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Research Article

Approximation Methods for Common Fixed Points of Mean Nonexpansive Mapping in Banach Spaces

Zhaohui Gu[1] and Yongjin Li[2]

[1] *Department of Foundation, Guangdong Finance and Economics College, Guangzhou 510420, China*
[2] *Institute of Logic and Cognition, Department of Mathematics, Sun Yat-Sen University, Guangzhou 510275, China*

Correspondence should be addressed to Yongjin Li, stslyj@mail.sysu.edu.cn

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Let X be a uniformly convex Banach space, and let $S,\ T$ be a pair of mean nonexpansive mappings. In this paper, it is proved that the sequence of Ishikawa iterations associated with S and T converges to the common fixed point of S and T.

1. Introduction and preliminaries

Let X be a Banach space and let S, T be mappings from X to X. The pair of mean nonexpansive mappings was introduced by Bose in [1]:

$$\|Sx - Ty\| \le a\|x - y\| + b\{\|x - Sx\| + \|y - Ty\|\} + c\{\|x - Ty\| + \|y - Sx\|\}, \tag{1.1}$$

for all $x, y \in X$, $a, b, c \in [0,1]$, $a + 2b + 2c \le 1$.

The Ishikawa iteration sequence $\{x_n\}$ of S and T was defined by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n Sx_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Ty_n, \end{aligned} \tag{1.2}$$

where $x_0 \in X$, $\alpha_n, \beta_n \in [0,1]$. The recursion formulas (1.2) were first introduced in 1994 by Rashwan and Saddeek [2] in the framework of Hilbert spaces.

In recent years, several authors (see [2–6]) have studied the convergence of iterations to a common fixed point for a pair of mappings. Rashwan has studied the convergence of Mann iterations to a common fixed point (see [5]) and proved that the Ishikawa iterations converge to a unique common fixed point in Hilbert spaces (see [2]). Recently, Ćirić has proved that if the sequence of Ishikawa iterations sequence $\{x_n\}$ associated with S and T converges to p, then p is the common fixed point of S and T (see [7]). In [4, 6], the authors studied the same problem. In [1], Bose defined the pair of mean nonexpansive mappings, and proved the existence of the fixed point in Banach spaces. In particular, he proved the following theorem.

Theorem 1.1 (see [1]). *Let X be a uniformly convex Banach space and K a nonempty closed convex subset of X, $S : K\to K$ and $T : K\to K$ are a pair of mean nonexpansive mappings, and $c \neq 0$. Then,*

(i) *S and T have a common fixed point u;*

(ii) *further, if $b \neq 0$, then*

  (a) *u is the unique common fixed point and unique as a fixed point of each S and T,*

  (b) *the sequence $\{x_n\}$ defined by $x_1 = Sx_0$, $x_2 = Tx_1$, $x_3 = Sx_2 \ldots$, for any $x_0 \in K$, converges strongly to u.*

It is our purpose in this paper to consider an iterative scheme, which converges to a common fixed point of the pair of mean nonexpansive mappings. Theorem 2.1 extends and improves the corresponding results in [1].

2. Main results

Now we prove the following theorem which is the main result of this paper.

Theorem 2.1. *Let X be a uniformly convex Banach space, $S : X\to X$ and $T : X\to X$ are a pair of mean nonexpansive with a nonempty common fixed points set; if $b > 0$, $0 < \alpha \le \alpha_n \le 1/2$, $0 \le \beta_n \le \beta < 1$, then the Ishikawa sequence $\{x_n\}$ converges to the common fixed point of S and T.*

Proof. First, we show that the sequence $\{x_n\}$ is bounded. For a common fixed point p of S and T, we have

$$\begin{aligned} \|Tx-p\| &= \|Tx-Sp\| \\ &\le a\|x-p\| + b\{\|x-Tx\| + \|p-Sp\|\} + c\{\|x-Sp\| + \|p-Tx\|\} \\ &\le a\|x-p\| + b\{\|x-p\| + \|p-Tx\|\} + c\{\|x-Sp\| + \|p-Tx\|\}. \end{aligned} \tag{2.1}$$

Let $L = (a+b+c)/(1-b-c)$, by $a+2b+2c \le 1$, it is easy to see that $a+b+c \le 1-b-c$, thus $0 \le L \le 1$ and $\|Tx-p\| \le L\|x-p\| \le \|x-p\|$.

Similarly, we have $\|Sx-p\| \le L\|x-p\| \le \|x-p\|$,

$$\begin{aligned} \|x_{n+1}-p\| &= \|(1-\alpha_n)x_n + \alpha_n Ty_n - p\| \\ &= \|(1-\alpha_n)(x_n-p) + \alpha_n(Ty_n-p)\| \\ &\le (1-\alpha_n)\|x_n-p\| + \alpha_n\|Ty_n-p\| \\ &\le (1-\alpha_n)\|x_n-p\| + \alpha_n L\|y_n-p\| \\ &\le (1-\alpha_n)\|x_n-p\| + \alpha_n\|(1-\beta_n)x_n + \beta_n Sx_n - p\| \\ &= (1-\alpha_n)\|x_n-p\| + \alpha_n\|(1-\beta_n)(x_n-p) + \beta_n(Sx_n-p)\| \\ &\le (1-\alpha_n)\|x_n-p\| + \alpha_n(1-\beta_n)\|x_n-p\| + \alpha_n\beta_n\|Sx_n-p\| \\ &\le (1-\alpha_n)\|x_n-p\| + \alpha_n(1-\beta_n)\|x_n-p\| + \alpha_n\beta_n\|x_n-p\| \\ &= ((1-\alpha_n) + \alpha_n(1-\beta_n) + \alpha_n\beta_n)\|x_n-p\| = \|x_n-p\|. \end{aligned} \tag{2.2}$$

So

$$\|x_{n+1} - p\| \le \|x_n - p\| \le \|x_{n-1} - p\| \le \cdots \le \|x_0 - p\|. \tag{2.3}$$

Hence, $\{x_n\}$ is bounded.

Second, we show that

$$\lim_{n\to\infty} \|x_n - Ty_n\| = 0. \tag{2.4}$$

We recall that Banach space X is called uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$, where the modulus $\delta(\varepsilon)$ of convexity of X is defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1,\ \|y\| \le 1,\ \|x - y\| \ge \varepsilon\right\}, \tag{2.5}$$

for every ε with $0 \le \varepsilon \le 2$. It is easy to see that Banach space X is uniformly convex if and only if for any $x_n, y_n \in B_X = \{x \mid \|x\| \le 1\}$, $\|x_n + y_n\| \to 2$ implies $\|x_n - y_n\| \to 0$.

Assume that $\lim_{n\to\infty} \|x_n - Ty_n\| \ne 0$, then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a real number $\varepsilon_0 > 0$, such that

$$\|x_{n_k} - Ty_{n_k}\| \ge \varepsilon_0, \quad k = 1, 2, 3, \dots. \tag{2.6}$$

On the other hand, for a common fixed point p of T and S, we have

$$\begin{aligned}
\|x_{n_k} - Ty_{n_k}\| &\le \|x_{n_k} - p\| + \|Ty_{n_k} - p\| \\
&\le \|x_{n_k} - p\| + L\|y_{n_k} - p\| \\
&= \|x_{n_k} - p\| + L\|(1 - \beta_{n_k})x_{n_k} + \beta_{n_k} S x_{n_k} - p\| \\
&= \|x_{n_k} - p\| + L\|(1 - \beta_{n_k})(x_{n_k} - p) + \beta_{n_k}(S x_{n_k} - p)\| \\
&\le \|x_{n_k} - p\| + (1 - \beta_{n_k})L\|x_{n_k} - p\| + \beta_{n_k} L\|S x_{n_k} - p\| \\
&\le (1 + (1 - \beta_{n_k})L + \beta_{n_k} L^2)\|x_{n_k} - p\| \\
&\le (1 + L)\|x_{n_k} - p\| \le 2\|x_{n_k} - p\|.
\end{aligned} \tag{2.7}$$

Thus,

$$\|x_{n_k} - p\| \ge \frac{1}{2}\|x_{n_k} - Ty_{n_k}\| \ge \frac{\varepsilon_0}{2} = \varepsilon_1 > 0. \tag{2.8}$$

Because

$$\begin{aligned}
\|Ty_n - p\| \le \|y_n - p\| &\le \|(1 - \beta_n)x_n + \beta_n S x_n - p\| \\
&= \|(1 - \beta_n)(x_n - p) + \beta_n(S x_n - p)\| \le (1 - \beta_n)\|x_n - p\| + \beta_n\|S x_n - p\| \\
&\le (1 - \beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \le \|x_n - p\|,
\end{aligned} \tag{2.9}$$

we know $\{x_n\}$ is bounded, then there exists $M > 0$, such that $\|x_n - p\| \le M$. Thus, $\|Ty_n - p\| \le \|x_n - p\| \le M$.

Furthermore, we have

$$\left\| \frac{x_{n_k}-p}{\|x_{n_k}-p\|} - \frac{Ty_{n_k}-p}{\|x_{n_k}-p\|} \right\| = \frac{\|x_{n_k}-Ty_{n_k}\|}{\|x_{n_k}-p\|} \geq \frac{\varepsilon_1}{M} > 0. \tag{2.10}$$

From

$$\left\| \frac{x_{n_k}-p}{\|x_{n_k}-p\|} \right\| = 1, \qquad \left\| \frac{Ty_{n_k}-p}{\|x_{n_k}-p\|} \right\| \leq L \leq 1, \tag{2.11}$$

and the fact that X is uniformly convex Banach space, there exists $\delta > 0$, such that

$$\left\| \frac{x_{n_k}-p}{\|x_{n_k}-p\|} + \frac{Ty_{n_k}-p}{\|x_{n_k}-p\|} \right\| \leq 2-\delta. \tag{2.12}$$

Thus,

$$\begin{aligned} \|x_{n_k+1}-p\| &= \|(1-\alpha_{n_k})x_{n_k}+\alpha_{n_k}Ty_{n_k}-p\| \\ &\leq (1-2\alpha_{n_k})\|x_{n_k}-p\| + \|\alpha_{n_k}(x_{n_k}-p)+\alpha_{n_k}(Ty_{n_k}-p)\| \\ &\leq (1-2\alpha_{n_k})\|x_{n_k}-p\| + \alpha_{n_k}\|x_{n_k}-p\| \cdot \left\| \frac{x_{n_k}-p}{\|x_{n_k}-p\|} + \frac{Ty_{n_k}-p}{\|x_{n_k}-p\|} \right\| \\ &\leq (1-2\alpha_{n_k})\|x_{n_k}-p\| + (2-\delta)\alpha_{n_k}\|x_{n_k}-p\| \leq (1-\delta\alpha_{n_k})\|x_{n_k}-p\| \\ &= \|x_{n_k}-p\| - \delta\alpha_{n_k}\|x_{n_k}-p\| \leq \|x_{n_k}-p\| - \delta\alpha\varepsilon_1. \end{aligned} \tag{2.13}$$

Using (2.3), we obtain that

$$\begin{aligned} \|x_{n_k+1}-p\| \leq \|x_{n_k}-p\| - \delta\alpha\varepsilon_1 &\leq \|x_{n_k-1}-p\| - \delta\alpha\varepsilon_1 \\ &\leq \|x_{n_k-2}-p\| - \delta\alpha\varepsilon_1 \leq \cdots \leq \|x_{n_{k-1}+1}-p\| - \delta\alpha\varepsilon_1 \\ &\leq \|x_{n_{k-1}}-p\| - 2\delta\alpha\varepsilon_1. \end{aligned} \tag{2.14}$$

So

$$\|x_{n_k}-p\| \leq \|x_{n_{k-1}}-p\| - \delta\alpha\varepsilon_1 \leq \|x_{n_{k-2}}-p\| - 2\delta\alpha\varepsilon_1 \leq \cdots \leq \|x_{n_1}-p\| - (k-1)\delta\alpha\varepsilon_1. \tag{2.15}$$

Let $k \to \infty$, then we have $\|x_{n_k}-p\| < 0$. It is a contradiction. Hence, $\lim_{n\to\infty}\|x_n - Ty_n\| = 0$.

Third, we show that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{2.16}$$

Since

$$\begin{aligned} \|x_n - Sx_n\| &\leq \|x_n - Ty_n\| + \|Ty_n - Sx_n\| \\ &\leq \|x_n - Ty_n\| + a\|x_n - y_n\| + b\{\|x_n - Sx_n\| + \|y_n - Ty_n\|\} \\ &\quad + c\{\|x_n - Ty_n\| + \|y_n - Sx_n\|\} \\ &= (1+c)\|x_n - Ty_n\| + a\|x_n - y_n\| + b\|x_n - Sx_n\| \\ &\quad + b\|y_n - Ty_n\| + c\|y_n - Sx_n\| \\ &= (1+c)\|x_n - Ty_n\| + a\|(1-\beta_n)x_n + \beta_n Sx_n - x_n\| \\ &\quad + b\|x_n - Sx_n\| + b\|(1-\beta_n)x_n + \beta_n Sx_n - Ty_n\| \\ &\quad + c\|(1-\beta_n)x_n + \beta_n Sx_n - Sx_n\| \\ &\leq (1+c)\|x_n - Ty_n\| + a\beta_n\|x_n - Sx_n\| \\ &\quad + b\|x_n - Sx_n\| + b\beta_n\|x_n - Sx_n\| + b\|x_n - Ty_n\| + c(1-\beta_n)\|x_n - Sx_n\| \\ &= (1+b+c)\|x_n - Ty_n\| + \left(a\beta_n + b + b\beta_n + c(1-\beta_n)\right)\|x_n - Sx_n\|, \end{aligned} \tag{2.17}$$

we have

$$(1 - a\beta_n - b - b\beta_n - c(1 - \beta_n))\|x_n - Sx_n\| \le (1 + b + c)\|x_n - Ty_n\|. \tag{2.18}$$

Let $M_1 = 1 - a\beta_n - b - b\beta_n - c(1 - \beta_n)$, then

$$\begin{aligned} M_1 &= 1 - a\beta_n - b - b\beta_n - c + c\beta_n = 1 - b - c - (a + b - c)\beta_n \\ &\ge a + b + c - (a + b - c)\beta_n = (a + b)(1 - \beta_n) + c(1 + \beta_n) \\ &\ge (a + b)(1 - \beta) + c > 0. \end{aligned} \tag{2.19}$$

So

$$\|x_n - Sx_n\| \le \frac{1 + b + c}{M_1}\|x_n - Ty_n\|. \tag{2.20}$$

Using (2.4), we get that

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{2.21}$$

Forth, we show that if the Ishikawa sequence $\{x_n\}$ converges to some point $p \in X$, then p is the common fixed point of S and T. By

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n Sx_n, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \end{aligned} \tag{2.22}$$

we have $x_n - Ty_n = (1/\alpha_n)(x_{n+1} - x_n)$. Since $\{x_n\}$ is a convergent sequence, we get $\lim_{n\to\infty}\|x_n - Ty_n\| = 0$. It is easy to see that $\|x_n - y_n\| = \beta_n\|x_n - Sx_n\|$ and $\|Sx_n - y_n\| = (1 - \beta_n)\|x_n - Sx_n\|$. On the other hand,

$$\|y_n - Ty_n\| = \|(1 - \beta_n)x_n + \beta_n Sx_n - Ty_n\| \le (1 - \beta_n)\|x_n - Ty_n\| + \beta_n\|Sx_n - Ty_n\|. \tag{2.23}$$

By (1.1), we obtain

$$\begin{aligned} \|Ty_n - Sx_n\| &\le a\|x_n - y_n\| + b\{\|x_n - Sx_n\| + \|y_n - Ty_n\|\} + c\{\|x_n - Ty_n\| + \|y_n - Sx_n\|\} \\ &\le a\beta_n\|x_n - Sx_n\| + b\|x_n - Sx_n\| + b(1 - \beta_n)\|x_n - Ty_n\| \\ &\quad + b\beta_n\|Sx_n - Ty_n\| + c\|x_n - Ty_n\| + c(1 - \beta_n)\|x_n - Sx_n\| \\ &= (a\beta_n + b + c(1 - \beta_n))\|x_n - Sx_n\| \\ &\quad + (b(1 - \beta_n) + c)\|x_n - Ty_n\| + b\beta_n\|Sx_n - Ty_n\|. \end{aligned} \tag{2.24}$$

Since

$$\|x_n - Sx_n\| \le \|Sx_n - Ty_n\| + \|x_n - Ty_n\|, \tag{2.25}$$

we get

$$\begin{aligned} \|Ty_n - Sx_n\| &\le (b(1 - \beta_n) + c + a\beta_n + b + c(1 - \beta_n))\|x_n - Ty_n\| \\ &\quad + (b\beta_n + a\beta_n + b + c(1 - \beta_n))\|Sx_n - Ty_n\|. \end{aligned} \tag{2.26}$$

So

$$(1-b-c-(a+b-c)\beta_n)\|Ty_n - Sx_n\| \le (b(1-\beta_n)+c+a\beta_n+b+c(1-\beta_n))\|x_n - Ty_n\|. \tag{2.27}$$

Let $M_2 = 1-b-c-(a+b-c)\beta_n$, Since $0 \le \beta_n \le \beta < 1$, we have

$$M_2 \ge a+b+c-(a+b-c)\beta_n \ge (a+b)(1-\beta_n)+c(1+\beta_n) \ge (a+b)(1-\beta)+c > 0. \tag{2.28}$$

It is easy to see that

$$b(1-\beta_n)+c+a\beta_n+b+c(1-\beta_n) > 0. \tag{2.29}$$

Note that $\lim_{n\to\infty}\|x_n - Ty_n\| = 0$, then we get

$$\lim_{n\to\infty}\|Sx_n - Ty_n\| = 0, \qquad \lim_{n\to\infty}\|y_n - Ty_n\| = 0. \tag{2.30}$$

So $\lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\beta_n\|Sx_n - x_n\| = 0$.

Let $p = \lim_{n\to\infty}x_n$, then $\lim_{n\to\infty}y_n = p$, $\lim_{n\to\infty}Sx_n = p$, $\lim_{n\to\infty}Ty_n = p$. By (1.1), we have

$$\|Sx_n - Tp\| \le a\|x_n - p\| + b\{\|x_n - Sx_n\| + \|p - Tp\|\} + c\{\|x_n - Tp\| + \|p - Sx_n\|\}. \tag{2.31}$$

Let $n\to\infty$, then we get

$$\|p - Tp\| \le (b+c)\|p - Tp\|. \tag{2.32}$$

Since $b + c < 1$, it follows that

$$\|p - Tp\| = 0, \quad \text{that is } Tp = p. \tag{2.33}$$

Similarly, we can prove that $Sp = p$. So p is the common fixed point of S and T.

Finally, we show that $\{Sx_n\}$ is a Cauchy sequence. For any $m, n \in N$,

$$\begin{aligned}
\|Sx_n - Sx_{n+m}\| &\le \|Sx_n - Ty_{n+m}\| + \|Sx_{n+m} - Ty_{n+m}\| \\
&\le a\|x_n - y_{n+m}\| + b\{\|x_n - Sx_n\| + \|y_{n+m} - Ty_{n+m}\|\} \\
&\quad + c\{\|x_n - Ty_{n+m}\| + \|y_{n+m} - Sx_n\|\} + \|Sx_{n+m} - Ty_{n+m}\| \\
&\le a\{\|x_n - Sx_n\| + \|Sx_n - Sx_{n+m}\| + \|Sx_{n+m} - y_{n+m}\|\} \\
&\quad + b\{\|x_n - Sx_n\| + \|y_{n+m} - Ty_{n+m}\|\} \\
&\quad + c\{\|x_n - Sx_n\| + \|Sx_n - Sx_{n+m}\| \\
&\qquad + \|Sx_{n+m} - Ty_{n+m}\| + \|y_{n+m} - Sx_{n+m}\| \\
&\qquad + \|Sx_{n+m} - Sx_n\|\} + \|Sx_{n+m} - Ty_{n+m}\|.
\end{aligned} \tag{2.34}$$

Since $b > 0$, thus we get $1 - a - 2c > 0$. Simplify, then we have

$$\begin{aligned}
\|Sx_n - Sx_{n+m}\| \le{}& A\|x_n - Sx_n\| + B\|y_{n+m} - Ty_{n+m}\| \\
&+ C\|y_{n+m} - Sx_{n+m}\| + D\|Sx_{n+m} - Ty_{n+m}\|,
\end{aligned} \tag{2.35}$$

where $A = (a+b+c)/(1-a-2c) \ge 0$, $B = b/(1-a-2c) \ge 0$, $C = (a+c)/(1-a-2c) \ge 0$, and $D = (1+c)/(1-a-2c) \ge 0$. By (2.16) and (2.30), we know that

$$\|x_n - Sx_n\| \longrightarrow 0, \qquad \|y_{n+m} - Ty_{n+m}\| \longrightarrow 0, \qquad \|Sx_{n+m} - Ty_{n+m}\| \longrightarrow 0. \tag{2.36}$$

So it is easy to see that $\|y_{n+m} - Sx_{n+m}\| \to 0$. Thus, $\|Sx_n - Sx_{n+m}\| \to 0$, that is $\{Sx_n\}$ is a Cauchy sequence. Hence, there exists p, such that $p = \lim_{n\to\infty} Sx_n$. We know that $p = \lim_{n\to\infty} x_n$ and p is the common fixed point of S and T. This completes the proof of the theorem. □

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