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Research Article

Composite Implicit General Iterative Process for a Nonexpansive Semigroup in Hilbert Space

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Let C be nonempty closed convex subset of real Hilbert space H. Consider C a nonexpansive semigroup $\Im = \{T(s) : s \geq 0\}$ with a common fixed point, a contraction f with coefficient $0 < \alpha < 1$, and a strongly positive linear bounded operator A with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \overline{\gamma}/\alpha$. It is proved that the sequence $\{x_n\}$ generated iteratively by $x_n = (I - \alpha_n A)(1/t_n)\int_0^{t_n} T(s)y_n ds + \alpha_n \gamma f(x_n), y_n = (I - \beta_n A)x_n + \beta_n \gamma f(x_n)$ converges strongly to a common fixed point $x^* \in F(\Im)$ which solves the variational inequality $\langle (\gamma f - A)x^*, z - x^* \rangle \leq 0$ for all $z \in F(\Im)$.

1. Introduction and preliminaries

Let C be a closed convex subset of a Hilbert space H, recall that $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) := \{x \in C : Tx = x\}$.

Recall that a family $\Im = \{T(s) \mid 0 \leq s < \infty\}$ of mappings from C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;

(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;

(iv) for all $x \in C, s \mid \to T(s)x$ is continuous.

We denote by $F(\Im)$ the set of all common fixed points of $\Im$, that is, $F(\Im) = \cap_{0 \leq s < \infty} F(T(s))$. It is known that $F(\Im)$ is closed and convex.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems (see, e.g., [1–5] and the references therein). A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C}\frac{1}{2}\langle Ax,x\rangle-\langle x,b\rangle, \tag{1.1}$$

where C is the fixed point set of a nonexpansive mapping T on H, and b is a given point in H. Assume that A is strongly positive, that is, there is a constant $\overline{\gamma}>0$ with the property

$$\langle Ax,x\rangle\geq\overline{\gamma}\|x\|^2\quad\forall x\in H. \tag{1.2}$$

It is well known that $F(T)$ is closed convex (cf. [6]). In[3] (see also [4]), it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0\in H$ chosen arbitrarily,

$$x_{n+1}=(I-\alpha_nA)Tx_n+\alpha_nb,\quad n\geq 0 \tag{1.3}$$

converges strongly to the unique solution of the minimization problem (1.1) provided that the sequence $\{\alpha_n\}$ satisfies certain conditions.

On the other hand, Moudafi [7] introduced the viscosity approximation method for nonexpansive mappings (see [8] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_0\in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1}=(1-\sigma_n)Tx_n+\sigma_nf(x_n),\quad n\geq 0, \tag{1.4}$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$. It is proved [7, 8] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.4) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle(I-f)x^*,x-x^*\rangle\geq 0,\quad x\in C. \tag{1.5}$$

Recently, Marino and Xu [9] combined the iterative method (1.3) with the viscosity approximation method (1.4) considering the following general iteration process:

$$x_{n+1}=(I-\alpha_nA)Tx_n+\alpha_n\gamma f(x_n),\quad n\geq 0, \tag{1.6}$$

and proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$

generated by (1.6) converges strongly to the unique solution of the variational inequality

$$\langle (A-\gamma f)x^*, x-x^*\rangle \geq 0, \quad x\in C, \tag{1.7}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C}\frac{1}{2}\langle Ax,x\rangle - h(x), \tag{1.8}$$

where h is a potential function for γf (i.e., $h'(x)=\gamma f(x)$, for $x\in H$).

In this paper, motivated and inspired by the idea of Marino and Xu [9], we introduce the composite implicit general iteration process (1.9) as follows:

$$\begin{aligned} x_n &= (I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds + \alpha_n\gamma f(x_n),\\ y_n &= (I-\beta_n A)x_n + \beta_n\gamma f(x_n), \end{aligned} \tag{1.9}$$

where $\{\alpha_n\},\{\beta_n\}\subset(0,1)$, and investigate the problem of approximating common fixed point of nonexpansive semigroup $\{T(s) : s\geq 0\}$ which solves some variational inequality. The results presented in this paper extend and improve the main results in Marino and Xu [9], and the methods of proof given in this paper are also quite different.

In what follows, we will make use of the following lemmas. Some of them are known; others are not hard to derive.

Lemma 1.1 (Marino and Xu [9]). *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma}>0$ and $0<\rho\leq\|A\|^{-1}$. Then $\|I-\rho A\|\leq 1-\overline{\gamma}$.*

Lemma 1.2 (Shimizu and Takashi [10]). *Let C be a nonempty bounded closed convex subset of H and let $\Im=\{T(s) : 0\leq s<\infty\}$ be a nonexpansive semigroup on C, then for any $h\geq 0$,*

$$\lim_{t\to\infty}\sup_{x\in C}\left\|\frac{1}{t}\int_0^t T(s)x ds - T(h)\left(\frac{1}{t}\int_0^t T(s)x ds\right)\right\| = 0. \tag{1.10}$$

Lemma 1.3. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\Im=\{T(t) : 0\leq t<\infty\}$ be a nonexpansive semigroup on C. If $\{x_n\}$ is a sequence in C satisfying the following properties:*

(i) $x_n \rightharpoonup z$;

(ii) $\limsup_{t\to\infty}\limsup_{n\to\infty}\|T(t)x_n - x_n\| = 0$,

where $x_n\rightharpoonup z$ denote that $\{x_n\}$ converges weakly to z, then $z\in F(\Im)$.

Proof. This lemma is the continuous version of Lemma 2.3 of Tan and Xu [11]. This proof given in [11] is easily extended to the continuous case. □

2. Main results

Lemma 2.1. *Let H be a Hilbert space, C a closed convex subset of H, let $\Im = \{T(s) : s \geq 0\}$ be a nonexpansive semigroup on C, $\{t_n\} \subset (0,\infty)$ is a sequence, then $I - (1/t_n)\int_0^{t_n} T(s)ds$ is monotone.*

Proof. In fact, for all $x, y \in H$,

$$\begin{aligned}
&\left\langle x-y, \left(I - \frac{1}{t_n}\int_0^{t_n} T(s)ds\right)x - \left(I - \frac{1}{t_n}\int_0^{t_n} T(s)ds\right)y\right\rangle \\
&\quad = \|x-y\|^2 - \left\langle x-y, \frac{1}{t_n}\int_0^{t_n} T(s)xds - \frac{1}{t_n}\int_0^{t_n} T(s)yds\right\rangle \\
&\quad \geq \|x-y\|^2 - \|x-y\|\frac{1}{t_n}\int_0^{t_n}\left\|T(s)x - T(s)y\right\|ds \\
&\quad \geq \|x-y\|^2 - \|x-y\|^2 = 0.
\end{aligned} \tag{2.1}$$

□

Theorem 2.2. *Let C be nonempty closed convex subset of real Hilbert space H, suppose that $f : C \to C$ is a fixed contractive mapping with coefficient $0 < \alpha < 1$, and $\Im = \{T(s) : s \geq 0\}$ is a nonexpansive semigroup on C such that $F(\Im)$ is nonempty, and A is a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{t_n\} \subset (0,\infty)$ are real sequences such that*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \beta_n = \circ(\alpha_n), \qquad \lim_{n\to\infty} t_n = \infty, \tag{2.2}$$

then for any $0 < \gamma < \overline{\gamma}/\alpha$, there is a unique $\{x_n\} \in C$ such that

$$\begin{gathered}
x_n = (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds + \alpha_n \gamma f(x_n), \\
y_n = (I - \beta_n A)x_n + \beta_n \gamma f(x_n),
\end{gathered} \tag{2.3}$$

and the iteration process $\{x_n\}$ converges strongly to the unique solution $x^* \in F(\Im)$ of the variational inequality $\langle (\gamma f - A)x^*, z - x^*\rangle \leq 0$ for all $z \in F(\Im)$.

Proof. Our proof is divided into five steps.

Since $\alpha_n \to 0$, $\beta_n \to 0$ as $n \to \infty$, we may assume, with no loss of generality, that $\alpha_n < \|A\|^{-1}$, $\beta_n < \|A\|^{-1}$ for all $n \geq 1$.

(i) $\{x_n\}$ is bounded.

Firstly, we will show that the mapping $T_n^f : C \to C$ defined by

$$T_n^f = (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)\left[(I - \beta_n A) + \beta_n \gamma f\right]ds + \alpha_n \gamma f \tag{2.4}$$

is a contraction. Indeed, from Lemma 1.1, we have for any $x, y \in C$ that

$$\begin{aligned}
\|T_n^f x - T_n^f y\| &\le \|I - \alpha_n A\| \frac{1}{t_n}\int_0^{t_n} \|T(s)[(I-\beta_n A)x + \beta_n \gamma f(x)] \\
&\quad - T(s)[(I-\beta_n A)y + \beta_n\gamma f(y)]\| ds + \alpha_n \gamma \|f(x) - f(y)\| \\
&\le (1-\alpha_n\overline{\gamma}) \|[(I-\beta_n A)x + \beta_n\gamma f(x)] - [(I-\beta_n A)y + \beta_n\gamma f(y)]\| + \alpha_n\gamma\alpha\|x-y\| \\
&\le (1-\alpha_n\overline{\gamma})[\|I-\beta_n A\|\|x-y\| + \beta_n\gamma\alpha\|x-y\|] + \alpha_n\gamma\alpha\|x-y\| \\
&\le (1-\alpha_n\overline{\gamma})[1-\beta_n(\overline{\gamma}-\gamma\alpha)]\|x-y\| + \alpha_n\gamma\alpha\|x-y\| \\
&= \{(1-\alpha_n\overline{\gamma})[1-\beta_n(\overline{\gamma}-\gamma\alpha)] + \alpha_n\gamma\alpha\}\|x-y\| \\
&= \{[1-\alpha_n(\overline{\gamma}-\gamma\alpha)] - (1-\alpha_n\overline{\gamma})\beta_n(\overline{\gamma}-\gamma\alpha)\}\|x-y\| \\
&< [1-\alpha_n(\overline{\gamma}-\gamma\alpha)]\|x-y\| < \|x-y\|.
\end{aligned} \tag{2.5}$$

Let $x_n \in C$ be the unique fixed point of T_n^f. Thus,

$$\begin{aligned}
x_n &= (I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s) y_n ds + \alpha_n\gamma f(x_n), \\
y_n &= (I-\beta_n A)x_n + \beta_n\gamma f(x_n)
\end{aligned} \tag{2.6}$$

is well defined. Next, we will show that $\{x_n\}$ is bounded.

Pick any $z \in F(\mathfrak{I})$ to obtain

$$\begin{aligned}
\|x_n - z\| &= \left\|(I-\alpha_n A)\left(\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - z\right) + \alpha_n(\gamma f(x_n) - Az)\right\| \\
&\le \|I-\alpha_n A\|\frac{1}{t_n}\int_o^{t_n}\|T(s)y_n - z\|ds + \alpha_n[\gamma\|f(x_n)-f(z)\| + \|\gamma f(z) - Az\|] \\
&\le (1-\alpha_n\overline{\gamma})\|y_n - z\| + \alpha_n[\gamma\|f(x_n)-f(z)\| + \|\gamma f(z) - Az\|],
\end{aligned} \tag{2.7}$$

$$\|x_n - z\| \le (1-\alpha_n\overline{\gamma})\|y_n - z\| + \alpha_n\gamma\alpha\|x_n - z\| + \alpha_n\|\gamma f(z) - Az\|. \tag{2.8}$$

Also

$$\begin{aligned}
\|y_n - z\| &\le \|I-\beta_n A\|\|x_n - z\| + \beta_n\|\gamma f(x_n) - Az\| \\
&\le (1-\beta_n\overline{\gamma})\|x_n - z\| + \beta_n\gamma\alpha\|x_n - z\| + \beta_n\|\gamma f(z) - Az\| \\
&= [1-\beta_n(\overline{\gamma}-\gamma\alpha)]\|x_n - z\| + \beta_n\|\gamma f(z) - Az\|.
\end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.8), we obtain that

$$\begin{aligned}\|x_n - z\| &\le (1-\alpha_n\overline{\gamma})\left[1-\beta_n(\overline{\gamma}-\gamma\alpha)\right]\|x_n - z\| + (1-\alpha_n\overline{\gamma})\beta_n\|\gamma f(z) - Az\| \\ &\quad + \alpha_n\gamma\alpha\|x_n - z\| + \alpha_n\|\gamma f(z) - Az\| \\ &= \left\{(1-\alpha_n\overline{\gamma})\left[1-\beta_n(\overline{\gamma}-\gamma\alpha)\right] + \alpha_n\gamma\alpha\right\}\|x_n - z\| \\ &\quad + \left[(1-\alpha_n\overline{\gamma})\beta_n + \alpha_n\right]\|\gamma f(z) - Az\| \\ &= \left\{1-(\overline{\gamma}-\gamma\alpha)\left[\alpha_n + (1-\alpha_n\overline{\gamma})\beta_n\right]\right\}\|x_n - z\| \\ &\quad + \left[\alpha_n + (1-\alpha_n\overline{\gamma})\beta_n\right]\|\gamma f(z) - Az\|, \\ (\overline{\gamma}-\gamma\alpha)\left[\alpha_n + (1-\alpha_n\overline{\gamma})\beta_n\right]\|x_n - z\| &\le \left[\alpha_n + (1-\alpha_n\overline{\gamma})\beta_n\right]\|\gamma f(z) - Az\|, \\ \|x_n - z\| &\le \frac{1}{\overline{\gamma}-\gamma\alpha}\|\gamma f(z) - Az\|.\end{aligned} \tag{2.10}$$

Thus $\{x_n\}$ is bounded.

(ii) $\lim_{n\to\infty}\|x_n - T(s)x_n\| = 0$.

Denote that $z_n := (1/t_n)\int_0^{t_n} T(s)y_n ds$, since $\{x_n\}$ is bounded, $\|z_n - z\| \le \|y_n - z\|$ and $\{Az_n\}$, $\{f(x_n)\}$ are also bounded, From (2.6) and $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\|x_n - z_n\| = \alpha_n\|\gamma f(x_n) - Az_n\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{2.11}$$

Let $K = \{w \in C : \|w - z\| \le (1/(\overline{\gamma}-\gamma\alpha))\|\gamma f(z) - Az\|\}$, then K is a nonempty bounded closed convex subset of C and $T(s)$-invariant. Since $\{x_n\} \subset K$ and K is bounded, there exists $r > 0$ such that $K \subset B_r$, it follows from Lemma 1.2 that

$$\lim_{n\to\infty}\|z_n - T(s)z_n\| = 0 \quad \forall s \ge 0. \tag{2.12}$$

From (2.11) and (2.12), we have

$$\lim_{n\to\infty}\|x_n - T(s)x_n\| = 0. \tag{2.13}$$

(iii) There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in F(\mathfrak{I})$ and x^* is the unique solution of the following variational inequality:

$$\langle (A-\gamma f)x^*, x^* - z\rangle \le 0 \quad \forall z \in F(\mathfrak{I}). \tag{2.14}$$

Firstly since

$$\|y_n - x_n\| = \beta_n\|\gamma f(x_n) - A(x_n)\|. \tag{2.15}$$

From condition $\beta_n \to 0$ and the boundedness of $\{x_n\}$, we obtain that $\|y_n - x_n\| \to 0$. Again by boundedness of $\{x_n\}$, we know that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \rightharpoonup x^*$. Then $y_{n_k} \rightharpoonup x^*$. From Lemma 1.3 and step (ii), we have that $x^* \in F(\mathfrak{I})$.

Next we will prove that x^* solves the variational inequality (2.14). Since

$$x_n = (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s) y_n ds + \alpha_n \gamma f(x_n), \tag{2.16}$$

we derive that

$$(A - \gamma f)x_n = -\frac{1}{\alpha_n}(I - \alpha_n A)\left(I - \frac{1}{t_n}\int_0^{t_n} T(s) ds\right) y_n + \frac{1}{\alpha_n}\left[(I - \alpha_n A) y_n - (I - \alpha_n A) x_n\right]. \tag{2.17}$$

It follows that, for all $z \in F(\mathfrak{I})$,

$$\begin{aligned}
\langle (A - \gamma f)x_n, y_n - z\rangle &= -\frac{1}{\alpha_n}\left\langle (I - \alpha_n A)\left(I - \frac{1}{t_n}\int_0^{t_n} T(s) ds\right) y_n, y_n - z\right\rangle \\
&\quad + \frac{1}{\alpha_n}\langle (I - \alpha_n A) y_n - (I - \alpha_n A) x_n, y_n - z\rangle \\
&= -\frac{1}{\alpha_n}\left\langle \left(I - \frac{1}{t_n}\int_0^{t_n} T(s) ds\right) y_n - \left(I - \frac{1}{t_n}\int_0^{t_n} T(s) ds\right) z, y_n - z\right\rangle \\
&\quad + \left\langle A\left(I - \frac{1}{t_n}\int_0^{t_n} T(s) ds\right) y_n, y_n - z\right\rangle \\
&\quad + \frac{1}{\alpha_n}\langle (I - \alpha_n A) y_n - (I - \alpha_n A) x_n, y_n - z\rangle.
\end{aligned} \tag{2.18}$$

Using Lemma 2.1, we have from (2.18) that

$$\begin{aligned}
\left\langle (A - \gamma f)x_n, y_n - z\right\rangle &\le \left\langle A\left(I - \frac{1}{t_n}\int_0^{t_n} T(s) ds\right) y_n, y_n - z\right\rangle + \frac{1}{\alpha_n}\langle (I - \alpha_n A)(y_n - x_n), y_n - z\rangle \\
&\le \left\langle A\left(I - \frac{1}{t_n}\int_0^{t_n} T(s) ds\right) y_n, y_n - z\right\rangle + \frac{1}{\alpha_n}\beta_n \|\gamma f(x_n) - A(x_n)\| \|y_n - z\|.
\end{aligned} \tag{2.19}$$

Now replacing n in (2.19) with n_k and letting $k \to \infty$, we notice that

$$\left(I - \frac{1}{t_{n_k}}\int_0^{t_{n_k}} T(s) ds\right) y_{n_k} \rightharpoonup 0, \tag{2.20}$$

and from condition $\beta_n = \circ(\alpha_n)$ and boundedness of $\{x_n\}$, we have

$$\frac{1}{\alpha_{n_k}}\beta_{n_k}\|\gamma f(x_{n_k}) - A(x_{n_k})\|\|y_{n_k} - z\| \longrightarrow 0. \tag{2.21}$$

For $x^* \in F(\Im)$, we obtain

$$\langle (A-\gamma f)x^*, x^* - z\rangle \le 0. \tag{2.22}$$

From [9, Theorem 3.2], we know that the solution of the variational inequality (2.14) is unique. That is, $x^* \in F(\Im)$ is a unique solution of (2.14).

(iv)

$$\limsup_{n\to\infty}\left\langle \frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - x^*, \gamma f(x^*) - Ax^*\right\rangle \le 0, \tag{2.23}$$

where x^* is obtained in step (iii).

To see this, there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\begin{aligned}\limsup_{n\to\infty}&\left\langle \frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - x^*, \gamma f(x^*) - Ax^*\right\rangle \\ &= \lim_{i\to\infty}\left\langle \frac{1}{t_{n_i}}\int_0^{t_{n_i}} T(s)y_{n_i} ds - x^*, \gamma f(x^*) - Ax^*\right\rangle,\end{aligned} \tag{2.24}$$

we may also assume that $x_{n_i} \rightharpoonup z$, then $(1/t_{n_i})\int_0^{t_{n_i}} T(s)y_{n_i} ds \rightharpoonup z$, note from step (ii) that $z \in F(\Im)$ in virtue of Lemma 1.2. It follows from the variational inequality (2.14) that

$$\limsup_{n\to\infty}\left\langle \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x^*, \gamma f(x^*) - Ax^*\right\rangle = \langle z - x^*, \gamma f(x^*) - Ax^*\rangle \le 0. \tag{2.25}$$

So (2.23) holds thank to (2.14).

(v) $x_n \to x^*$ $(n\to\infty)$.

Finally, we will prove $x_n \to x^*$. Since

$$\begin{aligned}\|y_n - x^*\|^2 &= \|(I-\beta_n A)(x_n - x^*) + \beta_n(\gamma f(x_n) - Ax^*)\|^2 \\ &\le \|I - \beta_n A\|\|x_n - x^*\|^2 + \beta_n\|\gamma f(x_n) - Ax^*\|^2 \\ &\le (1-\beta_n\overline{\gamma})\|x_n - x^*\|^2 + \beta_n\|\gamma f(x_n) - Ax^*\|^2.\end{aligned} \tag{2.26}$$

Next, we calculate

$$\begin{aligned}\|x_n - x^*\|^2 &= \left\|(I-\alpha_n A)\left(\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - x^*\right) + \alpha_n(\gamma f(x_n) - Ax^*)\right\|^2 \\ &= \left\|(I-\alpha_n A)\left(\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - x^*\right)\right\|^2 + \alpha_n^2\|(\gamma f(x_n) - Ax^*)\|^2 \\ &\quad + 2\alpha_n\langle (I-\alpha_n A)\left(\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - x^*\right), \gamma f(x_n) - Ax^*\rangle \\ &\le (1-\alpha_n\overline{\gamma})^2\|y_n - x^*\|^2 + \alpha_n^2\|(\gamma f(x_n) - Ax^*)\|^2 \\ &\quad + 2\alpha_n\langle (I-\alpha_n A)\left(\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - x^*\right), \gamma f(x_n) - Ax^*\rangle.\end{aligned} \tag{2.27}$$

Thus it follows from (2.26) that

$$\begin{aligned}\|x_n - x^*\|^2 &\le (1-\alpha_n\overline{\gamma})^2(1-\beta_n\overline{\gamma})\|x_n - x^*\|^2 + (1-\alpha_n\overline{\gamma})^2\beta_n\|\gamma f(x_n) - Ax^*\|^2 \\ &\quad + \alpha_n^2\|\gamma f(x_n) - Ax^*\|^2 + 2\alpha_n\left\langle \frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - x^*, \gamma f(x_n) - Ax^*\right\rangle \\ &\quad - 2\alpha_n^2\left\langle A\left(\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - x^*\right), \gamma f(x_n) - Ax^*\right\rangle \\ &\le (1-\alpha_n\overline{\gamma})^2(1-\beta_n\overline{\gamma})\|x_n - x^*\|^2 + (1-\alpha_n\overline{\gamma})^2\beta_n\|\gamma f(x_n) - Ax^*\|^2 \\ &\quad + \alpha_n^2\|\gamma f(x_n) - Ax^*\|^2 + 2\alpha_n\gamma\left\langle \frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - x^*, f(x_n) - f(x^*)\right\rangle \\ &\quad + 2\alpha_n\left\langle \frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - x^*, \gamma f(x^*) - Ax^*\right\rangle \\ &\quad - 2\alpha_n^2\left\langle A\left(\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - x^*\right), \gamma f(x_n) - Ax^*\right\rangle \\ &\le (1-\alpha_n\overline{\gamma})^2(1-\beta_n\overline{\gamma})\|x_n - x^*\|^2 + 2\alpha_n\gamma\alpha\|y_n - x^*\|\|x_n - x^*\| \\ &\quad + (1-\alpha_n\overline{\gamma})^2\beta_n\|\gamma f(x_n) - Ax^*\|^2 \\ &\quad + \alpha_n\Bigg[2\left\langle \frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - x^*, \gamma f(x^*) - Ax^*\right\rangle \\ &\qquad + \alpha_n\left(\|\gamma f(x_n) - Ax^*\|^2 + 2\left\|A\left(\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - x^*\right)\right\|\cdot\|\gamma f(x_n) - Ax^*\|\right)\Bigg]\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n\overline{\gamma})^2(1-\beta_n\overline{\gamma})\|x_n-x^*\|^2+2\alpha_n\gamma\alpha[1-\beta_n(\overline{\gamma}-\gamma\alpha)]\|x_n-x^*\|^2\\
&\quad+2\alpha_n\gamma\alpha\beta_n\|\gamma f(x^*)-Ax^*\|\|x_n-x^*\|+(1-\alpha_n\overline{\gamma})^2\beta_n\|\gamma f(x_n)-Ax^*\|^2\\
&\quad+\alpha_n\Bigg[2\left\langle\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*,\gamma f(x^*)-Ax^*\right\rangle\\
&\qquad+\alpha_n\Bigg(\|\gamma f(x_n)-Ax^*\|^2+2\left\|A\left(\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*\right)\right\|\cdot\|\gamma f(x_n)-Ax^*\|\Bigg)\Bigg]\\
&=(1-\alpha_n\overline{\gamma})^2(1-\beta_n\overline{\gamma})\|x_n-x^*\|^2+2\alpha_n\gamma\alpha(1-\beta_n\overline{\gamma})\|x_n-x^*\|\\
&\quad+2\alpha_n\beta_n\alpha^2\gamma^2\|x_n-x^*\|^2+2\alpha_n\gamma\alpha\beta_n\|\gamma f(x^*)-Ax^*\|\|x_n-x^*\|\\
&\quad+(1-\alpha_n\overline{\gamma})^2\beta_n\|\gamma f(x_n)-Ax^*\|^2\\
&\quad+\alpha_n\Bigg[2\left\langle\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*,\gamma f(x^*)-Ax^*\right\rangle\\
&\qquad+\alpha_n\Bigg(\|\gamma f(x_n)-Ax^*\|^2+2\left\|A\left(\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*\right)\right\|\cdot\|\gamma f(x_n)-Ax^*\|\Bigg)\Bigg]\\
&=\left[(1-\alpha_n\overline{\gamma})^2+2\alpha_n\gamma\alpha\right](1-\beta_n\overline{\gamma})\|x_n-x^*\|^2+2\alpha_n\beta_n\alpha^2\gamma^2\|x_n-x^*\|^2\\
&\quad+2\alpha_n\gamma\alpha\beta_n\|\gamma f(x^*)-Ax^*\|\|x_n-x^*\|+(1-\alpha_n\overline{\gamma})^2\beta_n\|\gamma f(x_n)-Ax^*\|^2\\
&\quad+\alpha_n\Bigg[2\left\langle\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*,\gamma f(x^*)-Ax^*\right\rangle\\
&\qquad+\alpha_n\Bigg(\|\gamma f(x_n)-Ax^*\|^2+2\left\|A\left(\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*\right)\right\|\cdot\|\gamma f(x_n)-Ax^*\|\Bigg)\Bigg]\\
&<\left[(1-\alpha_n\overline{\gamma})^2+2\alpha_n\gamma\alpha\right]\|x_n-x^*\|^2+2\alpha_n\beta_n\alpha^2\gamma^2\|x_n-x^*\|^2\\
&\quad+2\alpha_n\gamma\alpha\beta_n\|\gamma f(x^*)-Ax^*\|\|x_n-x^*\|+(1-\alpha_n\overline{\gamma})^2\beta_n\|\gamma f(x_n)-Ax^*\|^2\\
&\quad+\alpha_n\Bigg[2\left\langle\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*,\gamma f(x^*)-Ax^*\right\rangle\\
&\qquad+\alpha_n\Bigg(\|\gamma f(x_n)-Ax^*\|^2+2\alpha_n\left\|A\left(\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*\right)\right\|\\
&\qquad\qquad\cdot\|\gamma f(x_n)-Ax^*\|\Bigg)\Bigg]\\
&=\left[1-2(\overline{\gamma}-\gamma\alpha)\alpha_n\right]\|x_n-x^*\|^2+2\alpha_n\beta_n\alpha^2\gamma^2\|x_n-x^*\|^2\\
&\quad+2\alpha_n\gamma\alpha\beta_n\|\gamma f(x^*)-Ax^*\|\|x_n-x^*\|+(1-\alpha_n\overline{\gamma})^2\beta_n\|\gamma f(x_n)-Ax^*\|^2\\
&\quad+\alpha_n\Bigg[2\left\langle\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*,\gamma f(x^*)-Ax^*\right\rangle\\
&\qquad+\alpha_n\Bigg(\|\gamma f(x_n)-Ax^*\|^2+2\left\|A\left(\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*\right)\right\|\\
&\qquad\qquad\cdot\|\gamma f(x_n)-Ax^*\|+\overline{\gamma}^2\|x_n-x^*\|^2\Bigg)\Bigg].
\end{aligned}
\tag{2.28}
$$

Thus

$$\begin{aligned}
2(\overline{\gamma}-\gamma\alpha)\|x_n-x^*\|^2 &\leq 2\beta_n\alpha^2\gamma^2\|x_n-x^*\|^2+2\gamma\alpha\beta_n\|\gamma f(x^*)-Ax^*\|\cdot\|x_n-x^*\| \\
&\quad+(1-\alpha_n\overline{\gamma})^2\frac{\beta_n}{\alpha_n}\|\gamma f(x_n)-Ax^*\|^2 \\
&\quad+\Bigg[2\left\langle\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*,\gamma f(x^*)-Ax^*\right\rangle \\
&\qquad+\alpha_n\Bigg(\|\gamma f(x_n)-Ax^*\|^2+2\left\|A\left(\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*\right)\right\| \\
&\qquad\qquad\cdot\|\gamma f(x_n)-Ax^*\|+\gamma^2\|x_n-x^*\|^2\Bigg)\Bigg] \\
&\leq \beta_n[2\alpha^2\gamma^2\|x_n-x^*\|^2+2\gamma\alpha\|\gamma f(x^*)-Ax^*\|\cdot\|x_n-x^*\|] \\
&\quad+\frac{\beta_n}{\alpha_n}\|\gamma f(x_n)-Ax^*\|^2 \\
&\quad+\Bigg[2\left\langle\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*,\gamma f(x^*)-Ax^*\right\rangle \\
&\qquad+\alpha_n\Bigg(\|\gamma f(x_n)-Ax^*\|^2+2\left\|A\left(\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*\right)\right\| \\
&\qquad\qquad\cdot\|\gamma f(x_n)-Ax^*\|+\gamma^2\|x_n-x^*\|^2\Bigg)\Bigg].
\end{aligned} \tag{2.29}$$

Since $\{x_n\}$ is bounded, we can take a constant $L_1, L_2, L_3 > 0$ such that

$$\begin{aligned}
&L_1 \geq 2\alpha^2\gamma^2\|x_n-x^*\|^2+2\gamma\alpha\|\gamma f(x^*)-Ax^*\|\cdot\|x_n-x^*\|, \\
&L_2 \geq \|\gamma f(x_n)-Ax^*\|^2, \\
&L_3 \geq \|\gamma f(x_n)-Ax^*\|^2+2\left\|A\left(\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*\right)\right\|\cdot\|\gamma f(x_n)-Ax^*\|+\gamma^2\|x_n-x^*\|^2
\end{aligned} \tag{2.30}$$

for all $n \geq 0$. It then follows from (2.29) that

$$2(\overline{\gamma}-\gamma\alpha)\|x_n-x^*\|^2 \leq \beta_n L_1+\frac{\beta_n}{\alpha_n}L_2+2\left\langle\frac{1}{t_n}\int_0^{t_n}T(s)y_n ds-x^*,\gamma f(x^*)-Ax^*\right\rangle+\alpha_n L_3. \tag{2.31}$$

Then

$$2(\overline{\gamma}-\gamma\alpha)\limsup_{n\to\infty}\|x_n-x^*\|^2 \le \limsup_{n\to\infty}\left(\beta_n L_1+\frac{\beta_n}{\alpha_n}L_2+\alpha_n L_3\right)+2\limsup_{n\to\infty}\left\langle \frac{1}{t_n}\int_0^{t_n}T(s)y_n ds - x^*, \gamma f(x^*)-Ax^*\right\rangle. \tag{2.32}$$

From condition $\alpha_n \to 0$, $\beta_n = \circ(\alpha_n)$ and (2.23), we conclude that

$$\limsup_{n\to\infty}\|x_n-x^*\|^2 \le 0. \tag{2.33}$$

So $x_n \to x^*$. This completes the proof of the Theorem 2.2. □

It follows from the above proof that Theorem 2.2 is valid for nonexpansive mappings. Thus, we have that Corollaries 2.3 and 2.4 are two special cases of Theorem 2.2.

Corollary 2.3. *Let T be a nonexpansive mapping from nonempty closed convex subset C of a Hilbert space H to C, $\{x_n\}$ is generated by the following algorithm:*

$$\begin{aligned} x_n &= (I-\alpha_n A)Ty_n + \alpha_n\gamma f(x_n), \\ y_n &= (I-\beta_n A)x_n + \beta_n\gamma f(x_n), \end{aligned} \tag{2.34}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ are real sequences such that

$$\lim_{n\to\infty}\alpha_n = 0, \qquad \beta_n = \circ(\alpha_n), \tag{2.35}$$

then for any $0<\gamma<\overline{\gamma}/\alpha$, the sequence $\{x_n\}$ above converges strongly to the unique solution $x^* \in F(T)$ of the variational inequality $\langle(\gamma f - A)x^*, z-x^*\rangle \le 0$ for all $z \in F(T)$.

Corollary 2.4. *Let T be a nonexpansive mapping from nonempty closed convex subset C of a Hilbert space H to C, $\{x_n\}$ is generated by the following algorithm:*

$$\begin{aligned} x_n &= (I-\alpha_n A)Ty_n + \alpha_n\gamma f(x_n), \\ y_n &= (I-\beta_n A)x_n + \beta_n\gamma f(x_n), \end{aligned} \tag{2.36}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following condition: $\lim_{n\to\infty}\alpha_n = 0$, then the sequence $\{x_n\}$ converges strongly to the unique solution $x^* \in F(T)$ of the variational inequality $\langle(I-f)x^*, x^*-z\rangle \le 0$ for all $z \in F(T)$.

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