|dv \\
&\qquad\qquad +\int_{t_0}^{s} e^{-\int_v^s h(u)du}h(v)\left(\sum_{j=1}^{n}\int_{v-\tau_j(v)}^{v}\left|h_j(u)\right|\cdot\left|x(u)-y(u)\right|du\right)dv\Bigg)^2 \\
&\qquad +4(1+l)\sup_{s\geq t_0}\left\{e\int_{t_0}^{s} e^{-\int_v^s h(u)du}\left(\sum_{j=1}^{n}\left|c_j(v)\right|\cdot\left|x\left(v-\delta_j(v)\right)-y\left(v-\delta_j(v)\right)\right|\right)^2 dv\right\} \\
&\quad \leq e\sup_{s\geq m(t_0)}\left|x(s)-y(s)\right|^2 \\
&\qquad \cdot\sup_{s\geq t_0}\Bigg\{\left(1+\frac{1}{l}\right)\Bigg(\sum_{j=1}^{n}\left|q_j(s)\right|+\sum_{j=1}^{n}\int_{t_0}^{s} e^{-\int_v^s h(u)du}\left|h(v)\right|\int_{v-\tau_j(v)}^{v}\left|h_j(u)\right|du\,ds \\
&\qquad\qquad +\sum_{j=1}^{n}\int_{s-\tau_j(s)}^{s}\left|h_j(v)\right|dv \\
&\qquad\qquad +\sum_{j=1}^{n}\int_{t_0}^{s} e^{-\int_v^s h(u)du} \\
&\qquad\qquad\qquad \times\left|\left(h_j\left(v-\tau_j(v)\right)\left(1-\tau_j'(v)\right)+b_j(v)-q_j(v)h(v)\right|dv\right)^2 \\
&\qquad\qquad +4(1+l)\int_{t_0}^{s} e^{-2\int_v^s h(u)du}\left(\sum_{j=1}^{n}\left|c_j(v)\right|\right)^2 dv\Bigg\} \leq \left(\alpha^2+\varepsilon\right)e\sup_{s\geq m(t_0)}\left|x(s)-y(s)\right|^2.
\end{aligned}
\tag{2.20}
$$

Therefore, P is contraction mapping with contraction constant $\alpha^2+\varepsilon$. By the contraction mapping principle, P has a fixed point $x\in S$, which is a solution of (2.7) with $x(s)=\phi(s)$ on $[m(t_0),t_0]$ and $E|x(t)|^2\to 0$ as $t\to\infty$.

To obtain the mean square asymptotic stability, we need to show that the zero solution of (2.7) is mean square stable. Let $\epsilon>0$ be given and choose $\delta>0$ and $\delta<\epsilon$ satisfying the following condition:

$$4\delta K^2(1+L)e^{2\int_0^{t_0}H(u)du}+\left(\alpha^2+\varepsilon\right)\epsilon<\epsilon, \tag{2.21}$$

where $K=\sup_{t\ge 0}\{e^{-\int_0^t H(s)ds}\}$. If $x(t)=x(t,t_0,\phi)$ is a solution of (2.7) with $\|\phi\|^2<\delta$, then $x(t)=(Px)(t)$ defined in (2.13). We assume that $E|x(t)|^2<\epsilon$ for all $t\ge t_0$. Notice that $E|x(t)|^2=\|\phi\|^2<\epsilon$ for $t\in[m(t_0),t_0]$. If there exists $t^*>t_0$ such that $E|x(t^*)|^2=\epsilon$ and $E|x(t)|^2<\epsilon$ for $t\in[m(t_0),t^*)$, then (2.13) and (2.19) imply that

$$\begin{aligned}
E\left|x(t^*)\right|^2 &\le (1+L)\|\phi\|^2\left(1+\sum_{j=1}^n\left|q_j(t_0)\right|+\sum_{j=1}^n\int_{t_0-\tau_j(t_0)}^{t_0}\left|h_j(s)\right|ds\right)^2 e^{-2\int_{t_0}^{t^*}H(u)du}\\
&\quad+\epsilon\left(1+\frac{1}{L}\right)\left(\sum_{j=1}^n\left|q_j(t^*)\right|+\sum_{j=1}^n\int_{t^*-\tau_j(t^*)}^{t^*}\left|h_j(s)\right|ds\right.\\
&\qquad+\int_{t_0}^{t^*}e^{-\int_s^{t^*}H(u)du}\left(\sum_{j=1}^n\int_{s-\tau_j(s)}^{s}\left|h_j(u)\right|du\right)\left|H(s)\right|ds\\
&\qquad\left.+\int_{t_0}^{t^*}e^{-\int_s^{t^*}H(u)du}\sum_{j=1}^n\left|h_j(s-\tau_j(s))\left(1-\tau_j'(s)\right)+b_j(s)-q_j(s)H(s)\right|ds\right)^2\\
&\quad+\epsilon\int_{t_0}^{t^*}e^{-2\int_s^{t^*}H(u)du}\left(\sum_{j=1}^n\left|c_j(s)\right|\right)^2ds\\
&\le (1+L)\delta\left(1+\sum_{j=1}^n\left|q_j(t_0)\right|+\sum_{j=1}^n\int_{t_0-\tau_j(t_0)}^{t_0}\left|h_j(s)\right|ds\right)^2 e^{-2\int_{t_0}^{t^*}H(u)du}+\left(\alpha^2+\varepsilon\right)\epsilon<\epsilon,
\end{aligned} \tag{2.22}$$

which contradicts the definition of t^*. Thus, the zero solution of (2.7) is stable. It follows that the zero solution of (2.7) is mean square asymptotically stable if (2.11) holds.

Conversely, we suppose that (2.11) fails. From (i), there exists a sequence $\{t_n\}$ with $t_n\to\infty$ as $n\to\infty$ such that $\lim_{n\to\infty}\int_0^{t_n}H(u)du=\beta$, where $\beta\in\mathbb{R}$. Then, we can choose a constant $J>0$ satisfying $\int_0^{t_n}H(u)du\in[-J,J]$ for all $n\ge 1$. Denote

$$\omega(s)=\sum_{j=1}^n\left|\left(h_j(s-\tau_j(s))\left(1-\tau_j'(s)\right)+b_j(s)-q_j(s)H(s)\right|+\left|H(s)\right|\int_{s-\tau_j(s)}^{s}\left|h_j(u)\right|du \tag{2.23}$$

for all $s\ge 0$. From (ii), we have

$$\int_0^{t_n}e^{-\int_s^{t_n}H(u)du}\omega(s)ds\le\alpha, \tag{2.24}$$

which implies

$$\int_0^{t_n} e^{\int_0^s H(u)du}\omega(s)ds \le \alpha e^{\int_0^{t_n} H(u)du} \le e^J. \tag{2.25}$$

Therefore, the sequence $\{\int_0^{t_n} e^{\int_0^s H(u)du}\omega(s)ds\}$ has a convergent subsequence. Without loss of generality, we can assume that

$$\lim_{n\to\infty}\int_0^{t_n} e^{\int_0^s H(u)du}\omega(s)ds = \gamma \tag{2.26}$$

for some $\gamma > 0$. Let k be an integer such that

$$\int_{t_k}^{t_n} e^{\int_0^s H(u)du}\omega(s)ds < \frac{\delta_0}{8K} \tag{2.27}$$

for all $n \ge k$, where $\delta_0 > 0$ satisfies $8\delta_0 K^2 e^{2J} + (\alpha^2 + \varepsilon) < 1$.

Now we consider the solution $x(t) = x(t, t_k, \phi)$ of (2.7) with $\|\phi(t_k)\|^2 = \delta_0$ and $\|\phi(s)\|^2 < \delta_0$ for $s < t_k$. By the similar method in (2.22), we have $E|x(t)|^2 < 1$ for $t \ge t_k$. We may choose ϕ so that

$$G(t_k) := \phi(t_k) - \sum_{j=1}^{n} q_j(t_k)\phi(t_k - \tau_j(t_k)) - \sum_{j=1}^{n}\int_{t_k-\tau_j(t_k)}^{t_k} h_j(s)\phi(s)ds \ge \frac{1}{2}\delta_0. \tag{2.28}$$

It follows from (2.13) and (2.28) with $x(t) = (Px)(t)$ that for $n \ge k$,

$$\begin{aligned} &E\left|x(t_n) - \sum_{j=1}^{n} q_j(t_n)x(t_n - \tau_j(t_n)) - \sum_{j=1}^{n}\int_{t_n-\tau_j(t_n)}^{t_n} h_j(s)x(s)ds\right|^2 \\ &\quad\ge G^2(t_k)e^{-2\int_{t_k}^{t_n} H(u)du} - 2G(t_k)e^{-\int_{t_k}^{t_n} H(u)du}\int_{t_k}^{t_n} e^{-\int_s^{t_n} H(u)du}\omega(s)ds \\ &\quad\ge \frac{\delta_0}{2}e^{-2\int_{t_k}^{t_n} H(u)du}\left(\frac{\delta_0}{2} - 2K\int_{t_k}^{t_n} e^{\int_0^s H(u)du}\omega(s)ds\right) \ge \frac{\delta_0^2}{8}e^{-2J} > 0. \end{aligned} \tag{2.29}$$

If the zero solution of (2.7) is mean square asymptotic stable, then $E|x(t)|^2 = E|x(t, t_k, \phi)|^2 \to 0$ as $t \to 0$. Since $t_n - \tau_j(t_n) \to \infty$, $t_n - \delta_j(t_n) \to \infty$ as $n \to \infty$ and condition (ii) and (2.11) hold,

$$E\left|x(t_n) - \sum_{j=1}^{n} q_j(t_n)x(t_n - \tau_j(t_n)) - \sum_{j=1}^{n}\int_{t_n-\tau_j(t_n)}^{t_n} h_j(s)x(s)ds\right|^2 \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{2.30}$$

which contradicts (2.29). Therefore, (2.11) is necessary for Theorem 2.1. This completes the proof. □

Remark 2.2. Theorem 2.1 still holds if condition (ii) is satisfied for $t \ge t_a$ for some $t_a \in \mathbb{R}^+$.

Remark 2.3. Theorem 2.1 improves Theorem C under different conditions.

Corollary 2.4. *Suppose that τ_j is differential, the inverse function $g_j(t)$ of $t-\tau_j(t)$ exists, and there exists a constant $\alpha \in (0,1)$ such that for $t \geq 0$, $\liminf_{t\to\infty}\int_0^t Q(s)ds > -\infty$ and*

$$\sum_{j=1}^{n}|q_j(t)| + \sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}|b_j(g_j(s))|ds + \sum_{j=1}^{n}\int_0^t e^{-\int_s^t Q(u)du}|b_j(s)\tau_j'(s) - q_j(s)Q(s)|ds$$
$$+\sum_{j=1}^{n}\int_0^t e^{-\int_s^t Q(u)du}|Q(s)|\int_{s-\tau_j(s)}^{s}|b_j(g_j(u))|du\,ds+2\left(\int_0^t e^{-2\int_s^t Q(u)du}\left(\sum_{j=1}^{n}|c_j(s)|\right)^2 ds\right)^{1/2} \leq \alpha < 1, \tag{2.31}$$

where $Q(t)=\sum_{j=1}^{n} - b_j(g_j(t))$. Then the zero solution of (2.7) *is mean square asymptotically stable if and only if $\int_0^t Q(s)ds \to \infty$ as $t\to\infty$.*

Remark 2.5. When $h_j(t) = -b_j(g_j(t))$ for $j = 1\cdots n$, Theorem 2.1 reduces to Corollary 2.4. On the other hand, we choose $q_j(t)\equiv c_j(t)\equiv 0$ and $b_j \equiv -\overline{b}_j$ for $j = 1\cdots n$, then Corollary 2.4 reduces to Theorem B.

3. Two examples

In this section, we give two examples to illustrate applications of Theorem 2.1 and Corollary 2.4.

Example 3.1. Consider the following linear neutral stochastic delay differential equation:

$$d\left(x(t)-\frac{x(t-t/2)}{1000}\right)=\left(-\frac{x(t-t/2)}{16+16t}-\frac{3\sin t+4}{48+48t}x\left(t-\frac{t}{4}\right)\right)dt+\left(\frac{x(t)}{24\sqrt{3+4t}}-\frac{x(t-\sin t)}{12\sqrt{3+4t}}\right)dW(t). \tag{3.1}$$

Then the zero solution of (3.1) is mean square asymptotically stable.

Proof. Choosing $h_1(t) = 1/(8+16t)$ and $h_2(t) = 7/(48+64t)$ in Theorem 2.1, we have

$$H(t) = \frac{1}{8+16t}+\frac{7}{48+64t}, \quad \frac{11}{48+64t} \leq H(t) \leq \frac{13}{48+64t},$$
$$\sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t}|h_j(s)|ds = \int_{t/2}^{t}\frac{1}{8+16s}ds + \int_{3t/4}^{t}\frac{7}{48+64s}ds \longrightarrow 0.07479, \quad \text{as } t\longrightarrow\infty,$$
$$\sum_{j=1}^{2}\int_0^t e^{-\int_s^t H(u)du}|H(s)|\int_{s-\tau_j(s)}^{s}|h_j(u)|du\,ds \leq \int_0^t e^{-\int_s^t (11/(48+64u))du}\frac{13}{48+64s}\cdot 0.07479\,ds \leq 0.08839,$$
$$2\left(\int_0^t e^{-2\int_s^t H(u)du}\left(\sum_{j=1}^{2}|c_j(s)|\right)^2 ds\right)^{1/2} \leq 2\left(\int_0^t e^{-\int_s^t (11/(24+32u))du}\frac{1}{8(24+32s)}ds\right)^{1/2} \leq 0.21320,$$
$$\sum_{j=1}^{2}\int_0^t e^{-\int_s^t H(u)du}|(h_j(s-\tau_j(s))(1-\tau_j'(s)) + b_j(s) - q_j(s)H(s)|ds$$
$$\leq \int_0^t e^{-\int_s^t (11/(48+64u))du}\left(\frac{0.013}{48+64s}+\frac{17}{144+192s}\right)ds \leq \frac{0.013}{11}+\frac{17}{33} = 0.51634. \tag{3.2}$$

It easy to check that $\int_0^\infty H(s)ds = \infty$. Let $\alpha = 0.001 + 0.07479 + 0.08839 + 0.21320 + 0.51634$. Then, $\alpha = 0.89372 < 1$ and the zero solution of (3.1) is mean square asymptotically stable by Theorem 2.1. □

Example 3.2. Consider the following delay differential equation:

$$x'(t) = -\frac{1}{6+4t}x\left(t-\frac{t}{3}\right) - \frac{1}{12+4t}x\left(t-\frac{2}{3}t\right). \tag{3.3}$$

Then the zero solution of (3.3) is asymptotically stable.

Proof. Choosing $h_1(t) = h_2(t) = 1/(4+4t)$ in Theorem 2.1, we have $H(t) = 1/(2+2t)$ and

$$\begin{gathered}\sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t}|h_j(s)|ds = \int_{(2/3)t}^{t}\frac{1}{4+4s}\,ds + \int_{t/3}^{t}\frac{1}{4+4s}\,ds \longrightarrow \frac{1}{2}\ln 3 - \frac{1}{4}\ln 2 = 0.37602, \quad \text{as } t\longrightarrow\infty, \\ \sum_{j=1}^{2}\int_0^t e^{-\int_s^t H(u)du}|H(s)|\int_{s-\tau_j(s)}^{s}|h_j(u)|du\,ds \le \int_0^t e^{-\int_s^t(1/(2+2u))du}\frac{1}{2+2s}\cdot 0.37602\,ds \le 0.37602.\end{gathered} \tag{3.4}$$

Notice that $q_j(t) = c_j(t) \equiv 0$ and

$$\begin{aligned}&\sum_{j=1}^{2}\left|(h_j(s-\tau_j(s))(1-\tau_j'(s)) + b_j(s) - q_j(s)H(s)\right| \\ &\quad = \left|\frac{3}{12+8s}\cdot\frac{2}{3} - \frac{1}{6+4s}\right| + \left|\frac{3}{12+4s}\cdot\frac{1}{3} - \frac{1}{12+4s}\right| = 0.\end{aligned} \tag{3.5}$$

It is easy to see that all the conditions of Theorem 2.1 hold for $\alpha = 0.37602+0.37602 = 0.75204 < 1$. Thus, Theorem 2.1 implies that the zero solution of (3.3) is asymptotically stable.

However, Theorem B cannot be used to verify that the zero solution of (3.3) is asymptotically stable. In fact, $\overline{b}_1(t) = 1/(6+4t)$, $\overline{b}_2(t) = 1/(12+4t)$, $\overline{b}_1(g_1(t)) = 1/(6+6t)$, $\overline{b}_2(g_2(t)) = 1/(12+12t)$, and $|Q(t)| = 1/(4+4t)$. As $t\to\infty$,

$$\sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t}|\overline{b}_j(g_j(s))|ds \le \int_{(2/3)t}^{t}\frac{1}{6+6s}\,ds + \int_{t/3}^{t}\frac{1}{12+12s}\,ds \longrightarrow \frac{1}{4}\ln 3 - \frac{1}{6}\ln 2 = 0.15913. \tag{3.6}$$

Notice that

$$\sum_{j=1}^{2}\left|\overline{b}_j(s)\tau_j'(s) - q_j(s)Q(s)\right| = \frac{1}{18+12s} + \frac{1}{18+6s} \le \frac{1}{4+4s}. \tag{3.7}$$

It follows from (3.7) that

$$\sum_{j=1}^{2}\int_0^t e^{-\int_s^t Q(u)du}\left|\overline{b}_j(s)\tau_j'(s) - q_j(s)Q(s)\right|ds \le \int_0^t e^{-\int_s^t(1/(4+4u))du}\frac{1}{4+4s}\,ds \le 1. \tag{3.8}$$

From (3.6), we obtain

$$\sum_{j=1}^{2}\int_{0}^{t}e^{-\int_{s}^{t}Q(u)du}|Q(s)|\int_{s-\tau_j(s)}^{s}\left|\bar{b}_j(g_j(u))\right|du\,ds\le\int_{0}^{t}e^{-\int_{s}^{t}(1/(4+4u))du}\frac{1}{4+4s}\cdot 0.15913\,ds\le 0.15913. \tag{3.9}$$

Combining (3.6), (3.8), and (3.9), we see that the condition (2.4) of Theorem B does not hold with $\alpha = 1.31825$. □

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