

Research Article

Best Proximity Sets and Equilibrium Pairs for a Finite Family of Multimaps

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We establish the existence of a best proximity pair for which the best proximity set is nonempty for a finite family of multimaps whose product is either an \mathfrak{A}_ξ^k -multimap or a multimap $T : A \rightarrow 2^B$ such that both T and $S \circ T$ are closed and have the **KKM** property for each Kakutani multimap $S : B \rightarrow 2^A$. As applications, we obtain existence theorems of equilibrium pairs for free n -person games as well as for free 1-person games. Our results extend and improve several well-known and recent results.

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1. Introduction

Let $E := (E, \|\cdot\|)$ be a normed space, A a nonempty subset of E , and $T : A \rightarrow E$ a single-valued map. Whenever the equation $T(x) = x$ has no solution in A , it is natural to ask if there exists an approximate solution. Fan [1] provided sufficient conditions for the existence of an approximate solution $a \in A$ (called a best approximant) such that

$$\|a - T(a)\| = d(T(a), A) := \inf\{d(T(a), x) : x \in A\}, \quad (1.1)$$

where A is compact and convex and T is continuous. However, there is no guarantee that such an approximate solution is optimal. For suitable subsets A and B of E and multimap $T : A \rightarrow 2^B$, Sadiq Basha and Veeramani [2] provided sufficient conditions for the existence of an optimal solution $(a, T(a))$ (called a best proximity pair) such that

$$d(a, T(a)) = d(A, B) := \inf\{\|x - y\| : x \in A, y \in B\}. \quad (1.2)$$

Srinivasan and Veeramani [3, 4] extended these results and obtained existence theorems of equilibrium pairs for constrained generalized games. Kim and Lee [5, 6] generalized

Srinivasan and Veeramani results and obtained existence theorems of equilibrium pairs for free n -person games. Recently, Al-Thagafi and Shahzad [7] generalized and extended the above results to Kakutani multimaps.

In this paper, we establish the existence of a best proximity pair for which the best proximity set is nonempty for a finite family of multimaps whose product is either an $\mathfrak{A}_c^{\mathfrak{K}}$ -multimap or a multimap $T : A \rightarrow 2^B$ such that both T and $S \circ T$ are closed and have the **KKM** property for each Kakutani multimap $S : B \rightarrow 2^A$. As applications, we obtain existence theorems of equilibrium pairs for free n -person games as well as free 1-person games. Our results extend and improve several well-known and recent results.

2. Preliminaries

Throughout, $E := (E, \|\cdot\|)$ is a normed space, A and B are nonempty subsets of E , 2^A is the family of all subsets of A , $\text{co}A$ is the convex hull of A in E , $\text{int} A$ is the interior of A in E , $\mathfrak{C}(A, B)$ is the set of all continuous single-valued maps, $d(x, A) := \inf\{d(x, a) : a \in A\}$, and $d(A, B) := \inf\{\|a - b\| : a \in A \text{ and } b \in B\}$. A map $T : A \rightarrow 2^B$ is called a multimap (multifunction or correspondence) if $T(x)$ is nonempty for each $x \in A$. A multimap $T : A \rightarrow 2^A$ is said to have a fixed point $a \in A$ if $a \in T(a)$; the set of fixed points of T is denoted by $F(T)$. A multimap $T : A \rightarrow 2^B$ is said to be (a) upper semicontinuous if $\overline{T^{-1}(D)} = \{x \in A : T(x) \cap D \neq \emptyset\}$ is closed in A whenever D is closed in B , (b) compact if $\overline{T(A)}$ is compact in B , (c) closed if its graph $\text{Gr}(T) := \{(x, y) : x \in A \text{ and } y \in T(x)\}$ is closed in $A \times B$ and (d) compact-valued (resp., convex) if $T(x)$ is compact (resp., convex) in B for every $x \in A$. A map $f : A \rightarrow B$ is proper if $f^{-1}(K)$ is compact in A whenever K is compact in B . A map $f : A \rightarrow E$ is quasilinear if the set $Q(x) := \{a \in A : \|f(a) - x\| \leq r\}$ is convex for every $x \in E$ and $r \in [0, \infty)$.

Lemma 2.1 (see [8]). *Let A and B be nonempty subsets of a normed space E . If $T : A \rightarrow 2^B$ is an upper semicontinuous multimap with compact values, then T is closed.*

The set of all $a \in A$ such that $\|a - x\| = d(x, A)$, denoted by $P_A(x)$, is called the set of best approximations in A to $x \in E$. The multimap $P_A : E \rightarrow 2^A$ is called the metric projection on A . Whenever A is compact and convex, P_A is upper semicontinuous with compact and convex values (see [8]).

A polytope P in A is any convex hull of a nonempty finite subset D of A . Whenever \mathfrak{X} is a class of maps, denote the set of all finite compositions of maps in \mathfrak{X} by \mathfrak{X}_c and denote the set of all multimaps $T : A \rightarrow 2^B$ in \mathfrak{X} by $\mathfrak{X}(A, B)$. Let \mathfrak{A} be an abstract class of maps [9] satisfying the following properties:

- (1) \mathfrak{A} contains the class \mathfrak{C} of continuous single-valued maps;
- (2) each $T \in \mathfrak{A}_c$ is upper semicontinuous with compact values;
- (3) for any polytope P , each $T \in \mathfrak{A}_c(P, P)$ has a fixed point.

Let $T : A \rightarrow 2^B$. We say that (e) T is an $\mathfrak{A}_c^{\mathfrak{K}}$ -multimap [9] if for every compact set K in A , there exists an \mathfrak{A}_c -multimap $f : K \rightarrow 2^B$ such that $f(x) \subseteq T(x)$ for each $x \in K$, (f) T is a **K**-multimap (or Kakutani multimap) [10] if T is upper semicontinuous with compact and convex values, (g) $S : A \rightarrow 2^B$ is a generalized **KKM**-multimap with respect to T [11] if $T(\text{co}D) \subseteq S(D)$ for each finite subset D of A , (h) T has the **KKM** property [11] if, whenever $S : A \rightarrow 2^B$ is a generalized **KKM** multimap w.r.t. T , the family $\{\overline{S(x)} : x \in A\}$ has the finite

intersection property; (i) T is a **PK**-multimap [12] if there exists a multimap $g : A \rightarrow 2^B$ satisfying $A = \bigcup \{\text{int } g^{-1}(y) : y \in B\}$ and $\text{co}(g(x)) \subseteq T(x)$ for every $x \in A$. Note that each \mathfrak{A}_c^κ -multimap has the **KKM** property and each **K**-multimap (resp., \mathfrak{A}_c -multimap, **PK**-multimap) is an \mathfrak{A}_c^κ -multimap (see [9, 13, 14]).

Let A and B_i be nonempty subsets of a normed space E for each $i \in I_n := \{1, 2, \dots, n\}$. Define

$$\begin{aligned} A_i^0 &:= \{a \in A : \|a - b\| = d(A, B_i) \text{ for some } b \in B_i\}, \\ B_i^0 &:= \{b \in B_i : \|a - b\| = d(A, B_i) \text{ for some } a \in A\}, \end{aligned} \quad (2.1)$$

$A^0 := \bigcap_{i \in I_n} A_i^0$. For $n = 1$, let $A_0 := A_1^0 = A^0$ and $B_0 := B_1^0$.

The following result is a part of [7, Theorem 3.1].

Lemma 2.2. *Let A and B_i be nonempty subsets of E for each $i \in I_n$:*

- (a) $P_A(B_i^0) = P_{A_i^0}(B_i^0) = A_i^0$;
- (b) if A_i^0 and B_i are compact (resp., convex), then B_i^0 is compact (resp., convex);
- (c) if A_i^0 is nonempty, compact, and convex and B_i^0 is convex, then $P_{A_i^0|B_i^0}$ is a **K**-multimap.

Remark 2.3. We note, from part (a) of Lemma 2.2 and the definitions of A^0 , A_i^0 , and B_i^0 , that

- (a₁) A_i^0 is nonempty if and only if B_i^0 is nonempty;
- (a₂) $P_A(B_i^0) = A^0$ if and only if $A_i^0 = A^0$; so [5, Theorems 1, 2, and 4] by Kim and Lee are valid only whenever $A_i^0 = A^0$;
- (a₃) $\bigcap_{i=1}^n P_A(B_i^0) = \bigcap_{i=1}^n P_{A_i^0}(B_i^0) = \bigcap_{i=1}^n A_i^0 = A^0$. So $A^0 \neq \emptyset$ if and only if $\bigcap_{i=1}^n P_{A_i^0}(y_i) \neq \emptyset$ for some $(y_1, \dots, y_n) \in \prod_{i=1}^n B_i^0$.

Lemma 2.4 (see [11, 14]). *Let A be a nonempty convex subset of a normed space E . If $T : A \rightarrow 2^A$ is a closed and compact multimap having the **KKM** property, then T has a fixed point.*

Lemma 2.5 (see [15]). *For each $i \in I_n$, let B_i be a nonempty, compact, and convex subset of a normed space E , $P_i : \prod_{j=1}^n B_j \rightarrow 2^{B_i}$ a map such that*

- (a) $x_i \notin \text{co}P_i(x)$ for each $x = (x_1, \dots, x_n) \in B := \prod_{j=1}^n B_j$;
- (b) $P_i^{-1}(y)$ is open in B for each $y \in B_i$.

Then there exists $b \in B$ such that $P_i(b) = \emptyset$ for each $i \in I_n$.

Lemma 2.6 (see [5, 6, 15, 16]). *Let B be a nonempty, compact, and convex subset of a normed space E and $P : B \rightarrow 2^B$ a map such that*

- (a) $x \notin \text{co}P(x)$ for each $x \in B$.

Assume that one of the following conditions is satisfied:

- (b₁) if $z \in P^{-1}(y)$, then there exists some $y' \in B$ such that $z \in \text{int } P^{-1}(y')$;
- (b₂) $P^{-1}(y)$ is open in B for each $y \in B$.

Then there exists $b \in B$ such that $P(b) = \emptyset$.

3. Best proximity results

Lemma 3.1. *Let A and B_i be subsets of a normed space E such that A_i^0 (resp., B_i^0) are nonempty, compact (resp., closed), and convex for each $i \in I_n$. Suppose that $f : A^0 \rightarrow A^0$ is a continuous, proper, quasilinear, and surjective self-map, and $P : Y \rightarrow 2^{A^0}$ is a multimap defined by $P(y_1, \dots, y_n) := \bigcap_{i=1}^n P_{A_i^0}(y_i)$ for each $(y_1, \dots, y_n) \in Y := \prod_{i=1}^n B_i^0$. Then $f^{-1}P : Y \rightarrow 2^{A^0}$ is a \mathbf{K} -multimap.*

Proof. Fix $i \in I_n$. Since A_i^0 is compact and convex, then $P_{A_i^0} : E \rightarrow 2^{A_i^0}$ is a \mathbf{K} -multimap. As B_i^0 is closed, we conclude, from Lemma 2.2(c), that $P_{A_i^0}|_{B_i^0}$ is a \mathbf{K} -multimap and, hence, $P : Y \rightarrow 2^{A^0}$ is a \mathbf{K} -multimap. Let $S := f^{-1}P$. As f is surjective and

$$S(Y) = f^{-1}P(Y) \subseteq f^{-1}(A^0) = A^0, \quad (3.1)$$

then $S : Y \rightarrow 2^{A^0}$ is a multimap. To show that S is upper semicontinuous, let D be a closed subset of A^0 and let $\{y_m\}$ be a sequence in $S^{-1}(D)$ such that $y_m = (y_{m1}, \dots, y_{mn}) \rightarrow y = (y_1, \dots, y_n) \in Y$ as $m \rightarrow \infty$. Choose a sequence $\{x_m\}$ in D such that $x_m \in S(y_m)$. Then $f(x_m) \in P(y_m) \subseteq A^0$ for each $m \geq 1$. As D is compact, we may assume that $x_m \rightarrow x \in D$ as $m \rightarrow \infty$. The continuity of f and the compactness of A^0 imply that $f(x_m) \rightarrow f(x) \in A^0$ as $m \rightarrow \infty$. Since $f(x_m) \in P_{A_i^0}(y_{mi})$, it follows that

$$\begin{aligned} \|f(x) - y_i\| &\leq \|f(x) - f(x_m)\| + \|f(x_m) - y_{mi}\| + \|y_{mi} - y_i\| \\ &= \|f(x) - f(x_m)\| + d(y_{mi}, A_i^0) + \|y_{mi} - y_i\| \end{aligned} \quad (3.2)$$

for each m . Letting $m \rightarrow \infty$, we obtain $\|f(x) - y_i\| = d(y_i, A_i^0)$. This implies that $f(x) \in P_{A_i^0}(y_i)$ and hence $f(x) \in P(y)$. From this, we conclude that $x \in S(y) \cap D$ and $y \in S^{-1}(D)$. Therefore, $S^{-1}(D)$ is closed and hence S is upper semicontinuous.

Notice, as f is proper and $P(y)$ is compact, that $S(y)$ is compact. Also, as f is quasilinear, the set

$$Q(y_i) := \{a \in A^0 : \|f(a) - y_i\| = d(y_i, A_i^0)\} \quad (3.3)$$

is convex. For $a_1, a_2 \in S(y)$, we have $f(a_1), f(a_2) \in P(y)$ and hence $f(a_1), f(a_2) \in P_{A_i^0}(y_i)$. This implies that $a_1, a_2 \in Q(y_i)$ and, by the convexity of $Q(y_i)$, $y_\lambda := \lambda a_1 + (1 - \lambda)a_2 \in Q(y_i)$ for each $\lambda \in [0, 1]$. Thus $f(y_\lambda) \in P_{A_i^0}(y_i)$ and hence $f(y_\lambda) \in P(y)$. From this, we conclude that $y_\lambda \in S(y)$ and hence $S(y)$ is convex. Therefore, $S : Y \rightarrow 2^{A^0}$ is a \mathbf{K} -multimap. \square

Definition 3.2. Let A and B_i be nonempty subsets of a normed space E , $T_i : A \rightarrow 2^{B_i}$ a multimap for each $i \in I_n$, $f : A' \rightarrow A'$ a self-map of a nonempty subset A' of A , and $a \in A$. If $d(f(a), T_i(a)) = d(A, B_i)$, one says that $(f(a), T_i(a))$ is a best proximity pair. The best proximity set for the pair $(f(a), T_i(a))$ is given by

$$\mathfrak{T}_a^i(f) := \{b \in T_i(a) : d(f(a), T_i(a)) = \|f(a) - b\| = d(A, B_i)\}. \quad (3.4)$$

For $n = 1$, let $\mathfrak{T}_a(f) := \mathfrak{T}_a^1(f)$. Whenever f is the identity map, we write \mathfrak{T}_a^i instead of $\mathfrak{T}_a^i(f)$.

Definition 3.3. Let $T : A \rightarrow 2^B$ be a multimap. One says that T is a \mathbf{KKM}_0 -multimap if T and $S \circ T : A \rightarrow 2^A$ are closed and have the \mathbf{KKM} property for each \mathbf{K} -multimap $S : B \rightarrow 2^A$.

Theorem 3.4. Let A and B_i be subsets of a normed space E , A_i^0 (resp., B_i^0) nonempty, compact (resp., closed), and convex, and $T_i : A \rightarrow 2^{B_i}$ a multimap for each $i \in I_n$. Suppose that $\bigcap_{i=1}^n P_{A_i^0}(\mathbf{y}_i)$ is nonempty for each $(\mathbf{y}_1, \dots, \mathbf{y}_n) \in Y$ and $T : A^0 \rightarrow 2^Y$ is a \mathbf{KKM}_0 -multimap (resp., \mathfrak{A}_c^κ -multimap) where $T(x) := \prod_{i=1}^n T_i(x)$ for each $x \in A^0$ and $Y := \prod_{i=1}^n B_i^0$. Then, for each continuous, proper, quasiasffine, and surjective self-map $f : A^0 \rightarrow A^0$, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_a^i(f)$ is nonempty and closed.

Proof. Fix $i \in I_n$. Define $P : Y \rightarrow 2^{A^0}$ by $P(\mathbf{y}_1, \dots, \mathbf{y}_n) := \bigcap_{i=1}^n P_{A_i^0}(\mathbf{y}_i)$ for each $(\mathbf{y}_1, \dots, \mathbf{y}_n) \in Y$. Let $f : A^0 \rightarrow A^0$ be a continuous, proper, and quasiasffine self-map. As $\bigcap_{i=1}^n P_{A_i^0}(\mathbf{y}_i)$ is nonempty for each $(\mathbf{y}_1, \dots, \mathbf{y}_n) \in \prod_{i=1}^n B_i^0$, it follows from Lemma 3.1 that $f^{-1}P : Y \rightarrow 2^{A^0}$ is a \mathbf{K} -multimap. Now, assume that $T : A^0 \rightarrow 2^Y$ is a \mathbf{KKM}_0 -multimap. It follows from the definition of a \mathbf{KKM}_0 -multimap that $f^{-1}P \circ T : A^0 \rightarrow 2^{A^0}$ is a closed multimap having the \mathbf{KKM} property. As A^0 is a compact set, $f^{-1}P \circ T$ is a compact multimap. By Lemma 2.4, there exists $a \in A^0$ such that $a \in (f^{-1}P \circ T)(a)$ and hence $f(a) \in P(T(a))$. Thus, there exists $(b_1, \dots, b_n) \in T(a) = \prod_{i=1}^n T_i(a)$ such that $f(a) \in P(b_1, \dots, b_n) = \bigcap_{i=1}^n P_{A_i^0}(b_i) \subseteq A^0$. Hence, $f(a) \in P_{A_i^0}(b_i) \subseteq A_i^0$ and $b_i \in T_i(a) \subseteq B_i^0$. This implies that there exists $a'_i \in A_i^0$ such that $\|a'_i - b_i\| = d(A, B_i)$ and hence

$$d(A, B_i) \leq d(f(a), T_i(a)) \leq \|f(a) - b_i\| = d(b_i, A_i^0) \leq \|a'_i - b_i\| = d(A, B_i). \quad (3.5)$$

Thus $d(f(a), T_i(a)) = \|f(a) - b_i\| = d(A, B_i)$.

Next, assume that $T : A^0 \rightarrow 2^Y$ is an \mathfrak{A}_c^κ -multimap. Then, there exists an \mathfrak{A}_c -multimap $T' : A^0 \rightarrow 2^Y$ such that T' is upper semicontinuous with compact values and $T'(x) := \prod_{i=1}^n T'_i(x) \subseteq T(x)$ for each $x \in A^0$ for every $x \in A^0$. Since $f^{-1}P \circ T' : A^0 \rightarrow 2^{A^0}$ is an \mathfrak{A}_c^κ -multimap (hence, a multimap having the \mathbf{KKM} property) and $f^{-1}P \circ T'$ is closed, then $T' : A^0 \rightarrow 2^Y$ is a \mathbf{KKM}_0 -multimap. It follows from the previous paragraph that there exists $(a, b) \in A^0 \times Y$ such that $b = (b_1, \dots, b_n)$, $b_i \in T'_i(a)$, and

$$d(f(a), T'_i(a)) = \|f(a) - b_i\| = d(A, B_i). \quad (3.6)$$

As $d(A, B_i) \leq d(f(a), T_i(a)) \leq d(f(a), T'_i(a))$, we conclude that

$$d(f(a), T_i(a)) = \|f(a) - b_i\| = d(A, B_i). \quad (3.7)$$

Therefore, in both cases, the best proximity set $\mathfrak{T}_a^i(f)$ is nonempty and its closedness follows from the continuity of the norm. \square

Corollary 3.5. Let A and B_i be subsets of a normed space E such that A_i^0 (resp., B_i^0) is nonempty, compact (resp., closed), and convex. Suppose that $\bigcap_{i=1}^n P_{A_i^0}(\mathbf{y}_i)$ is nonempty for each $(\mathbf{y}_1, \dots, \mathbf{y}_n) \in Y := \prod_{i=1}^n B_i^0$ and $T_i : A^0 \rightarrow 2^{B_i^0}$ is an \mathfrak{A}_c^κ -multimap for each $i \in I_n$. Then, for each continuous, proper, quasiasffine, and surjective self-map $f : A^0 \rightarrow A^0$, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_a^i(f)$ is nonempty and closed.

Proof. Define $T : A^0 \rightarrow 2^Y$ by $T(x) := \prod_{i=1}^n T_i(x)$ for each $x \in A^0$. As $T : A^0 \rightarrow 2^Y$ is an \mathfrak{A}_c^κ -multimap, the result follows from Theorem 3.4. \square

Remark 3.6. Since each PK-multimap is an \mathfrak{A}_c^κ -multimap, Theorem 4.1 of [12] is a special case of Corollary 3.5.

Corollary 3.7. *Let A and B_i be subsets of a normed space E , A_i^0 (resp., B_i^0) nonempty, compact (resp., closed), and convex, and $T_i : A \rightarrow 2^{B_i}$ a multimap for each $i \in I_n$. Suppose that $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1, \dots, y_n) \in Y$ and $T : A^0 \rightarrow 2^Y$ is a \mathbf{KKM}_0 -multimap (resp., \mathfrak{A}_c^κ -multimap) where $T(x) := \prod_{i=1}^n T_i(x)$ for each $x \in A^0$ and $Y := \prod_{i=1}^n B_i^0$. Then, there exists $a \in A^0$ such that the best proximity set \mathfrak{T}_a^i is nonempty and closed.*

Theorem 3.8. *Let A and B_i be subsets of a normed space E , A_i^0 (resp., B_i^0) nonempty, compact (resp., closed), and convex, $T_i : A^0 \rightarrow 2^{B_i}$ an upper semicontinuous multimap with compact values, and $T_i(x) \cap B_i^0$ nonempty for each $x \in A^0$ for each $i \in I_n$. Suppose that $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1, \dots, y_n) \in Y := \prod_{i=1}^n B_i^0$. Then, for each continuous, proper, quasiasffine and, surjective self-map $f : A^0 \rightarrow A^0$, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_a^i(f)$ is nonempty and closed.*

Proof. Fix $i \in I_n$. Define $T'_i : A^0 \rightarrow 2^{B_i^0}$ by $T'_i(x) := T_i(x) \cap B_i^0$ for each $x \in A^0$. Thus $T'_i : A^0 \rightarrow 2^{B_i^0}$ is an upper semicontinuous multimap with compact values. Define $T : A^0 \rightarrow 2^Y$ by $T(x) := \prod_{i=1}^n T'_i(x)$ for each $x \in A^0$. As A^0 is compact and $T : A^0 \rightarrow 2^Y$ is an upper semicontinuous multimap with compact values, then T is an \mathfrak{A}_c^κ -multimap. It follows from Corollary 3.5 that there exists $(a, b) \in A^0 \times Y$ such that $b = (b_1, \dots, b_n) \in \prod_{i=1}^n T'_i(a)$ and

$$d(f(a), T'_i(a)) = \|f(a) - b_i\| = d(A, B_i). \quad (3.8)$$

As $d(A, B_i) \leq d(f(a), T_i(a)) \leq d(f(a), T'_i(a))$, the result follows as in Theorem 3.4. \square

Corollary 3.9. *Let A and B_i be subsets of a normed space E , A_i^0 (resp., B_i^0) nonempty, compact (resp., closed), and convex, $T_i : A^0 \rightarrow 2^{B_i}$ an upper semicontinuous multimap with compact values, and $T_i(x) \cap B_i^0$ nonempty for each $x \in A^0$ for each $i \in I_n$. Suppose that $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1, \dots, y_n) \in Y := \prod_{i=1}^n B_i^0$. Then, there exists $a \in A^0$ such that the best proximity set \mathfrak{T}_a^i is nonempty and closed.*

Remark 3.10. Corollary 3.9 extends and improves [7, Theorems 3.2 and 3.4] by Al-Thagafi and Shahzad, [5, Theorems 1 and 2] by Kim and Lee, [3, Theorem 3.4] by Srinivasan and Veeramani, and [4, Theorem 3.2] by Srinivasan and Veeramani.

4. Equilibrium pair results for free n -person games

A free n -person game is a family of ordered quadruples $(A, B_i, T_i, P_i)_{i \in I_n}$ such that A and B_i are nonempty subsets of a normed space E , $T_i : A \rightarrow 2^{B_i}$ is a constraint multimap, and $P_i : B \rightarrow 2^{B_i}$ is a preference map where $B := \prod_{j=1}^n B_j$ (see [5]). An equilibrium pair for $(A, B_i, T_i, P_i)_{i \in I_n}$ is a point $(a, b) \in A \times B$ such that $T_i(a) \cap P_i(b) = \emptyset$. For details on economic terminology (see [5, 16]).

Theorem 4.1. *Let $(A, B_i, T_i, P_i)_{i \in I_n}$ be a free n -person game such that A and B_i are nonempty subsets of a normed space E , $T_i : A \rightarrow 2^{B_i}$ is a constraint multimap, and $P_i : B \rightarrow 2^{B_i}$ is a preference map*

where $B := \prod_{j=1}^n B_j$. Assume that A^0 is nonempty, $T(x) := \prod_{i=1}^n T_i(x)$ for each $x \in A^0$, $Y := \prod_{i=1}^n B_i^0$, and for each $i \in I_n$,

- (a) A_i^0 and B_i are nonempty, compact, and convex;
- (b) $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1, \dots, y_n) \in \prod_{i=1}^n B_i^0$;
- (c) $T : A^0 \rightarrow 2^Y$ is a \mathbf{KKM}_0 -multimap (resp., \mathfrak{A}_c^k -multimap);
- (d) $x_i \notin \text{co}P_i(x)$ for each $x = (x_1, \dots, x_n) \in B$;
- (e) $P_i^{-1}(y)$ is open for each $y \in B_i$.

Then, there exists $b \in B$ such that $P_i(b) = \emptyset$ and, for each continuous, proper, quasiasffine, and surjective self-map $f : A^0 \rightarrow A^0$, there exists $a \in A^0$ such that the best proximity set $\mathfrak{A}_a^i(f)$ is nonempty and compact. If, in addition, $P_i(z)$ is nonempty for each $z \notin \prod_{i=1}^n \mathfrak{A}_a^i(f)$, then (a, b) is an equilibrium pair in $A^0 \times \prod_{i=1}^n \mathfrak{A}_a^i(f)$.

Proof. Fix $i \in I_n$. As A_i^0 and B_i are compact and convex, it follows from Lemma 2.2(b) that B_i^0 is compact and convex. By Theorem 3.4, there exists $a \in A^0$ such that the best proximity set $\mathfrak{A}_a^i(f)$ is nonempty and closed. By Lemma 2.5, there exists $b = (b_1, \dots, b_n) \in Y$ such that $P_i(b) = \emptyset$. As $P_i(z)$ is nonempty for each $z \notin \prod_{i=1}^n \mathfrak{A}_a^i(f)$, we conclude that $b = (b_1, \dots, b_n) \in \prod_{i=1}^n \mathfrak{A}_a^i(f)$. Thus $(a, b) \in A^0 \times Y$, $b = (b_1, \dots, b_n) \in \prod_{i=1}^n T_i(a)$, $T_i(a) \cap P_i(b) = \emptyset$ and $d(f(a), T_i(a)) = \|f(a) - b_i\| = d(A, B_i)$. Thus (a, b) is an equilibrium pair in $A^0 \times \prod_{i=1}^n \mathfrak{A}_a^i(f)$. \square

Corollary 4.2. Let $(A, B_i, T_i, P_i)_{i \in I_n}$ be a free n -person game such that A and B_i are nonempty subsets of a normed space E , $T_i : A \rightarrow 2^{B_i}$ is a constraint multimap, and $P_i : B \rightarrow 2^{B_i}$ is a preference map where $B := \prod_{j=1}^n B_j$. Assume that A^0 is nonempty, $T(x) := \prod_{i=1}^n T_i(x)$ for each $x \in A^0$, $Y := \prod_{i=1}^n B_i^0$, and for each $i \in I_n$,

- (a) A_i^0 and B_i are nonempty, compact, and convex;
- (b) $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1, \dots, y_n) \in \prod_{i=1}^n B_i^0$;
- (c) $T : A^0 \rightarrow 2^Y$ is a \mathbf{KKM}_0 -multimap (resp., \mathfrak{A}_c^k -multimap);
- (d) $x_i \notin \text{co}P_i(x)$ for each $x = (x_1, \dots, x_n) \in B$;
- (e) $P_i^{-1}(y)$ is open for each $y \in B_i$.

Then, there exists $b \in B$ such that $P_i(b) = \emptyset$ and there exists $a \in A^0$ such that the best proximity set \mathfrak{A}_a^i is nonempty and compact. If, in addition, $P_i(z)$ is nonempty for each $z \notin \prod_{i=1}^n \mathfrak{A}_a^i$, then (a, b) is an equilibrium pair in $A^0 \times \prod_{i=1}^n \mathfrak{A}_a^i$.

Theorem 4.3. Let $(A, B_i, T_i, P_i)_{i \in I_n}$ be a free n -person game such that A and B_i are subsets of a normed space E , $T_i : A \rightarrow 2^{B_i}$ is a constraint multimap, and $P_i : B \rightarrow 2^{B_i}$ is a preference map where $B := \prod_{j=1}^n B_j$. Assume that A^0 is nonempty, $Y := \prod_{i=1}^n B_i^0$, and for each $i \in I_n$,

- (a) A_i^0 and B_i are nonempty, compact, and convex;
- (b) $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1, \dots, y_n) \in \prod_{i=1}^n B_i^0$;
- (c) $T_i \mid A^0$ is an upper semicontinuous multimap with compact values and $T_i(x) \cap B_i^0$ is nonempty for each $x \in A^0$;
- (d) $x_i \notin \text{co}P_i(x)$ for each $x = (x_1, \dots, x_n) \in B$;
- (e) $P_i^{-1}(y)$ is open for each $y \in B_i$.

Then, there exists $b \in B$ such that $P_i(b) = \emptyset$ and, for each continuous, proper, quasiasffine, and surjective self-map $f : A^0 \rightarrow A^0$, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_a^i(f)$ is nonempty and compact. If, in addition, $P_i(z)$ is nonempty for each $z \notin \prod_{i=1}^n \mathfrak{T}_a^i(f)$, then (a, b) is an equilibrium pair in $A^0 \times \prod_{i=1}^n \mathfrak{T}_a^i(f)$.

Proof. Use Theorem 3.8 instead of Theorem 3.4 in the proof of Theorem 4.1. \square

Corollary 4.4. Let $(A, B_i, T_i, P_i)_{i \in I_n}$ be a free n -person game such that A and B_i are subsets of a normed space E , $T_i : A \rightarrow 2^{B_i}$ is a constraint multimap, and $P_i : B \rightarrow 2^{B_i}$ is a preference map where $B := \prod_{j=1}^n B_j$. Assume that A^0 is nonempty, $Y := \prod_{i=1}^n B_i^0$, and for each $i \in I_n$,

- (a) A_i^0 and B_i are nonempty, compact, and convex;
- (b) $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1, \dots, y_n) \in \prod_{i=1}^n B_i^0$;
- (c) $T_i \mid A^0$ is an upper semicontinuous multimap with compact values and $T_i(x) \cap B_i^0$ is nonempty for each $x \in A^0$;
- (d) $x_i \notin \text{co}P_i(x)$ for each $x = (x_1, \dots, x_n) \in B$;
- (e) $P_i^{-1}(y)$ is open for each $y \in B_i$.

Then, there exists $b \in B$ such that $P_i(b) = \emptyset$, and there exists $a \in A^0$ such that the best proximity set \mathfrak{T}_a^i is nonempty and compact. If, in addition, $P_i(z)$ is nonempty for each $z \notin \prod_{i=1}^n \mathfrak{T}_a^i$, then (a, b) is an equilibrium pair in $A^0 \times \prod_{i=1}^n \mathfrak{T}_a^i$.

Remark 4.5. Corollary 4.4 extends and improves [7, Theorem 4.1] by Al-Thagafi and Shahzad and [5, Theorem 4] by Kim and Lee.

Theorem 4.6. Let (A, B, T, P) be a free 1-person game such that A and B are subsets of a normed space E , $T : A \rightarrow 2^{B_0}$ is a constraint multimap, and $P : B \rightarrow 2^B$ is a preference map. Assume that

- (a) A_0 and B are nonempty, compact, and convex;
- (b) $T : A_0 \rightarrow 2^{B_0}$ is a \mathbf{KKM}_0 -multimap (resp., \mathfrak{A}_c^κ -multimap);
- (c) $x \notin \text{co}P(x)$ for each $x \in B$;
- (d) one of the following conditions is satisfied:
 - (d₁) if $z \in P^{-1}(y)$ for some $y \in B$, then there exists some $y' \in B$ such that $z \in \text{int} P^{-1}(y')$;
 - (d₂) for each $y \in B$, $P^{-1}(y)$ is open in B .

Then, there exists $b \in B$ such that $P(b) = \emptyset$ and, for each continuous, proper, quasiasffine, and surjective self-map $f : A_0 \rightarrow A_0$, there exists $a \in A_0$ such that the best proximity set $\mathfrak{T}_a(f)$ is nonempty and compact. If, in addition, $P(z)$ is nonempty for each $z \notin \mathfrak{T}_a(f)$, then (a, b) is an equilibrium pair in $A_0 \times \mathfrak{T}_a(f)$.

Proof. Since A_0 and B_0 are nonempty, compact, and convex, it follows from Theorem 3.4 that there exists $(a, c) \in A_0 \times B_0$ such that $c \in T(a)$ and $d(f(a), T(a)) = \|f(a) - c\| = d(A, B)$ and so $\mathfrak{T}_a(f)$ is nonempty. By Lemma 2.6, there exists $b \in B_0$ such that $P(b) = \emptyset$. As $P(z)$ is nonempty whenever $z \in B \setminus \mathfrak{T}_a(f)$, we conclude that $b \in \mathfrak{T}_a(f)$. So $(a, b) \in A_0 \times B_0$, $b \in T(a)$ and $d(f(a), T(a)) = \|f(a) - b\| = d(A, B)$. Thus (a, b) is an equilibrium pair in $A_0 \times \mathfrak{T}_a(f)$. \square

Corollary 4.7. *Let (A, T, P) be a free 1-person game such that A is a nonempty, compact, and convex subset of a normed space E , $T : A \rightarrow 2^A$ is a constraint multimap, and $P : A \rightarrow 2^A$ is a preference map. Assume that*

- (a) $T : A \rightarrow 2^A$ is a \mathbf{KKM}_0 -multimap (resp., \mathfrak{A}_c^k -multimap);
- (b) $x \notin \text{co}P(x)$ for each $x \in A$;
- (c) one of the following conditions is satisfied:
 - (c₁) if $z \in P^{-1}(y)$ for some $y \in A$, then there exists some $y' \in A$ such that $z \in \text{int } P^{-1}(y')$;
 - (c₂) for each $y \in A$, $P^{-1}(y)$ is open in A .

Then, there exists $b \in A$ such that $P(b) = \emptyset$ and, for each continuous, proper, quasiaffine, and surjective self-map $f : A \rightarrow A$, there exists $a \in A$ such that $f(a) = b$. If, in addition, $P(z)$ is nonempty for each $z \notin \{x \in A : f(x) \in T(x)\}$, then $f(a) \in T(a)$.

Remark 4.8. Corollary 4.7 extends and improves [7, Theorem 4.3] by Al-Thagafi and Shahzad and [5, Theorem 3] by Kim and Lee.

Corollary 4.7 follows also from the following result.

Theorem 4.9. *Let (A, B, T, P) be a free 1-person game such that A and B are subsets of a normed space E , $T : A \rightarrow 2^B$ is a constraint multimap, and $P : B \rightarrow 2^B$ is a preference map. Assume that*

- (a) A_0 and B are nonempty, compact, and convex;
- (b) $T \upharpoonright A_0$ is an upper semicontinuous multimap with compact values and $T(x) \cap B_0$ is nonempty for each $x \in A_0$;
- (c) $x \notin \text{co}P(x)$ for each $x \in B$;
- (d) one of the following conditions is satisfied:
 - (d₁) if $z \in P^{-1}(y)$ for some $y \in B$, then there exists some $y' \in B$ such that $z \in \text{int } P^{-1}(y')$;
 - (d₂) for each $y \in B$, $P^{-1}(y)$ is open in B .

Then, there exists $b \in B$ such that $P(b) = \emptyset$ and, for each continuous, proper, quasiaffine, and surjective self-map $f : A_0 \rightarrow A_0$, there exists $a \in A_0$ such that the best proximity set $\mathfrak{T}_a(f)$ is nonempty and compact. If, in addition, $P(z)$ is nonempty for each $z \notin \mathfrak{T}_a(f)$, then (a, b) is an equilibrium pair in $A_0 \times \mathfrak{T}_a(f)$.

Proof. Use Theorem 3.8 instead of Theorem 3.4 in the proof of Theorem 4.3. □

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